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# Product property for capacities in $\mathbb{C}^N$

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*Dedicated to Professor Józef Siciak*
*on the occasion of his 80th birthday*

**Abstract.** The paper deals with logarithmic capacities, an important tool in pluripotential theory. We show that a class of capacities, which contains the $L$-capacity, has the following product property:

$$C_\nu(E_1 \times E_2) = \min(C_{\nu_1}(E_1), C_{\nu_2}(E_2)),$$

where $E_j$ and $\nu_j$ are respectively a compact set and a norm in $\mathbb{C}^{N_j}$ $(j = 1, 2)$, and $\nu$ is a norm in $\mathbb{C}^{N_1+N_2}$, $\nu = \nu_1 \oplus_p \nu_2$ with some $1 \le p \le \infty$.

For a convex subset $E$ of $\mathbb{C}^N$, denote by $C(E)$ the standard $L$-capacity and by $\omega_E$ the minimal width of $E$, that is, the minimal Euclidean distance between two supporting hyperplanes in $\mathbb{R}^{2N}$. We prove that $C(E) = \omega_E/2$ for a ball $E$ in $\mathbb{C}^N$, while $C(E) = \omega_E/4$ if $E$ is a convex symmetric body in $\mathbb{R}^N$. This gives a generalization of known formulas in $\mathbb{C}$. Moreover, we show by an example that the last equality is not true for an arbitrary convex body.

**1. Introduction.** *Siciak's extremal function* is defined for a compact subset $E$ of $\mathbb{C}^N$ by the formula

$$\Phi_E(z) = \Phi(E, z) = \sup\{|P(z)|^{1/\deg P} : \deg P \ge 1,\ \|P\|_E \le 1\},\ \ z \in \mathbb{C}^N. \tag{1.1}$$

We refer to [S1, S2, S3, K2] for definitions and basic properties related to this important tool in pluripotential theory and its applications to approximation theory.

If $\nu$ is a complex norm in $\mathbb{C}^N$ then we define the *$\nu$-capacity* of $E$ as the quantity

$$\log C_\nu(E) = \liminf_{z\to\infty}(\log \nu(z) - \log \Phi(E, z)), \tag{1.2}$$

which is finite for any compact set $E \subset \mathbb{C}^N$. Here and subsequently, we write $z \to \infty$ when $\nu(z) \to \infty$. If $\nu(z) = \|z\|_2$ is the standard Euclidean norm then

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we get the so called *L-capacity* and we put $C_\nu(E) = C(E)$. Kołodziej [Ko] has proved that $C(E)$ is a Choquet capacity. Since any two norms $\nu$ and $\mu$ in $\mathbb{C}^N$ are equivalent, we have

$$\min_{\mu(z)=1} \nu(z) \le \frac{C_\nu(E)}{C_\mu(E)} \le \max_{\mu(z)=1} \nu(z).$$

By a well known result of Siciak (see [S1, S3]), $C_\nu(E) = 0$ if and only if $E$ is a pluripolar set.

Following [BC1, BC2] assume that there exists a norm $\nu = \nu_0$ (which depends on $E$) such that we can replace the lim inf in (1.2) by lim (we shall denote such a modification by $(1.2)^*$). If $\nu_1$ is another norm with this property then it must be a positive multiple of $\nu_0$. Indeed, if $\nu_0, \nu_1$ satisfy $(1.2)^*$ then the limit

$$\lim_{z\to\infty} \frac{\nu_1(z)}{\nu_0(z)} = \alpha \in (0, +\infty)$$

exists, whence

$$\alpha = \liminf_{z\to\infty} \frac{\nu_1(z)}{\nu_0(z)} = \min_{\nu_0(z)=1} \nu_1(z) = \limsup_{z\to\infty} \frac{\nu_1(z)}{\nu_0(z)} = \max_{\nu_0(z)=1} \nu_1(z),$$

which means $\nu_1 = \alpha\nu_0$. We shall call $\nu_0$ that satisfies $(1.2)^*$ and $C_{\nu_0}(E) = 1$ the *C-norm for* $E$ and denote it by $h_E$. So, if the C-norm exists, then for an arbitrary norm $\nu$ we have

$$C_\nu(E) = \min_{h_E(z)=1} \nu(z) = \frac{1}{\max_{\nu(z)=1} h_E(z)} = (\|\mathrm{Id} : (\mathbb{C}^N, \nu) \to (\mathbb{C}^N, h_E)\|)^{-1}$$

where $\|\cdot\|$ is the usual norm of linear mappings. Hence, if $L$ is a linear isomorphism of $\mathbb{C}^N$ we get the following connection between the $\nu$-capacity of $E$ and $L(E)$:

$$C_\nu(L(E)) = (\|L^{-1} : (\mathbb{C}^N, \nu) \to (\mathbb{C}^N, h_E)\|)^{-1}.$$

