ANNALES
POLONICI MATHEMATICI
106 (2012)

Radially symmetric plurisubharmonic functions

by Per Åhag (Umeå), Rafał Czyż (Kraków)
and Leif Persson (Umeå)

Dedicated to Professor Józef Siciak
on the occasion of his 80th birthday

Abstract. In this note we consider radially symmetric plurisubharmonic functions and the complex Monge–Ampère operator. We prove among other things a complete characterization of unitary invariant measures for which there exists a solution of the complex Monge–Ampère equation in the set of radially symmetric plurisubharmonic functions. Furthermore, we prove in contrast to the general case that the complex Monge–Ampère operator is continuous on the set of radially symmetric plurisubharmonic functions. Finally we characterize radially symmetric plurisubharmonic functions among the subharmonic ones using merely the laplacian.

1. Introduction. Let $\psi \geq 0$ be a smooth radially symmetric function defined in a neighborhood of the unit ball $\mathbb{B}$ in $\mathbb{C}^2$. In 1975, Kerzman noticed that the function u defined by

$$u(z) = -\frac{4}{\sqrt{\omega}} \int_{|z|}^{1} \Big[\int_{|z|<r} \psi(z)\, dV_4 \Big]^{1/2} \frac{dr}{r}, \qquad z \in \mathbb{B} \subset \mathbb{C}^2,$$

where ω is the area of the unit sphere in $\mathbb{C}^2$ and dV_4 is the Lebesgue measure in $\mathbb{C}^2$, is a radially symmetric plurisubharmonic function satisfying

$$\text{(1.1)} \qquad \begin{aligned} \det\left(\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}\right) &= \psi \quad \text{on } \mathbb{B}, \\ u &= 0 \quad \text{on } \partial\mathbb{B} \end{aligned}$$

(see [12]). In other words, Kerzman gave an explicit integral representation formula for the solution of the simple Dirichlet problem given in (1.1). Later

2010 *Mathematics Subject Classification*: Primary 32U05, 32W20; Secondary 31B05.
Key words and phrases: complex Monge–Ampère operator, plurisubharmonic function, Dirichlet problem, radially symmetric function, subharmonic function.

DOI: 10.4064/ap106-0-1

Monn generalized this result to the unit ball in $\mathbb{C}^n$ and to more general functions ψ (see [17, 18], and also Theorem 3.1). One cannot expect a similar formula for other domains like for example the polydisc. This is since the assumption of radial symmetry reduces the problem to a question concerning properties of one-variable convex functions. The representation formula is remarkably simple, and this makes it an efficient tool for constructing examples.

In this note we collect several results concerning radially symmetric plurisubharmonic functions, some of which, we believe, are known to experts but have never been written down. Let ∂, $\bar\partial$ be the usual differential operators, $d = \partial + \bar\partial$ and $d^c = i(\bar\partial - \partial)$. Furthermore, let $\mathcal{PSH}^R(\mathbb{B})$ denote the set of functions defined on $\mathbb{B}$ that are *non-positive*, radially symmetric, and plurisubharmonic, and let $\mathcal{M}^R$ denote the set of non-negative Radon measures μ defined on $\mathbb{B}$ such that there exists a function $u \in \mathcal{PSH}^R(\mathbb{B})$ with $(dd^cu)^n = \mu$. In Section 3 we prove the following characterization.

THEOREM 3.9. *Let μ be a unitary invariant measure defined on $\mathbb{B}$, and let $F(t) = (2\pi)^{-n}\mu(\mathbb{B}_t)$. Then the following assertions are equivalent:*

(1) $\mu \in \mathcal{M}^R$,

(2) $\displaystyle\int_{1/2}^{1} \sqrt[n]{F(t)}\, dt < \infty$.

In [6], Cegrell showed that the complex Monge–Ampère operator $(dd^c\,\cdot\,)^n$ is not continuous. But if we restrict our attention to $\mathcal{PSH}^R(\mathbb{B})$, then it is continuous in the following sense: let $\{u_j\}$, $u_j \in \mathcal{PSH}^R(\mathbb{B})$, be a sequence that converges pointwise to a function u in $\mathcal{PSH}^R(\mathbb{B})$; then the associated sequence $\{(dd^cu_j)^n\}$ tends to $(dd^cu)^n$ in the weak*-topology (Theorem 4.1). Furthermore, we note that in $\mathcal{PSH}^R(\mathbb{B})$ locally uniform convergence, pointwise convergence, weak convergence, convergence in the sense of distribution and convergence in capacity are all equivalent (Corollary 4.4).

Finally we characterize radially symmetric plurisubharmonic functions among the subharmonic ones using merely the laplacian. We prove the following theorem.

THEOREM 5.1. *Let u be a radially symmetric subharmonic function defined on $\mathbb{B} \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$, $n \geq 2$, and set $G(t) = \Delta u(\mathbb{B}_t)/\sigma(\partial\mathbb{B}_t)$. Then u is plurisubharmonic if, and only if, the function $t \mapsto tG(t)$ is increasing.*

It would be of great interest to obtain a characterization of those subharmonic functions that are plurisubharmonic without the assumption of radial symmetry.

For further information on classical and pluripotential theory see e.g. [2] and [9, 15, 16] which are all excellent references. Kiselman's historical survey [14] is highly recommended for those interested in the history of plurisubharmonic functions and potential theory in several complex variables.

2. Preliminaries. For $r > 0$ let $\mathbb{B}(z_0, r) = \{z \in \mathbb{C}^n : |z - z_0| < r\}$, and to simplify the notation set $\mathbb{B} = \mathbb{B}(0,1)$, and $\mathbb{B}_r = \mathbb{B}(0,r)$. A function $u : \mathbb{B} \to [-\infty, +\infty)$ is said to be *radially symmetric* if

$$u(z) = u(|z|) \quad \text{for all } z \in \mathbb{B}.$$

For each radially symmetric function $u : \mathbb{B} \to [-\infty, +\infty)$ we define the function $\tilde{u} : [0,1) \to [-\infty, +\infty)$ by

$$\tilde{u}(t) = u(|z|), \quad \text{where } t = |z|. \tag{2.1}$$

On the other hand, to every function $\tilde{v} : [0,1) \to [-\infty, +\infty)$ we can associate a radially symmetric function v through (2.1).

