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Markov's property for kth derivative

by Mirosław Baran (Kraków), Beata Milówka (Tarnów) and Paweł Ozorka (Tarnów)

Dedicated to Professor Józef Siciak on the occasion of his 80th birthday

Abstract. Consider the normed space $(\mathbb{P}(\mathbb{C}^N), \|\cdot\|)$ of all polynomials of N complex variables, where $\|\ \|$ a norm is such that the mapping $L_g : (\mathbb{P}(\mathbb{C}^N), \|\cdot\|) \ni f \mapsto gf \in (\mathbb{P}(\mathbb{C}^N), \|\cdot\|)$ is continuous, with g being a fixed polynomial. It is shown that the Markov type inequality

$$\left\|\frac{\partial}{\partial z_j} P\right\| \le M(\deg P)^m \|P\|, \quad j = 1, \dots, N,\ P \in \mathbb{P}(\mathbb{C}^N),$$

with positive constants M and m is equivalent to the inequality

$$\left\|\frac{\partial^N}{\partial z_1 \dots \partial z_N} P\right\| \le M'(\deg P)^{m'} \|P\|, \quad P \in \mathbb{P}(\mathbb{C}^N),$$

with some positive constants M' and m'. A similar equivalence result is obtained for derivatives of a fixed order $k \ge 2$, which can be more specifically formulated in the language of normed algebras. In addition, we give a nontrivial example of Markov's inequality in the Wiener algebra of absolutely convergent trigonometric series and show that the Banach algebra approach to Markov's property furnishes new tools in the study of polynomial inequalities.

1. Introduction. Let $m > 0$. A compact subset E of $\mathbb{K}^N$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) is called a *Markov set* with exponent m if for every $P \in \mathbb{P}(\mathbb{C}^N)$ the following *Markov inequality* holds:

$$(\mathcal{M}(m)) \qquad \big\||\mathrm{grad}\, P|\big\|_E \le M(\deg P)^m \|P\|_E,$$

where $\|f\|_E = \max\{|f(x)| : x \in E\}$, $|(z_1, \dots, z_N)| = (\sum_{j=1}^N |z_j|^2)^{1/2}$ and M is independent of P. If E is such a set, we shall write $E \in \mathcal{M}(m)$.

Since $|z| = \max_{|v|=1} |v_1 z_1 + \dots + v_N z_N|$, $(\mathcal{M}(m))$ is equivalent to the existence of N linearly independent vectors $v_1, \dots, v_N$ and positive constans

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m_j, M_j, $j = 1, \ldots, N$, such that $\|D_{v_j} P\|_E \le M_j (\deg P)^{m_j} \|P\|_E$ for $j = 1, \ldots, N$ and $m = \max_{1 \le j \le N} m_j$.

Let us recall the classical result: if $E = [-1, 1] \subset \mathbb{C}$, then Markov's inequality holds with $M = 1$ and $m = 2$. These constants are the best possible because for each n and $P = T_n$, where T_n is the nth Chebyshev polynomial of the first kind, we have $\|T_n\|_E = 1$ and $T_n'(1) = n^2$.

Markov's exponent of a Markov set E is, by definition, the best exponent in $(\mathcal{M}(m))$, i.e. $m(E) := \inf\{m > 0 : E \in \mathcal{M}(m)\}$. If E is not a Markov set, we put $m(E) := \infty$. Note also that there are examples such that $E \in \mathcal{M}(m(E) + \varepsilon)$ but $E \notin \mathcal{M}(m(E))$ (see [BBM]). Let us also remark that in the one-dimensional case the constants M and m are related to lower bounds of the logarithmic capacity of E (cf. [B-C1], [B-C2]).

The importance of Markov's property was explained by W. Pleśniak in [P1] (cf. [P2]). The notion of the Markov exponent was introduced in [BaPl] and we refer the reader to that paper for further properties of $m(E)$ (see also [BBM] and [M2]).