Let $\Psi(E, \cdot)$ be Siciak's homogeneous extremal function for $E$. Taking into account the results of [S1, S2, S3], it is not difficult to check the following

PROPOSITION 1.1. *Assume that a compact $E \subset \mathbb{C}^N$ possesses the C-norm. Then*

(a)

$$h_E(z) = \lim_{r\to\infty} \frac{1}{r} \Phi(E, rz) \ge \Psi(E, z), \qquad z \in \mathbb{C}^N. \tag{1.3}$$

(b) *If $F \subset \mathbb{C}^M$ also possesses the C-norm then so does $E \times F$ and*

$$h_{E\times F}(z, w) = \max(h_E(z), h_F(w)), \qquad (z, w) \in \mathbb{C}^{N+M}.$$

(c) *If $L$ is a linear isomorphism of $\mathbb{C}^N$ then $h_{L(E)} = h_E \circ L^{-1}$.*

(d) *If $E_R = \{z \in \mathbb{C}^N : \Phi(E, z) \le R\}$ then $h_{E_R} = (1/R)h_E$.*

EXAMPLE 1.2. If $q$ is a norm in $\mathbb{C}^N$ and $E = \{z \in \mathbb{C}^N : q(z) \le 1\}$ then $h_E(z) = q(z) = \Psi(E, z)$, $z \in \mathbb{C}^N$.

REMARK 1.3. If for a compact $E \subset \mathbb{C}^N$ the limit

$$f_E(z) = \lim_{r\to\infty} r^{-1}\Phi(E, rz)$$

exists for $z \in \mathbb{C}^N$ (as in (1.3)), then we shall call $f_E$ the *C-h-function of* $E$ (it is a positive homogeneous function) whenever additionally $f_E$ is continuous and $\lim_{z\to\infty} \Phi(E, z)/f_E(z) = 1$.

If $f_E$ is a C-h-function of $E$ and $\nu$ is an arbitrary complex norm in $\mathbb{C}^N$ then

$$C_\nu(E) = \frac{1}{\sup_{\nu(z)=1} f_E(z)} = \inf_{f_E(z)=1} \nu(z) =: \rho_\nu(S_{f_E}(0,1)) =: \frac{1}{2}\omega_\nu(S_{f_E}(0,1)),$$

where $S_{f_E}(0, 1) = \{z \in \mathbb{C}^N : f_E(z) = 1\}$.

We do not know when a C-h-function exists and when it is a norm in $\mathbb{C}^N$.

EXAMPLE 1.4. Let $Q = (Q_1, \ldots, Q_N) : \mathbb{C}^N \to \mathbb{C}^N$ be a polynomial mapping such that $\deg Q_j = d \ge 1$, $j = 1, \ldots, N$, and $Q = \widehat{Q} + R$, where $\widehat{Q} = (\widehat{Q_1}, \ldots, \widehat{Q_N})$ is the main (homogeneous) part of $Q$ of degree $d$, $\deg R < d$ and $\widehat{Q}^{-1}(\{0\}) = \{0\}$. Then, by the Klimek theorem [K1] (see also [K2, Thm. 5.3.1]), for an arbitrary compact $E$ we have

$$\Phi(Q^{-1}(E), z) = (\Phi(E, Q(z)))^{1/d}, \quad z \in \mathbb{C}^N.$$

If $E$ possesses the C-norm then $Q^{-1}(E)$ possesses a C-h-function and

$$f_{Q^{-1}(E)} = (h_E \circ \widehat{Q})^{1/d}.$$

REMARK 1.5. Note the following result of Klimek [K1]: if $F : \mathbb{C}^N \to \mathbb{C}^N$ is a polynomial mapping of degree $d \ge 1$ and $\liminf_{z\to\infty}(\|F(z)\|_2/\|z\|_2^d) > 0$ then for every compact set $E$ in $\mathbb{C}^N$,

$$C(F^{-1}(E)) \liminf_{z\to\infty} \|F(z)\|_2/\|z\|_2^d \le (C(E))^{1/d} \le C(F^{-1}(E)) \limsup_{z\to\infty} \|F(z)\|_2/\|z\|_2^d.$$

