On the Dirichlet problem in the Cegrell classes

by Rafał Czyż (Kraków) and Per Åhag (Taipei)

Abstract. Let $\mu$ be a non-negative measure with finite mass given by $\varphi(\ddc \psi)^n$, where $\psi$ is a bounded plurisubharmonic function with zero boundary values and $\varphi \in L^q((\ddc \psi)^n)$, $\varphi \geq 0$, $1 \leq q \leq \infty$. The Dirichlet problem for the complex Monge–Ampère operator with the measure $\mu$ is studied.

1. Introduction. Let $\Omega \subseteq \mathbb{C}^n$, $n \geq 2$, be a bounded hyperconvex domain, i.e., a bounded, connected and open set that admits a negative plurisubharmonic exhaustion function. A bounded plurisubharmonic function $u$ defined on $\Omega$ belongs to the class $\mathcal{E}_0$ if

$$\lim_{z \to \xi} u(z) = 0$$

for every $\xi \in \partial \Omega$, and

$$\int_\Omega (\ddc u)^n < \infty,$$

where $(\ddc \cdot)^n$ is the complex Monge–Ampère operator. Let the measure $\mu$ defined on $\Omega$ be given by

$$\mu = \varphi(\ddc \psi)^n,$$

where $\varphi \in L^q((\ddc \psi)^n)$, $\varphi \geq 0$, $1 \leq q \leq \infty$ and $\psi \in \mathcal{E}_0$. It is proved in Theorem 5.11 of [4] that every non-negative measure $\tilde{\mu}$ defined on $\Omega$ can be decomposed into

$$(1.1) \quad \tilde{\mu} = \varphi(\ddc \tilde{\psi})^n + \nu,$$

where $\tilde{\psi} \in \mathcal{E}_0$ and $\varphi \in L^1_{\text{loc}}((\ddc \tilde{\psi})^n)$, $\varphi \geq 0$. The non-negative measure $\nu$ is such that there exists a pluripolar set $A \subseteq \Omega$ such that $\nu(A) = \nu(\Omega)$. For $q = 1$, the measure $\mu$ has finite mass and it puts no mass on pluripolar sets, by (1.1). Lemma 5.14 in [4] implies that there exists a unique function $u \in \mathcal{F}$

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such that $(dd^cu)^n = \mu$ as measures defined on $\Omega$ (see Definition 2.1 for the class $\mathcal{F}$).

In this article some results concerning the solution of this Dirichlet problem will be obtained for the case $q > 1$. If $q > 1$ is such that $n(q - 1) \geq 1$, then our Theorem 2.3 states that the unique solution $u \in \mathcal{F}$ belongs to $\mathcal{F}_p$, where $p = n(q - 1)$ (see Definition 2.1 for the class $\mathcal{F}_p$). On the other hand, if $q > 1$ is such that $n(q - 1) < 1$, then it is proved in Theorem 2.4 that the unique solution $u \in \mathcal{F}$ is such that
\[
\int_{\Omega} (-u)^p (dd^cu)^n < \infty,
\]
where $p = n(q - 1)$. Moreover there exists a decreasing sequence $[u_j]_{j=1}^{\infty}$, $u_j \in \mathcal{E}_0$, which converges pointwise to $u$ on $\Omega$ as $j$ tends to $\infty$ and satisfies
\[
\sup_j \int_{\Omega} (-u_j)^p (dd^cu_j)^n < \infty, \quad \sup_j \int_{\Omega} (dd^cu_j)^n < \infty.
\]

If there can be no misinterpretation a sequence $[\cdot]_{j=1}^{\infty}$ will be denoted by $[\cdot]$. The results of Theorems 2.3 and 2.4 will be extended to the corresponding classes with continuous boundary values. The note ends by recalling a theorem for the case when $q = \infty$ (see Theorem 2.7). For an introduction to pluripotential theory we recommend [7].

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2. Dirichlet problem. A domain is an open and connected set. A domain $\Omega \subseteq \mathbb{C}^n$ is called hyperconvex if there exists a plurisubharmonic function $\varphi : \Omega \to (-\infty, 0)$ such that the closure of the set
\[
\{z \in \Omega : \varphi(z) < c\}
\]
is compact in $\Omega$ for every $c \in (-\infty, 0)$. Throughout this note $\Omega$ will be a bounded hyperconvex domain in $\mathbb{C}^n$, $n \geq 2$.