If we replace the uniform norm in the space $\mathbb{P}(\mathbb{C}^N)$ by an L_p norm with respect to a probability measure μ on E or by an arbitrary norm $\|\cdot\|$ in $\mathbb{P}(\mathbb{C}^N)$, then we may ask about Markov's property of polynomials and Markov's exponent with respect to such a norm. Then, in general, the situation becomes much more complicated. As an example, let us recall the classical Hille–Szegö–Tamarkin theorem [HST] for $E = [-1, 1]$ and $\mu = \frac{1}{2}\,dx$, $\|P\|_p = \big(\frac{1}{2}\int_E |P|^p\,d\mu\big)^{1/p}$, $1 \le p < \infty$, which reads as follows. For each $p \in [1, \infty)$, there is $C_p > 0$ such that for each $P \in \mathbb{P}(\mathbb{C})$ one has $\|P'\|_p \le C_p (\deg P)^2 \|P\|_p$. An exact value of C_p is known only for $p = 2$, and the fact that $\lim_{p\to\infty} C_p = 1$ was proved by Baran [B] 60 years after [HST]. A similar result to that of Hille–Szegö–Tamarkin is true for E being the closure of a domain in $\mathbb{R}^N$ with Lipschitz boundary (see [G]). However, if E is a fat domain with cusps, no example is known of E where Markov's exponent is calculated with respect to the L_p norm, for some $1 \le p < \infty$.

Milówka [M2] discussed a Markov type inequality for elements in Banach algebras.

Let x be an element of a normed complex algebra $(\mathcal{A}, \|\,\|)$ with unity e. Then x has Markov's property if there exist positive constants M, m such that for each polynomial P of one variable we have

$$(\mathcal{M}(m, x)) \qquad \|P'(x)\| \le M (\deg P)^m \|P(x)\|.$$

If e.g. E is a compact subset of $\mathbb{C}$ and $x = \mathrm{id}_E \in (\mathcal{C}(E), \|\,\|_E)$ then x has Markov's property iff $E \in \mathcal{M}(m)$. In the general case of an algebra $\mathcal{A}$ we define $m(x) = \inf\{m > 0 : x \in \mathcal{M}(m)\}$.

Now we shall give an interesting example (cf. [O]), which justifies investigating Markov's inequality in normed algebras.

Let $\|z\|_p = \|(z_1,\ldots,z_N)\|_p = (|z_1|^p + \cdots + |z_N|^p)^{1/p}$ for $1 \le p < \infty$ and $\|z\|_\infty = \lim_{p\to\infty}\|z\|_p = \max(|z_1|,\ldots,|z_N|)$. We denote by B_p^N (resp. S_p^{N-1}) the closed unit ball (the unit sphere resp.) in $(\mathbb{C}^N, \|\cdot\|_p)$. If $\operatorname{extr}(E)$ denotes the set of extreme points of a convex body $E \subset \mathbb{C}^N$, then for $1 < p < \infty$ we have $\operatorname{extr}(B_p^N) = S_p^{N-1}$. But in the case $p = 1$ one has $\operatorname{extr}(B_1^N) = \{\eta_1 e_1, \ldots, \eta_N e_N : |\eta_j| = 1, j = 1,\ldots,N\}$, where $e_1,\ldots,e_N$ is the canonical basis in $\mathbb{C}^N$. Moreover, in case $p = \infty$ we have a simple description: $\operatorname{extr}(B_\infty^N) = \mathbb{T}^N$, where $\mathbb{T}^N$ denotes the N-dimensional torus.

Let $P(z) = \sum_{j=0}^n a_j z^j = \sum_{j=0}^n a_j(P) z^j$, then we shall denote by $a(P)$ the vector

$$a(P) = \begin{bmatrix} a_0(P) \\ \vdots \\ a_{n-1}(P) \\ a_n(P) \end{bmatrix}.$$

Let $\mathcal{A}(\mathbb{T})$ denote the complex Wiener algebra of absolutely convergent trigonometric series $y = \sum_{k=-\infty}^{\infty} a_k e^{ikt}$ equipped with Wiener's norm $w_1(y) = \sum_{k=-\infty}^{\infty} |a_k|$.