REMARK. Let $U(n)$ denote the unitary group of degree n. A function u defined on $\mathbb{B}$ is radially symmetric if, and only if, it is unitary invariant, i.e. $u \circ T = u$ for all $T \in U(n)$.

Now let us recall some basic facts on convex functions. For further information we refer to [5] and [11]. Let $I \subset \mathbb{R}$ be an interval. A function $f : I \to [-\infty, \infty)$ is called *convex* if

$$f(\lambda x + (1-\lambda)x) \leq \lambda f(x) + (1-\lambda) f(y) \tag{2.2}$$

for all $x, y \in I$ and all $\lambda \in [0,1]$. We say that a function $f : [0,1) \to [-\infty, \infty)$ is *convex with respect to* $\ln r$ if the function $r \mapsto f(e^r)$ is convex. We would like to emphasize that the notion of being convex with respect to $\ln r$ is not the same as being logarithmically convex: the latter means that the function f is positive and $\ln f$ is convex.

LEMMA 2.1. *If $u \in \mathcal{PSH}^R(\mathbb{B})$, then*

1. *$\tilde{u}$ is an increasing function, and convex with respect to* $\ln r$,
2. *the limit* $\lim_{t \to 1^-} \tilde{u}(t)$ *exists,*
3. $\{z \in \mathbb{B} : u(z) = -\infty\} \subset \{0\}$, *and*
4. *u is continuous.*

Proof. For any plurisubharmonic function we can consider

$$t \mapsto \int\limits_{\partial \mathbb{B}(z,t)} u \, d\sigma_t = M(u, z, t),$$

where $d\sigma_t$ is the normalized Lebesgue measure on $\partial \mathbb{B}(z,t)$. This is an increasing function that is convex with respect to $\ln r$ (see e.g. [11]). By assumption

u is radially symmetric, and therefore

$$\tilde{u}(t) = M(u, 0, t).$$

Furthermore, since $\tilde{u}$ is increasing, and bounded above by 0, the limit $\lim_{t\to 1^-} \tilde{u}(t)$ exists. That $\tilde{u}$ is bounded above by 0 follows from the assumption that $u \in \mathcal{PSH}^R(\mathbb{B})$. Assertion (3) follows from the fact that $\{z \in \mathbb{B} : u(z) = -\infty\}$ is pluripolar, and u is radially symmetric. The function $r \mapsto \tilde{u}(e^r)$ is convex, hence (see e.g. [11]) is continuous, and therefore u is continuous on $\mathbb{B}$. ∎

Remark. Let $u \in \mathcal{PSH}^R(\mathbb{B})$. Then by property (2) of Lemma 2.1 there is no loss of generality to assume that

$$\liminf_{\substack{z\to\xi \\ z\in\mathbb{B}}} u(z) = \limsup_{\substack{z\to\xi \\ z\in\mathbb{B}}} u(z) = 0 \quad \text{for all } \xi \in \partial\mathbb{B}. \tag{2.3}$$

We shall therefore hereon assume (2.3).

Theorem 2.2. *Let u be a radially symmetric function defined on $\mathbb{B}$ that satisfies* (2.3). *Then the following assertions are equivalent:*

(1) $u \in \mathcal{PSH}^R(\mathbb{B})$,
(2) *$\tilde{u}$ is an increasing function that is convex with respect to* $\ln r$.

Proof. The implication (1)⇒(2) follows from Lemma 2.1.

(2)⇒(1). Note that u is plurisubharmonic at 0, since $\tilde{u}$ is an increasing function. For the moment assume that u is smooth, and let $\mathcal{L}(u, X)$ be the Levi form of u at $X \in \mathbb{C}^n$. Then for all $X \in \mathbb{C}^n$, and all $z \neq 0$, we have

$$\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(z) = u_{j\bar{k}}(z) = \frac{1}{4|z|^3}\bar{z}_j z_k\big(|z|\tilde{u}''(|z|) - \tilde{u}'(|z|)\big) + \delta_{jk}\frac{\tilde{u}'(|z|)}{2|z|}, \tag{2.4}$$

where δ_{jk} is the Kronecker delta, and

$$\begin{aligned}\mathcal{L}(u, X) = \sum_{j,k=1}^{n} u_{j\bar{k}}(z) X_j \overline{X}_k = {}& \frac{|z|\tilde{u}'(|z|) + |z|^2\tilde{u}''(|z|)}{4|z|^4}|\langle z, X\rangle|^2 \\ &+ \frac{\tilde{u}'(|z|)}{2|z|^3}(|z|^2|X|^2 - |\langle z, X\rangle|^2) \geq 0,\end{aligned}$$

since $\tilde{u}(t)$ is an increasing function, and convex with respect to $\ln r$. Thus, u is plurisubharmonic on $\mathbb{B}$. In the general case let $\{v_j\}$ be a sequence of smooth increasing functions that are convex with respect to $\ln r$ and such that $\{v_j\}$ decreases pointwise to $\tilde{u}$. The sequence $\{v_j\}$ can possibly be defined on smaller intervals $[0, 1 - 1/j)$. For the existence of such a sequence $\{v_j\}$ see e.g. [11]. We observe that $\{u_j\}$ defined by $u_j(z) = v_j(|z|)$ is a sequence of smooth plurisubharmonic radially symmetric functions decreasing to u. Thus, u is plurisubharmonic. ∎

3. The complex Monge–Ampère equation. In [18], Monn proved the following result (we give a proof here for completeness):

THEOREM 3.1. *Let $u \in \mathcal{PSH}^R(\mathbb{B}_{r_2} \setminus \bar{\mathbb{B}}_{r_1}) \cap \mathcal{C}^2(\mathbb{B}_{r_2})$ with $0 \le r_1 < r_2 \le 1$. Then for $r_1 \le r < s \le r_2$,*

$$\tilde{u}(r) - \tilde{u}(s) = \int_r^s \frac{-2}{t}\Big(2n\int_0^t x^{2n-1} f(x)\,dx\Big)^{1/n} dt, \tag{3.1}$$

where $f(z) = f(|z|)$ is given by $(dd^c u)^n = 4^n n! f(z) dV_{2n}$, and dV_{2n} is the $2n$-dimensional Lebesgue measure on $\mathbb{B}$.