Definition 2.1. Define $\mathcal{F} (= \mathcal{F}(\Omega))$ to be the class of plurisubharmonic functions $\varphi$ defined on $\Omega$ such that there exists a decreasing sequence $[\varphi_j]$, $\varphi_j \in \mathcal{E}_0$, which converges pointwise to $\varphi$ on $\Omega$ as $j$ tends to $\infty$, and
\[
\sup_j \int_{\Omega} (dd^c\varphi_j)^n < \infty.
\]

Let $p \geq 1$. If $[\varphi_j]$ can also be chosen such that
\[
\sup_j \int_{\Omega} (-\varphi_j)^p (dd^c\varphi_j)^n < \infty,
\]
then $\varphi$ is said to be in the class $\mathcal{F}_p (= \mathcal{F}_p(\Omega))$. 
The classes $\mathcal{F}_p$ were first introduced in [3] and $\mathcal{F}$ in [4]. These classes are two of the so-called Cegrell classes. For further information about the Cegrell classes see, e.g., [3]–[6] and [9].

**Lemma 2.2.** If $\phi, \psi \in \mathcal{E}_0$, then for each $p \geq 0$,

$$
\int_{\Omega} (-\phi)^{p+n}(dd^c \psi)^n \leq C \int_{\Omega} (-\phi)^{p}(dd^c \phi)^n,
$$

where $C \geq 0$ is a constant, depending only on $n, p$ and the supremum of $\psi$.

**Proof.** Cf. [1] (see also [2]).

**Theorem 2.3.** Let $\psi \in \mathcal{E}_0$ and $\varphi \in L^q((dd^c \psi)^n)$, $\varphi \geq 0$, $n \geq 2$. If $1 < q < \infty$ is such that $n(q-1) \geq 1$, then there exists a unique function $u \in \mathcal{F}_p$ such that

$$(dd^c u)^n = \varphi(dd^c \psi)^n,$$

where $p = n(q-1)$.

**Proof.** Let $\phi \in \mathcal{E}_0$. Hölder’s inequality implies that

$$
\int_{\Omega} (-\phi)^p \varphi(dd^c \psi)^n \leq \left( \int_{\Omega} \varphi^q(dd^c \psi)^n \right)^{1/q} \left( \int_{\Omega} (-\phi)^{pq/(q-1)}(dd^c \psi)^n \right)^{(q-1)/q} = C_1 \left( \int_{\Omega} (-\phi)^{p+n}(dd^c \psi)^n \right)^{p/(n+p)},
$$

where $C_1 \geq 0$ is a constant and $p = n(q-1)$. Since $p \geq 1$ by assumption, it follows from Lemma 2.2 that

$$
\int_{\Omega} (-\phi)^{p+n}(dd^c \psi)^n \leq C_2 \int_{\Omega} (-\phi)^p(dd^c \phi)^n,
$$

where $C_2 \geq 0$ is a constant. Inequalities (2.1) and (2.2) imply that there exists a constant $A$ such that

$$
\int_{\Omega} (-\phi)^p \varphi(dd^c \psi)^n \leq A \left( \int_{\Omega} (-\phi)^p(dd^c \phi)^n \right)^{p/(p+n)}
$$

for every $\phi \in \mathcal{E}_0$, hence Theorem 5.1 in [3] shows that there exists a unique $u \in \mathcal{F}_p$ such that

$$(dd^c u)^n = \varphi(dd^c \psi)^n.$$
where $p = n(q - 1)$. Moreover there exists a decreasing sequence $[u_j]$, $u_j \in \mathcal{E}_0$, which converges pointwise to $u$ on $\Omega$ as $j$ tends to $\infty$ and satisfies

$$\sup_j \int_{\Omega} (-u_j)^p (dd^c u_j)^n < \infty, \quad \sup_j \int_{\Omega} (dd^c u_j)^n < \infty.$$ 

**Proof.** Let $\phi \in \mathcal{E}_0$. By using the same technique as in the proof of Theorem 2.3 it follows that there exists a constant $A$ not depending on $\phi$ such that

$$\int_{\Omega} (\phi) \varphi(dd^c \psi)^n \leq A \left( \int_{\Omega} (\phi)^p (dd^c \phi)^n \right)^{p/(n+p)}.$$