The Chebyshev polynomials T_k are linearly independent in $\mathbb{P}(\mathbb{C})$. Hence if $P \in \mathbb{P}_n(\mathbb{C}) = \{P \in \mathbb{P}(\mathbb{C}) : \deg P \le n\}$, then $P(z) = \sum_{k=0}^n \alpha_k T_k(z)$. Thus

$$P\Big(\frac{1}{2}(e^{-it}+e^{it})\Big) = \sum_{k=0}^n \alpha_k T_k(\cos kt) = \sum_{k=0}^n \alpha_k \cos kt = \sum_{k=0}^n \alpha_k \frac{1}{2}(e^{-ikt}+e^{ikt}),$$

and for $x = \cos t = \frac{1}{2}(e^{-it} + e^{it}) \in \mathcal{A}(\mathbb{T})$ we obtain $w_1(P(x)) = \sum_{k=0}^N |\alpha_k|$.

Let $[\ell_{jk}(n)] \in \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$. Set $z^k = \sum_{j=0}^n \ell_{jk}(n) T_j(z)$, $k = 0,\ldots,n$. Then $a(z^k) = \sum_{j=0}^n \ell_{jk}(n) a(T_j(z))$, which means that

$$\begin{aligned} a_m(z^k) &= \sum_{j=0}^n \ell_{jk}(n) a_m(T_j(z)) \\ &= \sum_{j=0}^n a_m(T_j(z)) \ell_{jk}(n) = \delta_{mk}, \qquad k, l \in \{0,\ldots,n\}, \end{aligned}$$

whence $[\ell_{jk}(n)]^{-1} = [a_j(T_k(z))]$. In particular,

$$[\ell_{jk}(n)]^{-1} e_m = [a_j(T_k(z))] e_m = a(T_m),$$

and we have

$$P(z) = \sum_{k=0}^n a_k(P) \sum_{l=0}^n \ell_{lk}(n) T_l(z) = \sum_{l=0}^N \Big(\sum_{k=0}^n a_k(P) \ell_{lk}(n)\Big) T_l(z).$$

Thus

$$a_l(P) = \sum_{k=0}^{n} a_k(P)\ell_{lk}(n) \quad \text{and} \quad w_1(P(x)) = \sum_{l=0}^{n}\Big|\sum_{k=0}^{n} a_k(P)\ell_{lk}(n)\Big|.$$

Let $\widetilde{X}_n = (\mathbb{P}_n(\mathbb{C}), w_1(P(x)))$ and $\mathcal{B}^n(x) = \{P \in \widetilde{X}_n : w_1(P(x)) \le 1\}$.

Theorem 1.1. *For each $n \in \mathbb{Z}_+$,*

$$\operatorname{extr}(\mathcal{B}^n(x)) = \{\eta_0 T_0, \dots, \eta_n T_n : |\eta_j| = 1,\ j = 0, \dots, n\}.$$

Proof. If $P \in \operatorname{extr}(\mathcal{B}^n(x))$, then $[\ell_{jk}(n)]a(P) \in \operatorname{extr}(B_1^{n+1})$, whence $a(P) \in [\ell_{jk}(n)]^{-1}(\operatorname{extr}(B_1^{n+1}))$. Thus, for some $m \in \{0, \dots, n\}$, we must have

$$a(P) = [\ell_{jk}(n)]^{-1}\eta_m e_m = \eta_m a(T_m) \ \Leftrightarrow\ P = \eta_m T_m,\ |\eta_m| = 1. \ \blacksquare$$

Proposition 1.2. *For each $n \ge 1$ and all $P \in \mathbb{P}_n(\mathbb{C})$, we have*

$$w_1(P'(x)) \le n^2 w_1(P(x)) \quad \textit{with } x = \cos t,$$

where equality holds only for $P = \eta T_n$ with $|\eta| = 1$.