Proof. Note that if $u(z) = u(|z|) = \tilde{u}(t)$ is a smooth radially symmetric plurisubharmonic function, then for $t = |z|$ we have, by (2.4),

$$\det(u_{j\bar{k}}) = \frac{1}{4}\Big(\frac{\tilde{u}'(t)}{2t}\Big)^{n-1}\Big(\tilde{u}''(t) + \frac{\tilde{u}'(t)}{t}\Big). \tag{3.2}$$

Since

$$(dd^c u)^n = 4^n n! \det(u_{j\bar{k}}) dV_{2n},$$

it is enough to solve the following non-linear ordinary differential equation:

$$t\tilde{u}''(t)\tilde{u}'(t)^{n-1} + \tilde{u}'(t)^n = 2^{n+1} t^n f(t)$$

and therefore

$$\tilde{u}'(t)^n = \frac{n2^{n+1}}{t^n}\int_0^t x^{2n-1} f(x)\,dx.$$

Thus,

$$\tilde{u}(r) - \tilde{u}(s) = \int_r^s \frac{-2}{t}\Big(2n\int_0^t x^{2n-1} f(x)\,dx\Big)^{1/n} dt.$$

The function $\tilde{u}$ given above is increasing and convex with respect to $\ln r$, since

$$\tilde{u}(e^r) - \tilde{u}(e^s) = \int_r^s -2\Big(2n\int_0^{e^t} x^{2n-1} f(x)\,dx\Big)^{1/n} dt. \ \blacksquare$$

DEFINITION 3.2. Let $f : \mathbb{B} \to \mathbb{R}$ be a smooth, non-negative, radially symmetric function, and set $\mu = f dV_{2n}$. We define the function $F : [0, 1) \to \mathbb{R}$ by

$$F(t) = \frac{1}{2^{n-1}(n-1)!}\int_0^t x^{2n-1} f(x)\,dx. \tag{3.3}$$

REMARK. This construction yields $F(t) = (2\pi)^{-n}\mu(\mathbb{B}_t)$.

Now we prove the representation formula for all radially symmetric plurisubharmonic functions. Recall that since for all $u \in \mathcal{PSH}^R(\mathbb{B})$ we have

$\lim_{z\to\xi} u(z) = 0$, $\xi \in \partial\mathbb{B}$, it follows from [8] that the complex Monge–Ampère operator is well defined on $\mathcal{PSH}^R(\mathbb{B})$.

THEOREM 3.3. *If* $u \in \mathcal{PSH}^R(\mathbb{B})$ *with assumption* (2.3), *then*

$$\tilde{u}(r) = \int_r^1 -\frac{1}{t}\sqrt[n]{F(t)}\,dt, \tag{3.4}$$

where

$$F(t) = \frac{1}{(2\pi)^n}(dd^c u)^n(\mathbb{B}_t).$$

Proof. For $u \in \mathcal{PSH}^R(\mathbb{B})$ take a regularizing sequence $\{u_j\}$ with $u_j \in \mathcal{PSH}^R \cap \mathcal{C}^\infty(\mathbb{B}_{1-1/j})$ as in the proof of Theorem 2.2. Theorem 3.1 shows that for $0 \le r \le s \le 1 - 1/j$ we have

$$\tilde{u}_j(r) - \tilde{u}_j(s) = \int_r^s -\frac{1}{t}\sqrt[n]{F_j(t)}\,dt,$$

where

$$F_j(t) = \frac{1}{(2\pi)^n}(dd^c u_j)^n(\mathbb{B}_t).$$

Since F is an increasing function, it is differentiable almost everywhere. If F is differentiable at t then we get $(dd^c u)^n(\mathbb{B}_t) = (dd^c u)^n(\bar{\mathbb{B}}_t)$. The sequence $\{(dd^c u_j)^n\}$ tends to $(dd^c u)^n$ in the weak*-topology and therefore

$$\limsup_{j\to\infty}(dd^c u_j)^n(\bar{\mathbb{B}}_t) \le (dd^c u)^n(\bar{\mathbb{B}}_t) = (dd^c u)^n(\mathbb{B}_t) \le \liminf_{j\to\infty}(dd^c u_j)^n(\bar{\mathbb{B}}_t).$$

Hence, $\lim_{j\to\infty} F_j(t) = F(t)$ almost everywhere. By the dominated convergence theorem

$$\begin{aligned}\tilde{u}(r) - \tilde{u}(s) &= \lim_{j\to\infty}(\tilde{u}_j(r) - \tilde{u}_j(s)) \\ &= \lim_{j\to\infty}\int_r^s -\frac{1}{t}\sqrt[n]{F_j(t)}\,dt = \int_r^s -\frac{1}{t}\sqrt[n]{F(t)}\,dt.\end{aligned}$$

Letting $s \to 1^-$ we obtain (3.4). ∎

As a direct consequence of the above representation formula we get the following corollary.

COROLLARY 3.4. *There is at most one solution in* $\mathcal{PSH}^R(\mathbb{B})$ *for the complex Monge–Ampère equation* $(dd^c u)^n = \mu$.

It is well known that uniqueness of the solution of the complex Monge–Ampère equation fails when the measure has mass on a pluripolar set, even if we are considering the unit ball in $\mathbb{C}^n$ and $\mu = \delta_0$, the Dirac measure at the origin (see [3]). If we restrict our attention to $\mathcal{PSH}^R(\mathbb{B})$, then we do have uniqueness. We have even more: by Lemma 2.1, if the Monge–Ampère measure of such a function does charge a pluripolar set, then this set must be the origin. Now we prove, in an explicit way, that $u(z) = \ln|z|$ is the unique radial solution to $(dd^c u)^n = (2\pi)^n \delta_0$.