It follows from [BC1] that if for a compact $E \subset \mathbb{C}^N$ there exists a C-norm then

$$d_\infty(E) = d_\infty(B_{h_E}(0, 1)),$$

where $B_{h_E}(0, 1)$ is the unit ball for the C-norm and $d_\infty(E)$ denotes the transfinite diameter for $E$. By the Shestakov formula for $L(E)$, where $L$ is a linear isomorphism of $\mathbb{C}^N$ (cf. [BC1]), we have

$$d_\infty(L(E)) = \sqrt[N]{|\det L|}\, d_\infty(E).$$

A product property was first proved in [BC1] (see also [BC2, CM, BES]):

$$d_\infty(E_1 \times E_2) = d_\infty(E_1)^{\frac{N_1}{N_1+N_2}} d_\infty(E_2)^{\frac{N_2}{N_1+N_2}},$$

where $E_1 \subset \mathbb{C}^{N_1}, E_2 \subset \mathbb{C}^{N_2}$. We see that the given formula for $C_\nu(L(E))$ is much more complicated, but the product property for capacities is simpler, as we shall see later.

**2. Product property for capacities in $\mathbb{C}^N$.** If $f : \mathbb{C}^N \to \mathbb{R}$ is locally bounded and $\nu$ is a norm in $\mathbb{C}^N$, then we put

$$M_\nu(f, r) = \sup\{f(z) : \nu(z) \le r\}, \qquad r > 0.$$

Moreover, if $f$ is plurisubharmonic, then by the maximum principle we can write $M_\nu(f, r) = \sup_{\nu(z)=r} f(z)$.

For a locally bounded function $f : \mathbb{C}^N \to \mathbb{R}$ we denote by $f^*$ its upper regularization: $f^*(z) = \limsup_{w\to z} f(w)$, which is upper semicontinuous.

It is well known (Siciak's theorem, see e.g. [S3]) that if $C(E) = 0$ then $\Phi^*(E, z) \equiv +\infty$. In the case $C(E) > 0$ we have $\log \Phi^*(E, \cdot) \in \mathrm{PSH}(\mathbb{C}^N) \cap L^\infty_{\mathrm{loc}}(\mathbb{C}^N)$ and $\log \Phi^*(E, z) - \log \nu(z) = O(1)$ for an arbitrary norm $\nu$ as $\nu(z) \to \infty$.

PROPOSITION 2.1. *If $C(E) > 0$ then:*

(a) *for an arbitrary norm $\nu$ and for all $r > 0$,*

$$M_\nu(\Phi_E, r) = M_\nu(\Phi_E^*, r) = \sup_{\nu(z)=r} \Phi^*(E, z) = \sup_{\nu(z)=r} \Phi(E, z);$$

(b) *for each $r > 0$ exists $z_r \in \mathbb{C}^N$ such that $\nu(z_r) = r$ and $\Phi^*(E, z_r) = M_\nu(\Phi_E^*, r)$;*
(c) *$M_\nu(\log \Phi_E^*, e^t)$ is an increasing convex function in $\mathbb{R}$;*
(d) *$M_\nu(\log \Phi_E^*, e^t) - t = O(1)$ as $t \to +\infty$.*

*Proof.* Part (a) is a consequence of the maximum principle for plurisubharmonic functions (see [K2, Cor. 2.9.9]) and Bedford–Taylor theory (see [K2]): the set $\{z \in \mathbb{C}^N : \Phi(E, z) < \Phi^*(E, z)\}$ is pluripolar if it is non-empty. Since psh functions are upper semicontinuous, this implies (b). Assertion (c) is a special case of Prop. 1.4 in [LG]. The last part follows from Siciak's result (see [S2, K2]). ∎

LEMMA 2.2. *For any $a \in \mathbb{R}$, if $\varphi : (a, +\infty) \to \mathbb{R}$ is a convex function such that $\varphi(t) = o(t)$ as $t \to +\infty$, then $\varphi$ is a decreasing function.*