Proof. By Theorem 1.1, it is sufficient to consider the case of $P \in \mathbb{P}_n(\mathbb{C})$ of the form $P = \eta T_n$, $|\eta| = 1$. Then $P'(\cos t) = \eta n U_{n-1}(\cos t)$, where U_{n-1} is the Chebyshev polynomial of the second kind. One has

$$w_1(P'(x)) = n w_1(U_{n-1}(\cos t)) = n w_1\Big(\frac{\sin(nt)}{\sin t}\Big) = n w_1\Big(\frac{(e^{it})^n - (e^{-it})^n}{e^{it} - e^{-it}}\Big)$$
$$= n w_1\Big(\sum_{k=1}^{n} e^{it(n-2k+1)}\Big) = n^2 = n^2 w_1(P(x)). \ \blacksquare$$

Remark 1.3. If $f \in \mathcal{A}(\mathbb{T})$ is a trigonometric polynomial of degree n, that is, $f = \sum_{k=-n}^{n} a_k e^{ikt}$, $|a_n| + |a_{-n}| > 0$, then we have the bounds

$$\|f\|_\infty = \max_{|t|\le\pi} |f(t)| \le w_1(f) \le (2n+1)^{1/2}\|(a_{-n}, \dots, a_n)\|_2$$
$$= (2n+1)^{1/2}\|f\|_2 \le (2n+1)^{1/2}\|f\|_\infty.$$

Since $\|f\|_\infty = \rho(f)$, the radial norm in $\mathcal{A}(\mathbb{T})$, we can write $\rho(f) \le w_1(f) \le (2n+1)^{1/2}\rho(f)$. If $x = \cos t$, we get

$$\rho(P'(x)) \le w_1(P'(x)) \le (\deg P)^2 w_1(P(x))$$
$$\le (2\deg P + 1)^{1/2}(\deg P)^2 \rho(P(x)),$$

which is Markov's inequality for $E = [-1, 1]$ with exponent $5/2$. It is probably one of the simplest proofs establishing Markov's property of an interval.

Definition 1.4. Let $\mathcal{T} \subset \mathbb{N}^N \setminus \{0\}$ be a finite (test) set. A set $E \subset \mathbb{C}^N$ is said to have *Markov's property with exponent m with respect to $\mathcal{T}$*, briefly

$E \in \mathcal{M}(\mathcal{T}, m)$, if there exist positive constants M and m such that for all $P \in \mathbb{P}(\mathbb{C}^N)$,

$$(\mathcal{M}(\mathcal{T}, m)) \qquad \|D^\alpha P\| \le M^{|\alpha|}(\deg P)^{|\alpha| m}\|P\|, \quad \alpha \in \mathcal{T},$$

where $\|\cdot\|$ is a given norm in $\mathbb{P}(\mathbb{C}^N)$. One can also consider a similar property in a normed algebra $\mathcal{A}$ by replacing $\|P^{(k)}\|$ and $\|P\|$ with $\|P^{(k)}(x)\|$ and $\|P(x)\|$, where x is a fixed element of $\mathcal{A}$.

Remark 1.5. If E has Markov's property (with respect to a norm $\|\cdot\|$) and Markov's inequality $(\mathcal{M})$ holds with constants M, m then Markov's inequality $(\mathcal{M}(\mathcal{T}, m))$ holds with the same constants and thus E has Markov's property with respect to $\mathcal{T}$. Note that standard Markov's property is with respect to the set $\mathcal{T}_1 = \{e_1, \ldots, e_N\}$.

By [M1], if E admits Markov's inequality in the uniform norm with constants M and m with respect to $\mathcal{T}_k = \{ke_1, \ldots, ke_n\}$, then it admits $(\mathcal{M})$ with constants MC_k and m_k, where C_k and m_k depend only on k. Let us briefly recall the argument leading to this result.