PROPOSITION 3.5. *Let δ_0 denote the Dirac measure at the origin in $\mathbb{C}^n$. Then there exists a uniquely determined function u in $\mathcal{PSH}^R(\mathbb{B})$ such that $(dd^c u)^n = (2\pi)^n \delta_0$. Furthermore, the function is given explicitly by $u(z) = \ln|z|$.*

Proof. First note that for $\mathbb{B} \ni z \neq 0$ the measure

$$\frac{1}{(2\pi)^n} dd^c \max(\ln|z|, \ln|w|) \wedge (dd^c \ln|w|)^{n-1}$$

is the normalized Lebesgue measure on the sphere $|w| = |z|$. Then for any $u \in \mathcal{PSH}^R(\mathbb{B})$ we have, using integration by parts,

$$\begin{aligned} u(z) &= \frac{1}{(2\pi)^n} \int_{\mathbb{B}} u(w) dd^c \max(\ln|z|, \ln|w|) \wedge (dd^c \ln|w|)^{n-1} \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{B}} \max(\ln|z|, \ln|w|)\, dd^c u \wedge (dd^c \ln|w|)^{n-1} \\ &\geq \ln|z| \frac{1}{(2\pi)^n} \int_{\mathbb{B}} dd^c u \wedge (dd^c \ln|w|)^{n-1}, \end{aligned}$$

which means that there exists a constant $c = c(u)$ such that $u(z) \geq c \ln|z|$. We shall proceed by proving that $dd^c u \wedge (dd^c \ln|z|)^{n-1} = (2\pi)^n \delta_0$. We have (see [7, 9])

$$\int_{\mathbb{B}} dd^c u \wedge (dd^c \ln|z|)^{n-1} \leq \Big(\int_{\mathbb{B}} (dd^c u)^n \Big)^{1/n} \Big(\int_{\mathbb{B}} (dd^c \ln|z|)^n \Big)^{(n-1)/n} = (2\pi)^n.$$

Hence,

$$u(z) \geq \ln|z| \frac{1}{(2\pi)^n} \int_{\mathbb{B}} dd^c u \wedge (dd^c \ln|w|)^{n-1} \geq \ln|z|.$$

Furthermore, using one of Demailly's comparison principles (see e.g. [10] or Lemma 4.1 in [1]) we get

$$\begin{aligned} (2\pi)^n = \int_{\{0\}} (dd^c u)^n &\leq \int_{\{0\}} dd^c u \wedge (dd^c \ln|w|)^{n-1} \\ &\leq \int_{\{0\}} (dd^c \ln|w|)^n = (2\pi)^n. \end{aligned}$$

Hence, the measure $dd^c u \wedge (dd^c \ln|z|)^{n-1}$ is carried by $\{0\}$. Thus,

$$dd^c u \wedge (dd^c \ln|z|)^{n-1} = (2\pi)^n \delta_0.$$

To conclude this proof note that the assumption that $u \in \mathcal{PSH}^R(\mathbb{B})$ implies that for all $z \neq 0$,

$$\begin{aligned} u(z) &= \frac{1}{(2\pi)^n} \int\limits_{\mathbb{B}} \max(\ln|z|, \ln|w|)\, dd^c u \wedge (dd^c \ln|w|)^{n-1} \\ &= \frac{1}{(2\pi)^n} \int\limits_{\{0\}} \max(\ln|z|, \ln|w|)\, dd^c u \wedge (dd^c \ln|w|)^{n-1} \\ &\quad + \frac{1}{(2\pi)^n} \int\limits_{\mathbb{B}\setminus\{0\}} \max(\ln|z|, \ln|w|)\, dd^c u \wedge (dd^c \ln|w|)^{n-1} \\ &= \frac{1}{(2\pi)^n} \int\limits_{\{0\}} \ln|z|\, dd^c u \wedge (dd^c \ln|w|)^{n-1} = \ln|z|. \quad \blacksquare \end{aligned}$$

We need the following lemma.

LEMMA 3.6. *Let $F : [0,1) \to [0,\infty)$ be a non-decreasing function that is left-continuous and such that*

$$\int\limits_{1/2}^{1} \sqrt[n]{F(t)}\, dt < \infty.$$

Let

$$\tilde{u}(r) = \int\limits_{r}^{1} -\frac{1}{t} \sqrt[n]{F(t)}\, dt.$$

Then the function u defined on $\mathbb{B}$ by $u(z) = \tilde{u}(|z|)$ is in $\mathcal{PSH}^R(\mathbb{B})$, and $(dd^c u)^n$ is the unique unitary invariant measure that satisfies

$$\frac{1}{(2\pi)^n} (dd^c u)^n(\mathbb{B}_t) = F(t).$$

Proof. Since F is non-negative, $\tilde{u}$ is non-decreasing. Furthermore, F is a non-decreasing function that is left-continuous, and it follows that for $r < 0$ we have

$$v(r) = \tilde{u}(e^r) = \int\limits_{e^r}^{1} \frac{-1}{t} \sqrt[n]{F(t)}\, dt = \int\limits_{r}^{0} -\sqrt[n]{F(e^s)}\, ds.$$

From the fact that the left derivative $v_l'(r) = \sqrt[n]{F(e^r)}$ is a non-decreasing function we deduce that v is a convex function. Thus, $\tilde{u}$ is convex with respect to $\ln r$, and therefore $u(z) = \tilde{u}(|z|) \in \mathcal{PSH}^R(\mathbb{B})$.

For the second part, let us first assume that F is a smooth function.

Recall that the *Lelong number* (see e.g. [13]) of a plurisubharmonic function α at 0 is defined by

$$\nu(\alpha,0)=\lim_{r\to 0}\frac{1}{(2\pi)^n}\int\limits_{\mathbb{B}_r} dd^c\alpha\wedge(dd^c\ln|z|)^{n-1}=\lim_{r\to 0}\frac{\sup_{\mathbb{B}_r}\alpha}{\ln r}.$$

Therefore

$$\lim_{r\to 0}\frac{\sup_{\mathbb{B}_r} u}{\ln r}=\lim_{r\to 0}\frac{\int_r^1 -t^{-1}\sqrt[n]{F(t)}\,dt}{\ln r}=\sqrt[n]{F(0)}. \tag{3.5}$$

If we assume that $F(0)=0$ then by (3.5) we have $\nu(u,0)=0$. Moreover, from the proof of Proposition 3.5 it follows that there exists a constant A such that $u(z)\geq A\ln|z|$, and then it is a well known fact that $(dd^cu)^n(\{0\})=0$ (see e.g. [7]).