*Proof.* Let $a < t_1 < t_2 < t$ and $t_2 = (1 - \alpha)t_1 + \alpha t$. Then $\alpha = \frac{t_2 - t_1}{t - t_1}$ and, by convexity of $\varphi$,

$$\varphi(t_2) \le (1 - \alpha)\varphi(t_1) + \alpha\varphi(t) = (1 - o(1))\varphi(t_1) + o(1),$$

which gives $\varphi(t_2) \le \varphi(t_1)$. ∎

THEOREM 2.3. *Let $\nu$ be a norm in $\mathbb{C}^N$ and $C(E) > 0$. Put*

$$\Lambda_\nu(E,t) := M_\nu(\log \Phi_E^*, e^t) = M_\nu(\log \Phi_E, e^t).$$

*Then*

(a) *$\Lambda_\nu(E,t) - t$ is a convex decreasing function on $\mathbb{R}$;*
(b) *for all $t \in \mathbb{R}$,*

$$\Lambda_\nu(E,t) - t \geq -\log C_\nu(E),$$
$$\lim_{t\to\infty}(\Lambda_\nu(E,t) - t) = \inf_{t\in\mathbb{R}}(\Lambda_\nu(E,t) - t) = -\log C_\nu(E).$$

*Proof.* $\Lambda_\nu(E,t) - t$ is a convex function on $\mathbb{R}$ as a sum of two convex functions. By Proposition 2.1(d) and Lemma 2.2 this function is decreasing, which implies (a) and (b) except the fact that $\Lambda_\nu(E,t) - t \to -\log C_\nu(E)$. To show this, we need to prove the crucial fact

$$C_\nu(E) = \lim_{r\to\infty} \frac{r}{M_\nu(\Phi_E^*, r)}.$$

By Proposition 2.1(b), we have

$$\lim_{r\to\infty} \frac{r}{M_\nu(\Phi_E^*, r)} = \lim_{r\to\infty} \frac{\nu(z_r)}{\Phi^*(E, z_r)}$$
$$\geq \liminf_{\nu(z)\to\infty} \frac{\nu(z)}{\Phi^*(E,z)} \geq \liminf_{\nu(z)\to\infty} \frac{\nu(z)}{\sup_{\nu(w)\leq\nu(z)} \Phi^*(E,w)} = \lim_{r\to\infty} \frac{r}{M_\nu(\Phi_E^*, r)},$$

which completes the proof. ∎

COROLLARY 2.4.

$$C_\nu(E) = \lim_{r\to\infty} \frac{r}{M_\nu(\Phi_E, r)} = \sup_{r>0} \frac{r}{M_\nu(\Phi_E, r)} \geq \frac{1}{\sup_{\nu(z)=1} \Phi^*(E,z)}.$$

COROLLARY 2.5. *For all $r > 0$,*

$$\sup_{\nu(z)=r} \Phi(E,z) \geq \frac{r}{C_\nu(E)}.$$

Let $n$ be a norm in $\mathbb{R}^2$ such that $n(x_1,x_2) = n(x_2,x_1) = n(|x_1|,|x_2|)$, for each $r \geq 0$ the function $n(r,\cdot)$ is increasing on $\mathbb{R}_+$, $n(1,0) = n(0,1) = 1$ and $n(x_1,x_2) \geq \max(|x_1|,|x_2|)$. If $\nu_1$ is a norm in $\mathbb{C}^N$ and $\nu_2$ is a norm in $\mathbb{C}^M$ then $\nu(z,w) = n(\nu_1(z),\nu_2(w))$ is a norm in $\mathbb{C}^N \times \mathbb{C}^M$ and $\{(z,w) \in \mathbb{C}^{N+M} : \nu(z,w) = r\} = \{(r_1 z, r_2 w) : \nu_1(z) = 1 = \nu_2(w),\ r_j \geq 0,\ n(r_1,r_2) = r\}$. Note that the norms $n$ with the above properties form a convex set.

THEOREM 2.6 (Product property). *If $E$ and $F$ are compact subsets of $\mathbb{R}^N$ and $\mathbb{R}^M$ respectively, then for the norm $\nu$ defined above*

$$M_\nu(\Phi_{E\times F}, r) = \max(M_{\nu_1}(\Phi_E, r), M_{\nu_2}(\Phi_F, r)), \qquad r > 0,$$

*and*

$$C_\nu(E \times F) = \min(C_{\nu_1}(E), C_{\nu_2}(F)).$$

*Proof.* It suffices to prove the first part, which easily implies the second. We have

$$\begin{aligned}\sup_{\nu(z,w)=r} \Phi(E\times F,(z,w)) &= \sup_{\nu(z,w)=r} \max(\Phi(E,z),\Phi(F,w))\\ &= \sup_{n(r_1,r_2)=r,\,\nu_1(z)=1=\nu_2(w)} \max(\Phi(E,r_1z),\Phi(F,r_2w))\\ &= \sup_{n(r_1,r_2)=r} \max(M_{\nu_1}(\Phi_E,r_1),M_{\nu_2}(\Phi_F,r_2))\\ &\le \max(M_{\nu_1}(\Phi_E,r),M_{\nu_2}(\Phi_F,r)).\end{aligned}$$