In [M1], Milówka has discovered the following polynomial identity

$$(\mathcal{I}) \qquad (P')^k = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j \binom{k}{j} P^j (P^{k-j})^{(k)},$$

where P is an arbitrary polynomial of one variable. It has been applied to show the following two results:

Proposition 1.6 ([M1, Thm. 3.5]). *If $E \subset \mathbb{C}$ has the Markov property with respect to the kth derivative, then E has Markov's property $(\mathcal{M})$. More precisely, if*

$$\|P^{(k)}\|_E \le C^k n^{km} \|P\|_E \quad \text{for all } P \in \mathbb{P}_n(\mathbb{C}) \text{ and a fixed } k \in \mathbb{Z}_+,$$

then

$$\|P'\|_E \le 6Ck^{m-1} n^m \|P\|_E.$$

Proposition 1.7 ([M1, Thm. 3.9]). *Let E be a compact set in $\mathbb{C}^N$. Then E has Markov's property if and only if there exists an integer $k \ge 1$ and positive constants C, m such that*

$$\|D^\alpha P\|_E \le C^k n^{km} \|P\|_E \quad \text{for } P \in \mathbb{P}_n(\mathbb{C}^N)$$

for all α with $|\alpha| = k$. Moreover, $m(E, k) = km(E, 1) = km(E)$, where $m(E, k) = \inf\{s : \|D^\alpha P\|_E \le \mathrm{const} \cdot n^s \|P\|_E$ for $P \in \mathbb{P}_n(\mathbb{C}^N)$ and $|\alpha| = k\}$.

Applying $(\mathcal{I})$, by a similar argument to that of [M1] one can extend Proposition 1.7 as follows.

Theorem 1.8. *Let $(\mathcal{A}, \|\,\|)$ be a normed algebra, and $x \in \mathcal{A}$ be such that $\rho(P(x)) \ge (K \deg P + 1)^{-\kappa} \|P(x)\|$ with nonnegative constants K, κ. Assume* *that for some* $k \geq 1$, $\|P^{(k)}(x)\| \leq C^k (\deg P)^{km} \|P(x)\|$. *Then*

$$\rho(P'(x)) \leq 6Ck^{m-1} (\deg P)^m \|P(x)\|,$$

and consequently

$$\|P'(x)\| \leq 6Ck^{m-1} (\deg P)^m (K \deg P + 1)^\kappa \|P(x)\|,$$
$$\rho(P'(x)) \leq 6Ck^{m-1} (\deg P)^m (K \deg P + 1)^\kappa \rho(P(x))).$$

Hence $m(x,1) \leq \kappa + m(x,k)/k$.

REMARK 1.9. An inspection of the proof of Proposition 1.6 in [M1] shows that the constant 6 in the above three statements can be replaced by $2e$.

Using Theorem 1.1 and Proposition 1.6 we get Markov's inequality for $E = [-1,1]$ with exponent $m = 2$ and constant $M = e^2$. The proof is surprisingly simple. The most difficult point of it is the well-known fact that

$$\|T_n^{(k)}\|_{[-1,1]} = T_n^{(k)}(1) = \frac{n^2(n^2-1)\cdots(n^2-(k-1)^2)}{1\cdot 3\cdots(2k-1)} \leq \frac{n^{2k}}{(2k-1)!!}$$

(cf. [BE, Exercise E2 c, p. 256]).

Now, if $P \in \mathbb{P}_n(\mathbb{C})$, $n \geq 1$, then by Theorem 1.1 and Remark 1.3,

$$\begin{aligned} w_1(P^{(k)}(x)) &\leq \max_{0\leq l\leq n} w_1(T_l^{(k)}(x)) w_1(P(x)) \\ &\leq \sqrt{2n+1} \max_{0\leq l\leq n} \|T_l^{(k)}\|_{[-1,1]} w_1(P(x)) \leq \sqrt{2n+1} \frac{n^{2k}}{(2k-1)!!} w_1(P(x)), \end{aligned}$$

where $x = \cos t \in \mathcal{A}(\mathbb{T})$. Again, by Remark 1.3, we get

$$\begin{aligned} \|P^{(k)}\|_{[-1,1]} &\leq \sqrt{2n+1}\, w_1(P^{(k)}(x)) \leq (2n+1) \frac{n^{2k}}{(2k-1)!!} \|P\|_{[-1,1]} \\ &\leq 4 \frac{n^{2k+1}}{(2k-1)!!} \|P\|_{[-1,1]}. \end{aligned}$$