Now for $F(0)>0$ we have

$$\begin{aligned} u(z)&=\int\limits_{|z|}^{1}-\frac{1}{t}\sqrt[n]{F(t)}\,dt=\int\limits_{|z|}^{1}-\frac{1}{t}\sqrt[n]{F(0)}\,dt+\int\limits_{|z|}^{1}-\frac{1}{t}(\sqrt[n]{F(t)}-\sqrt[n]{F(0)})\,dt\\ &=\sqrt[n]{F(0)}\ln|z|+u_1(z), \end{aligned}$$

and by the argument above $(dd^cu_1)^n(\{0\})=0$. Thus, we have proved that, at 0,

$$(dd^cu)^n=(2\pi)^nF(0)\delta_0.$$

For $z\neq 0$ it follows from (3.2) that

$$\begin{aligned} (dd^cu)^n&=4^nn!\frac{1}{4}\left(\frac{\tilde{u}'(|z|)}{2|z|}\right)^{n-1}\left(\tilde{u}''(|z|)+\frac{\tilde{u}'(|z|)}{|z|}\right)dV_{2n}\\ &=2^{n-1}(n-1)!F'(|z|)\frac{1}{|z|^{2n-1}}\,dV_{2n}. \end{aligned}$$

Furthermore,

$$\begin{aligned} \frac{1}{(2\pi)^n}(dd^cu)^n(\mathbb{B}_t)&=\frac{1}{(2\pi)^n}(dd^cu)^n(\{0\})\\ &\quad+\frac{1}{(2\pi)^n}\int\limits_{\mathbb{B}_t\setminus\{0\}}2^{n-1}(n-1)!F'(|z|)\frac{1}{|z|^{2n-1}}\,dV_{2n}\\ &=F(0)+\int\limits_0^tF'(x)\,dx=F(t). \end{aligned}$$

For an arbitrary function F let $\{F_j\}$ be a decreasing sequence of smooth, non-decreasing functions converging to F. Then

$$u_j(z)=\tilde{u}_j(|z|)\in\mathcal{PSH}^R(\mathbb{B}),\qquad \frac{1}{(2\pi)^n}(dd^cu_j)^n(\mathbb{B}_t)=F_j(t),$$

and by the monotone convergence theorem $\{u_j\}$ increases to $u \in \mathcal{PSH}^R(\mathbb{B})$, and

$$\tilde{u}(r) = \int_r^1 -\frac{1}{t}\sqrt[n]{F(t)}\,dt.$$

Let $G(t) = (2\pi)^{-n}(dd^c u)^n(\mathbb{B}_t)$. As in the proof of Theorem 3.3, the weak*-convergence of the sequence $\{(dd^c u_j)^n\}$ to the limit $(dd^c u)^n$ implies that $G(t) = F(t)$ almost everywhere. Thus, $F = G$, since both functions F and G are left-continuous. ∎

From the proofs of Proposition 3.5 and Lemma 3.6 we obtain the following decomposition of a radially symmetric plurisubharmonic function.

Corollary 3.7. *For any $u \in \mathcal{PSH}^R(\mathbb{B})$ there exist a constant $c > 0$ and a function $v \in \mathcal{PSH}^R(\mathbb{B})$ with Monge–Ampère measure $(dd^c v)^n$ vanishing on pluripolar sets, such that*

$$u(z) = c\ln|z| + v(z),$$

where

$$c = \frac{1}{(2\pi)^n}\int_{\{0\}} dd^c u \wedge (dd^c \ln|w|)^{n-1} = \sqrt[n]{F(0)} = \nu(u,0),$$

$$\begin{aligned} v(z) &= \frac{1}{(2\pi)^n}\int_{\mathbb{B}\setminus\{0\}} \max(\ln|z|, \ln|w|)\, dd^c u \wedge (dd^c \ln|w|)^{n-1} \\ &= \int_{|z|}^1 -\frac{1}{t}\big(\sqrt[n]{F(t)} - \sqrt[n]{F(0)}\big)\,dt. \end{aligned}$$

Here, $F(t) = (2\pi)^{-n}(dd^c u)^n(\mathbb{B}_t)$, and $\nu(u,0)$ is the Lelong number of u at 0.

Definition 3.8. Let $\mathcal{M}^R$ denote the set of non-negative Radon measures μ defined on $\mathbb{B}$ such that there exists a function $u \in \mathcal{PSH}^R(\mathbb{B})$ with $(dd^c u)^n = \mu$.

Remark. Let $T \in U(n)$, $u \in \mathcal{PSH}^R(\mathbb{B})$ and $E \subset \mathbb{B}$ be a Borel set. Then

$$(dd^c(u\circ T))^n(E) = |\mathrm{Jac}\, T|^2 (dd^c u)^n(T(E)) = (dd^c u)^n(T(E)),$$

and as a consequence every $\mu \in \mathcal{M}^R$ is a unitary invariant measure.

Theorem 3.9. *Let μ be a unitary invariant measure defined on $\mathbb{B}$, and let $F(t) = (2\pi)^{-n}\mu(\mathbb{B}_t)$. Then the following assertions are equivalent:*

(1) $\mu \in \mathcal{M}^R$,

(2) $\int_{1/2}^1 \sqrt[n]{F(t)}\,dt < \infty$.

Proof. (1)⇒(2): This implication follows from the representation formula (3.1).

(2)⇒(1): This implication follows from Lemma 3.6, since

$$u(z) = \tilde{u}(|z|) = \int_{|z|}^{1} -\frac{1}{t}\sqrt[n]{F(t)}\,dt$$

is a well defined radially symmetric plurisubharmonic function such that $(dd^c u)^n = \mu$. ■

As a direct consequence of Lemma 3.6 we have the following corollary.