On the other hand,

$$\begin{aligned}&\sup_{n(r_1,r_2)=r} \max(M_{\nu_1}(\Phi_E,r_1),M_{\nu_2}(\Phi_F,r_2))\\ &\quad\ge \max(\max(M_{\nu_1}(\Phi_E,r),M_{\nu_1}(\Phi_E,0)),\max(M_{\nu_2}(\Phi_F,r),M_{\nu_2}(\Phi_F,0)))\\ &\quad= \max(M_{\nu_1}(\Phi_E,r),M_{\nu_2}(\Phi_F,r)),\end{aligned}$$

which completes the proof. ■

COROLLARY 2.7. *If $E\subset\mathbb{C}^N$ and $F\subset\mathbb{C}^M$ are compact sets then*

$$C(E\times F)=\min(C(E),C(F)).$$

*Moreover for $E=E_1\times\cdots\times E_N$, $E_j\subset\mathbb{C}$, we have $C(E)\le d_\infty(E)$, with equality if and only if $C(E_1)=\cdots=C(E_N)$.*

For $1\le p\le\infty$ we take

$$\|z\|_p=(|z_1|^p+\cdots+|z_N|^p)^{1/p},\qquad \|z\|_\infty=\max(|z_1|,\ldots,|z_N|).$$

If $\nu_1=\|\cdot\|_p$, $\nu_2=\|\cdot\|_p$, $n(x_1,x_2)=\|(x_1,x_2)\|_p$ then we put $C_p(E)=C_{\|\,\|_p}(E)$.

COROLLARY 2.8. *If $E\subset\mathbb{C}^N$ and $F\subset\mathbb{C}^M$ are compact sets then*

$$C_p(E\times F)=\min(C_p(E),C_p(F)).$$

**3. $L$-capacity for convex sets in $\mathbb{C}^N$.** If $E$ is a convex body in $\mathbb{K}^N$, $\mathbb{K}=\mathbb{C}$ or $\mathbb{K}=\mathbb{R}$, and if we put

$$\rho_\nu(E):=\sup_{a\in\operatorname{int}_{\mathbb{K}}E}\sup\{r\ge 0: B_\nu(a,r)\subset E\},$$

then $\rho_\nu(E)$ is the $\nu$ *inner radius* of $E$ and $\omega_\nu(E)=2\rho_\nu(E)$ is the $\nu$ *width* of $E$ in $\mathbb{K}^N$.

EXAMPLE 3.1. Let $q$ be a norm in $\mathbb{C}^N$ and $E=\{z\in\mathbb{C}^N: q(z)\le 1\}$ be its closed unit ball. Then ([S1]) $\Phi(E,z)=\max(1,q(z))$, which gives $h_E(z)=q(z)$. Thus for every norm $\nu$,

$$C_\nu(E)=\rho_\nu(E)=\tfrac{1}{2}\omega_\nu(E).$$

In the next example we shall use dual norms in $\mathbb{C}^N$. If $\nu$ is a complex norm then $\nu^*(z) := \sup\{|\langle z, w\rangle| : \nu(w) \le 1\}$ where $\langle z, w\rangle := z_1\overline{w_1} + \cdots + z_N\overline{w_N}$.

EXAMPLE 3.2. Let $q$ be a norm in $\mathbb{R}^N$ and $E = \{x \in \mathbb{R}^N : q(x) \le 1\}$. Then ([Lu], see also [B1], [K2, Th. 5.4.6])

$$\Phi(E, z) = \max_{w\in E^*} h\Big(\tfrac{1}{2}|\langle z, w\rangle + 1| + \tfrac{1}{2}|\langle z, w\rangle - 1|\Big),$$

where $E^* = \{x \in \mathbb{R}^N : |\langle x, y\rangle| \le 1\ \forall y \in E\}$ is the dual ball and $h(t) = t + \sqrt{t^2 - 1}$, $t \ge 1$. Thus we get

$$h(\max(1, \check{q}(z))) \le \Phi(E, z) \le h(1 + \check{q}(z)), \qquad \check{q}(z) = \max_{w\in E^*} |\langle z, w\rangle|$$