Finally, by applying Proposition 1.6 with $C = \sqrt[k]{4/(2k-1)!!}$, $m = 2 + 1/k$, we obtain

$$\|P'\|_{[-1,1]} \leq 2e\sqrt[k]{4}(k/\sqrt[k]{(2k-1)!!}) n^{2+1/k} \|P\|_{[-1,1]}$$

and, since $\lim_{k\to\infty} k/\sqrt[k]{(2k-1)!!} = e/2$, letting $k \to \infty$ gives

$$\|P'\|_{[-1,1]} \leq e^2 n^2 \|P\|_{[-1,1]}.$$

2. Identities for derivatives of polynomials. Unfortunately the polynomial identity $(\mathcal{I})$ cannot be applied outside normed algebras. To omit this difficulty we shall find other identities that give relations between the first derivative of a polynomial and some derivatives of higher orders. This way we obtain results weaker than in the case of normed algebras, but the new tools will work in a very general situation.

Our first basic lemma is

LEMMA 2.1. *Fix $k \in \mathbb{N}, k \geq 2$. If P is a polynomial of one variable then*

$$(\mathcal{I}_1) \qquad P'(x) = \frac{(-1)^{k-1}}{k!} \sum_{l=0}^{k-1} \frac{(-1)^l}{l!} (P(x)x^l)^{(k)} (x^{k-1})^{(l)} = \frac{(-1)^{k-1}}{k!} \sum_{l=0}^{k-1} \frac{(-1)^l}{l!} (Q_l(x))^{(k)} (x^{k-1})^{(l)}.$$

Proof. Applying Cauchy's integral formula we can write

$$\begin{aligned} P'(x) &= \frac{1}{2\pi i} \oint_{\mathbb{T}} P(\zeta)(\zeta - x)^{-2}\, d\zeta = \frac{1}{2\pi i} \oint_{\mathbb{T}} (\zeta - x)^{k-1} P(\zeta)(\zeta - x)^{-k-1}\, d\zeta \\ &= (-1)^{k-1} x^{k-1} \frac{1}{2\pi i} \oint_{\mathbb{T}} \left(1 - \frac{\zeta}{x}\right)^{k-1} P(\zeta)(\zeta - x)^{-k-1}\, d\zeta \\ &= (-1)^{k-1} x^{k-1} \sum_{l=0}^{k-1} \binom{k-1}{l} (-1)^l x^{-l} \frac{1}{2\pi i} \oint_{\mathbb{T}} P(\zeta)\zeta^l (\zeta - x)^{-k-1}\, d\zeta. \end{aligned}$$

Hence, by Leibniz's formula, we get

$$\begin{aligned} P'(x) &= \frac{(-1)^{k-1}}{k!} \sum_{l=0}^{k-1} \binom{k-1}{l} (-1)^l x^{k-l-1} (P(x)x^l)^{(k)} \\ &= \frac{(-1)^{k-1}}{k!} \sum_{l=0}^{k-1} \frac{(-1)^l}{l!} (P(x)x^l)^{(k)} (x^{k-1})^{(l)}. \ \blacksquare \end{aligned}$$

PROPOSITION 2.2. *Fix $1 \leq j \leq N$ and $k \geq 2$ and put $\alpha = ke_j \in \mathbb{N}^N$. If $P \in \mathbb{P}(\mathbb{C}^N)$ and $Q_{j,l}(x) = P(x)x_j^l$, then*

$$(\mathcal{I}_1') \qquad D_j P(x) = \frac{(-1)^{k-1}}{k!} \sum_{l=0}^{k-1} \frac{(-1)^l}{l!} D^\alpha (Q_{j,l}(x)) (x_j^{k-1})^{(l)}.$$

The next lemma will be applied later in the case where $f(x) = P(x)$ is a polynomial of N variables.