COROLLARY 3.10. *Let μ be a unitary invariant measure on $\mathbb{B}$, let $F(t) = (2\pi)^{-n}\mu(\mathbb{B}_t)$, and let $(dd^c u)^n = d\mu$. Then u is bounded if, and only if,*

$$\frac{1}{t}\sqrt[n]{F(t)} \in L^1([0,1]).$$

COROLLARY 3.11. *If $\mu \in \mathcal{M}^R$ and ν is a unitary invariant measure on $\mathbb{B}$ such that $\nu \le \mu$, then $\nu \in \mathcal{M}^R$.*

EXAMPLE 3.12. Take $\mu = (1-|z|^{2n})^{-\alpha} dV_{2n}$. Then

$$F(t) = \begin{cases} c_n(1-\alpha)^{-1}(1-(1-t^{2n})^{1-\alpha}) & \text{for } \alpha \neq 1, \\ -d_n \ln(1-t^{2n}) & \text{for } \alpha = 1, \end{cases}$$

where c_n and d_n are constants. Therefore $\sqrt[n]{F}$ is integrable in the neighborhood of the point $t = 1$ if, and only if, $\alpha < n+1$. ■

DEFINITION 3.13. Let μ be the measure of the form

$$\mu = \sum_{k=1}^{\infty} a_k\, d\sigma_{b_k},$$

where a_k and b_k are sequences of real numbers with $a_k \ge 0$, $b_k > 0$, and $\{b_k\}$ increasing to 1. Here $d\sigma_{b_k}$ is the normalized Lebesgue measure on the sphere $\partial\mathbb{B}_{b_k}$. Denote the class of those measures by $\mathcal{M}_\sigma$.

THEOREM 3.14. *Let $\mu \in \mathcal{M}_\sigma$. Then $\mu \in \mathcal{M}^R$ if, and only if,*

$$\sum_{k=1}^{\infty}(a_1 + \cdots + a_k)^{1/n} \ln\frac{b_k}{b_{k+1}} > -\infty. \tag{3.6}$$

Proof. Let $\mu = \sum_{k=1}^{\infty} a_k\, d\sigma_{b_k} \in \mathcal{M}_\sigma$ and define

$$\mu_k = \sum_{l=1}^{k} a_l d\sigma_{b_l} \quad \text{and} \quad \tilde{u}_k(r) = \frac{1}{2\pi}\int_r^1 \frac{-1}{t}(\mu_k(\mathbb{B}_t))^{1/n}\,dt.$$

The sequence $\{u_k\}$ defined by $u_k(z) = \tilde{u}_k(|z|)$ is a decreasing sequence of plurisubharmonic functions that tends to $u \neq -\infty$ if, and only if, there exists

$r > 0$ such that $u(r) > -\infty$. Taking $0 < r < b_1$ we have

$$\lim_{k\to\infty} \frac{1}{2\pi} \int_0^1 \frac{-1}{t} (\mu_k(\mathbb{B}_t))^{1/n}\, dt = \frac{1}{2\pi} \sum_{k=1}^{\infty} (a_1 + \cdots + a_k)^{1/n} \ln \frac{b_k}{b_{k+1}} > -\infty. \ \blacksquare$$

Remark. Note that condition (3.6) is satisfied if $\sum_{k=1}^{\infty} a_k < \infty$, i.e. if the measure μ is finite.

4. Continuity of the complex Monge–Ampère operator. The aim of this section is to prove that the complex Monge–Ampère operator is continuous on $\mathcal{PSH}^R(\mathbb{B})$.

Theorem 4.1. *Let $\{u_j\}$, $u_j \in \mathcal{PSH}^R(\mathbb{B})$, be a sequence that converges pointwise to a function u in $\mathcal{PSH}^R(\mathbb{B})$. Then the associated sequence $\{(dd^c u_j)^n\}$ tends to $(dd^c u)^n$ in the weak*-topology.*

Proof. The continuity of the complex Monge–Ampère operator is a consequence of basic properties of convex functions. Let $\{u_j\}$ and u be as in the statement and in addition assume that they are bounded. Then the functions defined by

$$v_j(r) = u_j(e^r) \quad \text{and} \quad v(r) = u(e^r)$$

are non-decreasing, and convex on $(-\infty, 0]$. Furthermore, $\{v_j\}$ converges pointwise to v. Therefore, $\{v_j\}$ converges locally uniformly to v (see e.g. [11]). Hence, $\{u_j\}$ converges locally uniformly to u on $\mathbb{B} \setminus \{0\}$. We can now deduce that $\{(dd^c u_j)^n\}$ tends to $(dd^c u)^n$ in the weak*-topology on $\mathbb{B} \setminus \{0\}$. But since u_j, u are bounded, $(dd^c u_j)^n(\{0\}) = (dd^c u_j)^n(\{0\}) = 0$. Thus, the convergence of $\{(dd^c u_j)^n\}$ is valid on the whole $\mathbb{B}$.

In the general case, let $\{u_j\}$ and u be as in the statement without any additional assumption. Then for any $k < 0$:

(1) $\max(u_j, k) \to \max(u, k)$, $j \to \infty$, so
$$(dd^c \max(u_j, k))^n \to (dd^c \max(u, k))^n$$
in the weak*-topology (by the argument above),

(2) $\max(u_j, k) \to u_j$ as $k \to -\infty$, so $(dd^c \max(u_j, k))^n \to (dd^c u_j)^n$ in the weak*-topology, and

(3) $\max(u, k) \to u$ as $k \to -\infty$, so $(dd^c \max(u, k))^n \to (dd^c u)^n$ in the weak*-topology.

Conditions (2) and (3) follow from the fact that the complex Monge–Ampère operator is continuous under decreasing sequences (see e.g. [3]).

Hence, $\{(dd^c u_j)^n\}$ converges to $(dd^c u)^n$ in the weak*-topology in $\mathbb{B}$. $\blacksquare$

REMARK. Recall that the C_n-*capacity*, introduced by Bedford and Taylor in [4], of a Borel set $A \subset \Omega \Subset \mathbb{C}^n$ is defined by

$$C_n(A, \Omega) = \sup\Big\{ \int_A (dd^c u)^n : u \in \mathcal{PSH}(\Omega),\ -1 \le u \le 0\Big\}.$$

A sequence $\{u_j\}$ of functions defined in Ω is said to *converge in capacity to* u if for any $t > 0$ and $K \Subset \Omega$,

$$\lim_{j\to\infty} C_n(K \cap \{|u - u_j| > t\}, \Omega) = 0.$$

Theorem 4.1 is valid if we change pointwise convergence to any of the following:

(1) weak convergence,
(2) convergence in the sense of distributions,
(3) convergence in capacity.