In particular, this shows that the measure $\varphi(dd^c \psi)^n$ vanishes on pluripolar sets. Lemma 5.14 in [4] implies that there exists a unique $u \in \mathcal{F}$ such that

$$(dd^c u)^n = \varphi(dd^c \psi)^n,$$

since the given measure has finite total mass. Let $u_j \in \mathcal{E}_0$ be such that

$$(dd^c u_j)^n = \min(j, \varphi)(dd^c \psi)^n.$$

The comparison principle shows that the sequence $[u_j]$ is decreasing and converges pointwise to $u$ on $\Omega$ as $j$ tends to $\infty$. Inequality (2.3) implies that

$$\int_{\Omega} (-u_j)^p (dd^c u_j)^n = \int_{\Omega} (-u_j)^p \min(j, \varphi)(dd^c \psi)^n \leq \int_{\Omega} (-u_j)^p (dd^c u)^n \leq A \left( \int_{\Omega} (-u_j)^p (dd^c u_j)^n \right)^{p/(n+p)}.$$

This yields

$$\int_{\Omega} (-u_j)^p (dd^c u_j)^n \leq A^{(p+n)/n},$$

hence

$$\sup_j \int_{\Omega} (-u_j)^p (dd^c u_j)^n \leq A^{(p+n)/n}.$$ 

Inequality (2.3) implies that

$$\int_{\Omega} (-u_j)^p (dd^c u)^n \leq A \left( \int_{\Omega} (-u_j)^p (dd^c u_j)^n \right)^{p/(n+p)} \leq A^{(p+n)/n}.$$ 

By the monotone convergence theorem,

$$\lim_{j \to \infty} \int_{\Omega} (-u_j)^p (dd^c u)^n = \int_{\Omega} (-u)^p (dd^c u)^n.$$

Combining (2.4) and (2.5) yields

$$\int_{\Omega} (-u)^p (dd^c u)^n \leq A^{(p+n)/n}.$$
The next step is to generalize the results in Theorems 2.3 and 2.4 to some more general classes of bounded plurisubharmonic functions. Recall that the Perron–Bremermann envelope is defined by

\[ PB_f(z) := \sup \{ w(z) : w \in PSH(\Omega), \limsup_{\zeta \to \xi} w(\zeta) \leq f(\xi) \ \forall \xi \in \partial \Omega \}, \]

where \( PSH(\Omega) \) denotes the class of plurisubharmonic functions defined on \( \Omega \) and \( f : \partial \Omega \to \mathbb{R} \) is a given function. Recall that if \( \Omega \) is a bounded hyperconvex domain and \( f : \partial \Omega \to \mathbb{R} \) is a continuous function, then \( PB_f \in PSH(\Omega) \).

**Definition 2.5.** Let \( \mathcal{K} \in \{ \mathcal{E}_0, \mathcal{F}_p, \mathcal{F} \} \) and \( f : \partial \Omega \to \mathbb{R} \) be a continuous function such that

\[ \lim_{z \to \xi} PB_f(z) = f(\xi), \]

for every \( \xi \in \partial \Omega \). A plurisubharmonic function \( u \) defined on \( \Omega \) belongs to the class \( \mathcal{K}(f) (= \mathcal{K}(\Omega, f)) \) if there exists a function \( \varphi \in \mathcal{K} \) such that

\[ PB_f \geq u \geq \varphi + PB_f. \]

**Remark.** If \( \mathcal{K} \in \{ \mathcal{E}_0, \mathcal{F}_p, \mathcal{F} \} \), then \( \mathcal{K}(0) = \mathcal{K} \).

**Remark.** Theorem 2.3 is also valid for the Cegrell class \( \mathcal{F}_p(f) \).

The classes \( \mathcal{E}_0(f) \) and \( \mathcal{F}_p(f) \) were first introduced in [3] and the class \( \mathcal{F}(f) \) in [9].