For $j = 1, \ldots, N$, define $\varphi_j(x) = x_1 \cdots \widehat{x_j} \cdots x_N$. We shall denote by $\mathcal{P}_j(N)$ the set of all monomials $\psi(x)$ such that $\deg_j \psi(x) = 0$ and $\deg_i \psi(x) \leq 1$ for all i (so $\deg \psi(x) \leq N-1$). If $\psi(x) \in \mathcal{P}_j(N)$, then there is exactly one $\psi^*(x) \in \mathcal{P}_j(N)$ such that $\psi(x)\psi^*(x) = \varphi_j(x)$, $j = 1, \ldots, N$.

LEMMA 2.3. *Let $f \in \mathcal{O}(\mathbb{C}^N)$ and $\alpha = (1, \ldots, 1) = e_1 + \cdots + e_N \in \mathbb{N}^N$. Then*

$$(\mathcal{I}_2) \qquad D_j f(x) = \sum_{\psi \in \mathcal{P}_j(N)} (-1)^{\deg \psi} D^\alpha (f(x)\psi(x)) \cdot \psi^*(x), \qquad j = 1, \ldots, N.$$

Proof. We start with Cauchy's integral formula in $\mathbb{C}^N$. If $\alpha=(\alpha_1,\dots,\alpha_N)\in\mathbb{N}^N$ and x is chosen from the unit polydisc, we have

$$D^\alpha f(x) = \alpha!\left(\frac{1}{2\pi i}\right)^N \oint_{\mathbb{T}^N} f(\zeta_1,\dots,\zeta_N)(\zeta_1-x_1)^{-\alpha_1-1}\cdots(\zeta_N-x_N)^{-\alpha_N-1}\,d\zeta_1\dots d\zeta_N.$$

If now $\alpha=e_j$, we can write

$$\begin{aligned}D_jf(x_1,\dots,x_N) &= \left(\frac{1}{2\pi i}\right)^N \oint_{\mathbb{T}^N} f(\zeta_1,\dots,\zeta_N)(\zeta_j-x_j)^{-1}\cdot\prod_{1\le l\le N}(\zeta_l-x_l)^{-1}\,d\zeta_1\dots d\zeta_N\\ &= \left(\frac{1}{2\pi i}\right)^n \oint_{\mathbb{T}^N} f(\zeta_1,\dots,\zeta_N)\prod_{1\le l\le N,\,l\ne j}(\zeta_l-x_l)\cdot\prod_{1\le l\le N}(\zeta_l-x_l)^{-2}\,d\zeta_1\dots d\zeta_N.\end{aligned}$$

Hence, as

$$\prod_{1\le l\le N,\,l\ne j}(\zeta_l-x_l)=\sum_{\psi\in\mathcal{P}_j(N)}(-1)^{\deg\psi}\psi(\zeta)\cdot\psi^*(x),$$

we get

$$\begin{aligned}D_jf(x) &= \sum_{\psi\in\mathcal{P}_j(N)}(-1)^{\deg\psi}\left(\frac{1}{2\pi i}\right)^N \oint_{\mathbb{T}^N} f(\zeta)\psi(\zeta)\prod_{1\le l\le N}(\zeta_l-x_l)^{-2}\,d\zeta_1\dots d\zeta_N\cdot\psi^*(x)\\ &= \sum_{\psi\in\mathcal{P}_j(N)}(-1)^{\deg\psi}D^\alpha(f(x)\psi(x))\cdot\psi^*(x),\end{aligned}$$

with $\alpha=e_1+\dots+e_N$. ∎

3. Markov's property for the derivative of order k. Consider a seminorm $\|\cdot\|$ in $\mathbb{P}(\mathbb{C}^N)$ such that for all polynomials f, one has $\|fz_j\|\le A_j\|f\|$, $j=1,\dots,N$, where A_j are constants. Then for a fixed polynomial g, there exists a constant $A(g)$ such that $\|fg\|\le A(g)\|f\|$ for all polynomials f. Such a seminorm $\|\cdot\|$ will be called an *admissible seminorm* in $\mathbb{P}(\mathbb{C}^N)$.