It is well known that for plurisubharmonic functions weak convergence is equivalent to the convergence in the sense of distributions, and that convergence in capacity implies weak convergence (see [11, 15]). To see the remaining implication suppose that $\{u_j\}$, $u_j \in \mathcal{PSH}^R(\mathbb{B})$, is a sequence that is weakly convergent to a function u in $\mathcal{PSH}^R(\mathbb{B})$. Recall that it is sufficient to prove that for any subsequence $\{u_{j_k}\}$ of $\{u_j\}$ there exists a subsequence $\{u_{j_{k_l}}\}$ such that $\{(dd^c u_{j_{k_l}})^n\}$ tends to $(dd^c u)^n$ in the weak*-topology. But this follows from Theorem 4.1, since for any subsequence $\{u_{j_k}\}$ of $\{u_j\}$ there exists another subsequence $\{u_{j_{k_l}}\}$ of $\{u_{j_k}\}$ such that $\{u_{j_{k_l}}\}$ converges to u almost everywhere. All functions are continuous and convex with respect to $\ln r$ and therefore the convergence is pointwise. Theorem 4.1 shows now that $\{(dd^c u_{j_{k_l}})^n\}$ tends to $(dd^c u)^n$ in the weak*-topology.

Next we prove stability of the solution of the complex Monge–Ampère equation.

THEOREM 4.2. *Let $\{\mu_j\}$, $\mu_j \in \mathcal{M}^R$, be a sequence that tends in the weak*-topology to a measure μ in $\mathcal{M}^R$. Then $\{u(\mu_j)\}$ converges pointwise to $u(\mu)$, where $u(\nu) \in \mathcal{PSH}^R(\mathbb{B})$ is the unique solution of the complex Monge–Ampère equation $(dd^c u(\nu))^n = \nu$.*

Proof. On $[0, 1)$ we define the following functions:

$$F_j(t) = \frac{1}{(2\pi)^n}\mu_j(\mathbb{B}_t) \quad \text{and} \quad F(t) = \frac{1}{(2\pi)^n}\mu(\mathbb{B}_t),$$

and then, by Lemma 3.6,

$$u(\mu_j)(z) = \int_{|z|}^1 -\frac{1}{t}\sqrt[n]{F_j(t)}\,dt \quad \text{and} \quad u(\mu)(z) = \int_{|z|}^1 -\frac{1}{t}\sqrt[n]{F(t)}\,dt.$$

The functions defined by $v(r) = u(e^r)$ and $v_j(r) = u_j(e^r)$ are non-decreasing and convex on $(-\infty, 0]$. We shall prove that $\{v_j\}$ converges pointwise to v. It is enough to show that for any subsequence $\{v_{j_k}\}$ of v_j there exists a subsequence $\{v_{j_{k_l}}\}$ that converges pointwise to v. Take any subsequence $\{v_{j_k}\}$ of v_j; then there exist a subsequence $\{v_{j_{k_l}}\}$ of $\{v_{j_k}\}$ and a convex function $\tilde{v}$ such that $\{v_{j_{k_l}}\}$ converges locally uniformly to $\tilde{v}$ (see [11]). But since μ_j tends to μ in the weak*-topology, the sequence of left-continuous functions $\{F_j\}$ converges almost everywhere to the left-continuous function F (see the proof of Theorem 3.3). Hence, $\tilde{v} = v$. ∎

Corollary 4.3. *The complex Monge–Ampère operator*

$$\mathcal{PSH}^R(\mathbb{B}) \ni u \mapsto (dd^c u)^n \in \mathcal{M}^R$$

is a continuous bijection with continuous inverse.

Corollary 4.4. *In $\mathcal{PSH}^R(\mathbb{B})$, locally uniform convergence, pointwise convergence, weak convergence, convergence in the sense of distribution and convergence in capacity are equivalent.*

5. Radially symmetric subharmonic functions. Let $\mathcal{SH}^R(\mathbb{B})$ denote the set of *non-positive*, radially symmetric, and subharmonic functions defined in the unit ball $\mathbb{B}$ in $\mathbb{R}^N$. The aim of this section is the following theorem.

Theorem 5.1. *Let u be a radially symmetric subharmonic function defined on $\mathbb{B} \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$, $n \geq 2$, and set $G(t) = \Delta u(\mathbb{B}_t)/\sigma(\partial\mathbb{B}_t)$. Then u is plurisubharmonic if, and only if, the function $t \mapsto tG(t)$ is increasing.*

Before starting we need some preliminaries. We shall need counterparts of Lemma 2.1, Theorem 2.2 and Theorem 3.1 for radially symmetric subharmonic functions.

Lemma 5.2. *If $u \in \mathcal{SH}^R(\mathbb{B})$, then*

(1) *$\tilde{u}$ is an increasing function, and convex with respect to r^{2-N},*
(2) *$\lim_{t\to 1^-} \tilde{u}(t)$ exists,*
(3) *$\{x \in \mathbb{B} : u(x) = -\infty\} \subset \{0\}$, and*
(4) *u is continuous.*

Proof. The proof is the same as that of Lemma 2.1 (see also [2]). ∎

We have the following characterization of radially symmetric subharmonic functions.

Theorem 5.3. *Let u be a radially symmetric function defined on $\mathbb{B}$ that satisfies* (2.3). *Then the following assertions are equivalent:*

(1) *$u \in \mathcal{SH}^R(\mathbb{B})$,*
(2) *$\tilde{u}$ is an increasing function that is convex with respect to r^{2-N}.*

Proof. See [2]. ∎

For a smooth subharmonic function $u(x) = u(|x|) = \tilde{u}(t)$ note that

$$u_{jk}(x) = \frac{1}{t^3} x_j x_k \big(t\tilde{u}''(t) - \tilde{u}'(t)\big) + \delta_{jk} \frac{\tilde{u}'(t)}{t},$$

where $t = |x|$, and δ_{jk} is the Kronecker delta. Therefore,

$$\Delta u(x) = \frac{1}{t}(t\tilde{u}''(t) + (N-1)\tilde{u}'(t))$$

and the following theorem holds.