(see [B4] for the properties of $E^*$ and $\check{q}$). Hence we get $h_E(z) = 2\check{q}(z)$ and for every norm $\nu$,

$$C_\nu(E) = \frac{1}{2}\inf_{\check{q}(z)=1} \nu(z) = \frac{1}{2}\frac{1}{\sup_{\nu(z)=1}\check{q}(z)}.$$

Denote $q^*_{\mathbb{K}}(z) = \sup_{w\in E}|\langle z, w\rangle|$, $z \in \mathbb{K}^N$. Then

$$\begin{aligned}
C_\nu(E) &= \frac{1}{2}\frac{1}{\max_{\nu(z)=1}\max_{w\in E^*}|\langle z, w\rangle|} = \frac{1}{2}\frac{1}{\max_{w\in E^*}\max_{\nu(z)=1}|\langle z, w\rangle|}\\
&= \frac{1}{2}\frac{1}{\max_{w\in E^*}\max_{\nu(z)=1}|\langle w, z\rangle|} = \frac{1}{2}\frac{1}{\max_{w\in E^*}\nu^*(w)}\\
&= \frac{1}{2}\frac{1}{\max_{(\nu^*|_{\mathbb{R}^N})^*_{\mathbb{R}}(x)=1}\max_{w\in E^*}|\langle x, w\rangle|} = \frac{1}{2}\inf_{q(x)=1}(\nu^*|_{\mathbb{R}^N})^*_{\mathbb{R}}(x)\\
&= \frac{1}{2}\rho_{(\nu^*|_{\mathbb{R}^N})^*_{\mathbb{R}}}(E) = \frac{1}{4}\omega_{(\nu^*|_{\mathbb{R}^N})^*_{\mathbb{R}}}(E).
\end{aligned}$$

If $\nu$ satisfies $(\nu^*|_{\mathbb{R}^N})^*_{\mathbb{R}}(x) = \nu(x)$ for all $x \in \mathbb{R}^N$ then we get

$$C_\nu(E) = \tfrac{1}{4}\omega_\nu(E).$$

The norms $\nu(z) = \|z\|_p$, $p \ge 1$, and in particular the Euclidean norm, have this property.

Now we shall consider the case when $\nu$ is the Euclidean norm, i.e. $C_\nu = C$ is the standard $L$-capacity.

EXAMPLE 3.3. We shall denote by $S^{N-1}_{\mathbb{K}}$ the unit Euclidean sphere in $\mathbb{K}^N$. If $E$ is a convex body in $\mathbb{K}^N$ and

$$H^0_\xi(E) = \Big\{z \in \mathbb{K}^N : \operatorname{Re}\langle z, \xi\rangle = \min_{z\in E}\operatorname{Re}\langle z, \xi\rangle =: a_\xi(E)\Big\},$$

$$H^1_\xi(E) = \Big\{z \in \mathbb{K}^N : \operatorname{Re}\langle z, \xi\rangle = \max_{z\in E}\operatorname{Re}\langle z, \xi\rangle =: b_\xi(E)\Big\}$$

are supporting hyperplanes then

$$\rho_\xi(E) := \inf\{\|z - w\|_2 : z \in H^0_\xi,\ w \in H^1_\xi\} = b_\xi(E) - a_\xi(E)$$

is the width of $E$ in direction $\xi$, and

$$\omega_E = \inf_{\xi \in S_{\mathbb{K}}^{N-1}} \rho_\xi(E)$$

is the minimal width of $E$. If $E$ is a compact subset of $\mathbb{R}^N$ then we put $\omega_E := \omega_{\mathrm{conv}(E)}$.

Note the following property: *if $E$ is a convex body in $\mathbb{K}^N$ and $F$ is a convex body in $\mathbb{R}^M$ then*

$$\omega_{E\times F} = \min(\omega_E, \omega_F).$$

Indeed, it is easily seen that $\omega_{E\times F} \le \min(\omega_E, \omega_F)$. Next, observe that

$$a_{(\xi_1,\xi_2)}(E \times F) = \|\xi_1\|_2 a_{\xi_1/\|\xi_1\|_2}(E) + \|\xi_2\|_2 a_{\xi_2/\|\xi_1\|_2}(F),$$
$$b_{(\xi_1,\xi_2)}(E \times F) = \|\xi_1\|_2 b_{\xi_1/\|\xi_1\|_2}(E) + \|\xi_2\|_2 b_{\xi_2/\|\xi_1\|_2}(F),$$
$$\rho_{(\xi_1,\xi_2)}(E \times F) = \|\xi_1\|_2 \rho_{\xi_1/\|\xi_1\|_2}(E) + \|\xi_2\|_2 \rho_{\xi_2/\|\xi_1\|_2}(F),$$
$$\omega_{E\times F} \ge \min_{0\le\alpha\le 1} \alpha\omega_E + \sqrt{1-\alpha^2}\,\omega_F = \min(\omega_E, \omega_F).$$