**Theorem 2.6.** Assume that \( f : \partial \Omega \to \mathbb{R} \) is a continuous function such that

\[ \lim_{z \to \xi} PB_f(z) = f(\xi) \]

for every \( \xi \in \partial \Omega \), and \( PB_f + PB_{-f} \in \mathcal{E}_0 \). Let \( \psi \in \mathcal{E}_0 \) and \( \varphi \in L^q((dd^c\psi)^n) \), \( \varphi \geq 0, \ n \geq 2 \). If \( 1 < q < \infty \) is such that \( n(q - 1) < 1 \), then there exists a unique function \( u \in \mathcal{F}(f) \) such that \( (dd^c u)^n = \varphi (dd^c \psi)^n \) and

\[ \int_{\Omega} (-u - PB_{-f})^p (dd^c u)^n < \infty, \]

where \( p = n(q - 1) \). Moreover there exists a decreasing sequence \( \{u_j\}, u_j \in \mathcal{E}_0(f) \), which converges pointwise to \( u \) on \( \Omega \) as \( j \) tends to \( \infty \) and satisfies

\[ \sup_j \int_{\Omega} (-u_j - PB_{-f})^p (dd^c u_j)^n < \infty, \ \sup_j \int_{\Omega} (dd^c u_j)^n < \infty. \]

**Proof.** From Theorem 2.4 it follows that there exists \( v \in \mathcal{F} \) such that

\[ (dd^c v)^n = \varphi (dd^c \psi)^n, \ \int_{\Omega} (-v)^p (dd^c v)^n < \infty. \]
where \( p = n(q - 1) \). Theorem 7.4 in [9] implies that there exists \( u \in \mathcal{F}(f) \) such that \((dd^cu)^n = \varphi(dd^c\psi)^n\). The function \( v + \text{PB}_f \) belongs to \( \mathcal{F}(f) \) and
\[
(dd^cu)^n = (dd^cv)^n \leq (dd^c(v + \text{PB}_f))^n,
\]
hence
\[
v + \text{PB}_f \leq u,
\]
by Corollary 7.7 in [9]. Thus
\[
\int_{\Omega} (-u - \text{PB}_{-f})^p(dd^cu)^n \leq \int_{\Omega} (-v - \text{PB}_f - \text{PB}_{-f})^p(dd^cv)^n < \infty.
\]

Now for the second part of the theorem. Theorem 4.10 in [9] together with the assumption that \( \text{PB}_f + \text{PB}_{-f} \in \mathcal{E}_0 \) implies that there exists \( u_j \in \mathcal{E}_0(f) \) such that
\[
(dd^cu_j)^n = \min(j, \varphi)(dd^c\psi)^n.
\]
Moreover \([u_j]\) is a decreasing sequence which converges pointwise to \( u \) on \( \Omega \) as \( j \to \infty \), and
\[
\sup_j \int_{\Omega} (dd^cu_j)^n < \infty.
\]
Furthermore, \( u_j + \text{PB}_{-f} \in \mathcal{E}_0 \). The assertion then follows by repeating the argument in the proof of Theorem 2.4. \( \blacksquare \)

**Theorem 2.7.** Let \( \psi \in \mathcal{E}_0 \) and \( \varphi \in L^q((dd^c\psi)^n) \), \( \varphi \geq 0 \), \( n \geq 2 \). If \( q = \infty \), then there exists a unique \( u \in \mathcal{E}_0(f) \) such that \((dd^cu)^n = \varphi(dd^c\psi)^n\). In particular, if \( f = 0 \), then there exists a unique \( u \in \mathcal{E}_0 \) such that \((dd^cu)^n = \varphi(dd^c\psi)^n\).

**Proof.** There exists a constant \( c \geq 0 \) such that
\[
\varphi(dd^c\psi)^n \leq c(dd^c\psi)^n = (dd^c(c^{1/n}\psi))^n,
\]
hence there exists a unique \( u \in \mathcal{E}_0(f) \) such that \((dd^cu)^n = \varphi(dd^c\psi)^n\), by the proof of Theorem 7.4 in [9]. \( \blacksquare \)

**Remark.** Theorem 2.7 is a special case of Theorem A in [8].

**References**


Institute of Mathematics
Jagiellonian University
Reymonta 4
30-059 Kraków, Poland
E-mail: Rafał.Czyż@im.uj.edu.pl

Institute of Mathematics
Academia Sinica
Taipei 11529, Taiwan
E-mail: per@math.sinica.edu.tw

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