Theorem 5.4. *Let $u \in \mathcal{SH}^R(\mathbb{B}_{r_2} \setminus \bar{\mathbb{B}}_{r_1}) \cap \mathcal{C}^2(\mathbb{B}_{r_2})$ with $0 \le r_1 < r_2 \le 1$. Then for $r_1 \le r < s \le r_2$,*

$$\tilde{u}(r) - \tilde{u}(s) = \int_s^r \frac{1}{t^{N-1}} \int_0^t y^{N-1} f(y)\, dy\, dt,$$

where $f(x) = f(|x|)$ is given by $\Delta u = f(x) dV_N$, and dV_N is the N-dimensional Lebesgue measure on $\mathbb{B}$.

Following the proof of Theorem 3.3, with the use of Theorem 5.4 instead of Theorem 3.1 one can prove

Theorem 5.5. *If $u \in \mathcal{SH}^R(\mathbb{B})$, then*

$$\tilde{u}(r) - \tilde{u}(1) = \int_r^1 -G(t)\, dt, \quad \text{where} \quad G(t) = \frac{\Delta u(\mathbb{B}_t)}{\sigma(\partial \mathbb{B}_t)}.$$

We are now able to prove Theorem 5.1.

Proof of the Theorem 5.1. Assume at first that u is smooth. To prove that u is plurisubharmonic it is enough to check that $\tilde{u}$ is convex with respect to $\ln t$, i.e.

$$s\big(\tilde{u}'(s) + s\tilde{u}''(s)\big) = s(s\tilde{u}'(s))' \ge 0, \qquad \text{where } s = e^t. \tag{5.1}$$

By Theorem 5.5, condition (5.1) is equivalent to $(tG(t))' \ge 0$.

In the general case, we use an approximation procedure. Let $\{\tilde{u}_j\}$ be the smooth approximation sequence, defined in Theorem 2.2, that converges to $\tilde{u}$. Then the corresponding functions $tG_j(t)$ are increasing. Since the sequence $\{G_j\}$ converges to G except for a countable set (see the proof of Theorem 3.3), and G is a left-continuous function, we see that $tG(t)$ is also increasing. On the other hand, if $tG(t)$ is increasing and left-continuous, then the function

$$\tilde{u}(r) = \int_r^1 -G(t)\, dt$$

is convex with respect to $\ln r$, since

$$\tilde{u}(e^r) = \int_{e^r}^{1} -G(t)\,dt = \int_{0}^{r} e^s G(e^s)\,ds.$$

Thus, $u \in \mathcal{PSH}^R(\mathbb{B})$. ∎

Acknowledgments. This research was partially supported by the Lars Hierta Memorial Foundation. The second named author was partially supported by NCN grant 2011/01/B/ST1/00879.

References

[1] P. Åhag, U. Cegrell, R. Czyż and H. H. Phạm, *Monge–Ampère measures on pluripolar sets*, J. Math. Pures Appl. 92 (2009), 613–627.

[2] D. H. Armitage and S. J. Gardiner, *Classical Potential Theory*, Springer Monogr. Math., Springer, London, 2001.

[3] E. Bedford and B. A. Taylor, *The Dirichlet problem for a complex Monge–Ampère equation*, Invent. Math. 37 (1976), 1–44.

[4] E. Bedford and B. A. Taylor, *A new capacity for plurisubharmonic functions*, Acta Math. 149 (1982), 1–40.

[5] J. M. Borwein and J. D. Vanderwerff, *Convex Functions: Constructions, Characterizations and Counterexamples*, Encyclopedia Math. Appl. 109, Cambridge Univ. Press, Cambridge, 2010.

[6] U. Cegrell, *Discontinuité de l'opérateur de Monge–Ampère complexe*, C. R. Acad. Sci. Paris Sér. I Math. 296 (1983), 869–871.

[7] U. Cegrell, *The general definition of the complex Monge–Ampère operator*, Ann. Inst. Fourier (Grenoble) 54 (2004), 159–179.

[8] U. Cegrell, *A general Dirichlet problem for the complex Monge–Ampère operator*, Ann. Polon. Math. 94 (2008), 131–147.

[9] R. Czyż, *The complex Monge–Ampère operator in the Cegrell classes*, Dissertationes Math. 466 (2009).

[10] J.-P. Demailly, *Monge–Ampère operators, Lelong numbers and intersection theory*, in: Complex Analysis and Geometry, Univ. Ser. Math., Plenum Press, New York, 1993.

[11] L. Hörmander, *Notions of Convexity*, Progr. Math. 127, Birkhäuser Boston, Boston, MA, 1994.

[12] N. Kerzman, *A Monge–Ampère equation in complex analysis*, in: Several Complex Variables (Williamstown, MA, 1975), Proc. Sympos. Pure Math. 30, Part 1, Amer. Math. Soc., Providence, RI, 1977, 161–167.

[13] C. O. Kiselman, *Densité des fonctions plurisousharmoniques*, Bull. Soc. Math. France 107 (1979), 295–304.

[14] C. O. Kiselman, *Plurisubharmonic functions and potential theory in several complex variables*, in: Development of Mathematics 1950–2000, Birkhäuser, Basel, 2000, 655–714.

[15] M. Klimek, *Pluripotential Theory*, London Math. Soc. Monogr. 6, Oxford Sci. Publ., Clarendon Press, New York, 1991.

[16] S. Kołodziej, *The complex Monge–Ampère equation and pluripotential theory*, Mem. Amer. Math. Soc. 178 (2005), no. 840.

[17] D. R. Monn, *Regularity of the complex Monge–Ampère equation for radially symmetric functions of the unit ball*, Ph.D. Thesis, Univ. of North Carolina at Chapel Hill, 1985.

[18] D. R. Monn, *Regularity of the complex Monge–Ampère equation for radially symmetric functions of the unit ball*, Math. Ann. 275 (1986), 501–511.

Per Åhag, Leif Persson
Department of Mathematics
and Mathematical Statistics
Umeå University
SE-901 87 Umeå, Sweden
E-mail: Per.Ahag@math.umu.se
Leif.Persson@math.umu.se

Rafał Czyż
Faculty of Mathematics
and Computer Science
Jagiellonian University
Łojasiewicza 6
30-348 Kraków, Poland
E-mail: Rafal.Czyz@im.uj.edu.pl

Received 16.11.2011
and in final form 16.5.2012 (2611)