If $E = \{x \in \mathbb{R}^N : q(x) \le 1\}$ is a ball then for $\xi \in S_{\mathbb{R}}^{N-1}$ we have

$$\rho_\xi(E) = 2\,\mathrm{dist}(0, H_\xi^1) = 2\sup_{x\in E} |\langle x, \xi\rangle| = 2q^*(\xi),$$
$$\inf_{\xi\in S_{\mathbb{R}}^{N-1}} q^*(\xi) = \frac{1}{\sup_{q^*(x)=1} \|x\|_2} = 2C(E),$$

which gives a generalization of the one-dimensional case of an interval,

$$C(E) = \tfrac{1}{4}\omega_E.$$

If $E = \{z \in \mathbb{C}^N :\ q(z) \le 1\}$ is a complex ball then for $\xi \in S_{\mathbb{C}}^{N-1}$ we have

$$\rho_\xi(E) = 2\,\mathrm{dist}(0, H_\xi^1) = 2\sup_{z\in E} |\langle z, \xi\rangle| = 2q^*(\overline{\xi}),$$
$$\inf_{\xi\in S_{\mathbb{R}}^{N-1}} q^*(\overline{\xi}) = \frac{1}{\sup_{q^*(x)=1} \|x\|_2} = C(E),$$

which gives a generalization of the one-dimensional case of a disc,

$$C(E) = \tfrac{1}{2}\omega_E.$$

Example 3.4. Let now $E$ be a convex body in $\mathbb{R}^N$. Then we have the lower bound

$$\begin{aligned}\Phi(E,z) &\ge \sup_{\xi\in S_{\mathbb{R}}^{N-1}} \Phi([a_\xi(E), b_\xi(E)], \langle z, \xi\rangle)\\ &= \sup_{\xi\in S_{\mathbb{R}}^{N-1}} \Phi\biggl([-1,1], \frac{2\langle z,\xi\rangle}{b_\xi(E)-a_\xi(E)} - \frac{b_\xi(E)+a_\xi(E)}{b_\xi(E)-a_\xi(E)}\biggr), \qquad z \in \mathbb{C}^N,\end{aligned}$$

with equality if $z \in \mathbb{R}^N$ (see [BCL]). This gives

$$\sup_{\|z\|_2 \le r} \Phi(E,z) \ge h\bigg(\max\bigg(\sup_{\xi \in S_{\mathbb{R}}^{N-1}} \bigg|\frac{b_\xi(E)+a_\xi(E)}{b_\xi(E)-a_\xi(E)}\bigg|, \frac{2r}{\omega_E}\bigg)\bigg),$$

and therefore

$$C(E) \le \tfrac{1}{4}\omega_E.$$

It is known that we have equality in the above bound if $E$ is a symmetric ($E=-E$) convex body. There are some other cases when this is also true, e.g. when $E$ is the standard simplex $S_N = \{x \in \mathbb{R}^N : x_j \ge 0,\ x_1 + \cdots + x_N \le 1\}$. In this case we have (see [B1])

$$\Phi(S_N, z) = h(|z_1| + \cdots + |z_N| + |z_1 + \cdots + z_N - 1|), \qquad z \in \mathbb{C}^N.$$

Hence we can easily deduce that

$$h_{S_N}(z) = 2(|z_1| + \cdots + |z_N| + |z_1 + \cdots + z_N|),$$
$$C(S_N) = \frac{1}{\max_{z \in S_{\mathbb{C}}^{N-1}} h_{S_N}(z)} = \frac{1}{4\sqrt{N}} = \frac{1}{4}\omega_{S_N}.$$

Example 3.5. Now we shall present a counterexample to the equality $C(E) = \frac{1}{4}\omega_E$ for a convex body.

Let $L : \mathbb{C}^3 \to \mathbb{C}^3$ be given by

$$L(z_1,z_2,z_3) = (z_2, 2z_2 + z_3, z_1), \qquad L^{-1}(z) = (z_3, z_1, z_2 - 2z_1)$$

and put $E = L(S_3)$. Then it is easy to check that $\frac{1}{4}\omega_E = \frac{1}{4\sqrt{5}}$.

On the other hand, $C(E) = 1/\|L^{-1} : (\mathbb{C}^3, \|\ \|_2) \to (\mathbb{C}^3, h_{S_3})\|$. Hence

$$\begin{aligned}
&1/C(E)\\
&= 2\max\{|z_3| + |z_1| + |z_2 - 2z_1| + |z_3 + z_2 - z_1| : |z_1|^2 + |z_2|^2 + |z_3|^2 = 1\}\\
&= 2\max\bigg\{2|w_3| + \frac{|w_1+w_2|}{2} + \frac{|3w_1+w_2|}{3} + |w_1| : \frac{|w_1|^2}{2} + \frac{|w_2|^2}{2} + |w_3|^2 = 1\bigg\}\\
&= 2\max\{2r_3 + 3r_1 + r_2 : \tfrac{1}{2}r_1^2 + \tfrac{1}{2}r_2^2 + r_3^2 = 1\} = 4\sqrt{6},
\end{aligned}$$

which gives

$$\frac{1}{4\sqrt{6}} = C(E) < \frac{1}{4\sqrt{5}} = \frac{1}{4}\omega_E.$$

This example also proves that the equality $C(E) = \frac{1}{4}\omega_E$ for a convex body in $\mathbb{R}^N$ is not invariant under linear maps if $N \ge 3$. The situation for $N = 2$ is not clear.

However, we can show that if $E = S_2$ and $L(x,y) = (ax+by, cx+dy)$, $ad - bc \ne 0$, then

$$\omega_{L(E)} = |ad-bc| \min\left\{ \frac{1}{\sqrt{(c-d)^2+(a-b)^2}}, \frac{1}{\sqrt{a^2+c^2}}, \frac{1}{\sqrt{b^2+d^2}} \right\},$$

$$\|L^{-1} : (\mathbb{C}^2, \|\,\|_2) \to (\mathbb{C}^2, h_{S_2})\| = \frac{2}{|ad-bc|} \max\{\sqrt{(c-d)^2+(a-b)^2}, \sqrt{a^2+c^2}, \sqrt{b^2+d^2}\},$$

which implies $C(L(E)) = \frac{1}{4}\omega_{L(E)}$ and the equality $C(T) = \frac{1}{4}\omega_T$ holds for an arbitrary triangle $T \subset \mathbb{R}^2$.

**4. Capacities in $\mathbb{R}^N$.** Fix a norm $\nu$ in $\mathbb{R}^N$. For a compact set $E \subset \mathbb{R}^N$ define

$$C_{\nu,\mathbb{R}}(E) = \liminf_{\nu(x)\to\infty} \frac{\nu(x)}{\sup_{\nu(w)\le\nu(x)} \Phi(E,w)}$$

and in the case $\nu(x) = \|x\|_2$ we put $C_{\mathbb{R}}(E) := C_{\nu,\mathbb{R}}(E)$. Note that $\frac{1}{2}C_{\mathbb{R}}(E) \le C(E) \le C_{\mathbb{R}}(E)$ (see [Sz]) and $C_{\mathbb{R}}(E) = 0$ if and only if $E$ is a pluripolar subset of $\mathbb{C}^N$.

EXAMPLE 4.1. If $E$ is an arbitrary convex body in $\mathbb{R}^N$ then $C_{\mathbb{R}}(E) = \frac{1}{4}\omega_E$ (see [BCL]).

REMARK 4.2. We can formulate a few problems for $C_{\nu,\mathbb{R}}$:

(1) Is $C_{\nu,\mathbb{R}}$ continuous with respect to sequences of sets $E_j$ with $E_j \supset E_{j+1}$?

(2) Does the limit

$$\lim_{\nu(x)\to\infty} \frac{\nu(x)}{\sup_{\nu(w)\le\nu(x)} \Phi(E,w)}$$

exist for any compact set $E \subset \mathbb{R}^N$?

(3) Is the ratio $\frac{r}{\sup_{\nu(w)\le r} \Phi(E,w)}$ an increasing function of $r$?

(4) When does the equality $C(E) = C_{\mathbb{R}}(E)$ hold? In particular, for which convex bodies is it true?

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