Any L_p norm with respect to a probability measure on a compact subset E of $\mathbb{C}^N$ is admissible, since we can take $A(g)=\max_E|g|$. Another example may be obtained by considering a normed algebra $\mathcal{A}$ with fixed elements $\omega_1,\dots,\omega_N\in\mathcal{A}$ and $\|f\|=\|f(\omega_1,\dots,\omega_N)\|$, where the right-hand side norm is the norm of $\mathcal{A}$; we can put now $A(g)=\|g\|$.

The optimal value of $A(g)$ is $\|L_g\|$, where $L_g:(\mathbb{P}(\mathbb{C}^N),\|\cdot\|)\ni f\mapsto fg\in(\mathbb{P}(\mathbb{C}^N),\|\cdot\|)$. As an example of a norm that is not admissible, we can take $\|f\|=\int_{\mathbb{C}^N}|f(z)|e^{-|z|}\,dV(z)$, where $dV(z)$ is the Lebesgue measure. Let us also note (cf. [B-C2]) that if $E\subset\mathbb{C}$, the (Schur type) inequality $\|P\|_{L_p(E)}\le$

$M(\deg P)^m \|P(x-a)\|_{L_p(E)}$, with M, m independent of $a \in \mathbb{C}$, is equivalent to Markov's inequality with respect to the L_p norm. An interesting question arises whether Schur's inequality is equivalent to Markov's inequality in a more general case.

An immediate consequence of Corollary 2.2 is

PROPOSITION 3.1. *Let $k \geq 2$ and consider an admissible seminorm $\|\cdot\|$ in $\mathbb{P}(\mathbb{C}^N)$ such that $\|fg\| \leq A(g)\|f\|$. If there exist positive constants C_k, m_k such that $\|D^{ke_j}P\| \leq C_k(\deg P)^{m_k}\|P\|$, then*

$$\|D_jP\| \leq B_j(\deg P+1)^{m_k}\|P\|, \quad j=1,\dots,N,$$

where $B_j = \frac{C_k}{k!}A(x_j)\sum_{l=0}^{k-1}\frac{1}{l!}A((x_j^{k-1})^{(l)})$.

Analogously, applying Lemma 2.3 with $\alpha = e_1 + \dots + e_N \in \mathbb{N}^N$ gives

PROPOSITION 3.2. *If there exist positive constants M', m' such that*

$$\left\|\frac{\partial^N}{\partial z_1 \dots \partial z_N}P\right\| \leq M'(\deg P)^{m'}\|P\|, \quad P \in \mathbb{P}(\mathbb{C}^N),$$

we have

$$\|D_jP\| \leq M'N^{m'}\sum_{\psi\in\mathcal{P}_j(n)}A(\psi)A(\psi^*)(\deg P)^{m'}\|P\|, \quad j=1,\dots,N.$$

By combining Propositions 3.1 and 3.2, we get the main result of this paper:

THEOREM 3.3.

(a) *Let $\mathcal{T} = \{k_1e_1,\dots,k_Ne_N\}$, where $k_j \in \mathbb{Z}_+$, $k_j \geq 2$, $1 \leq j \leq N$. Then*

$$E \in \mathcal{M}(\mathcal{T}) \Leftrightarrow E \in \mathcal{M}.$$

(b) *If $\alpha = e_1 + \dots + e_N$ and $\mathcal{T} = \{\alpha\}$, then*

$$E \in \mathcal{M}(\mathcal{T}) \Leftrightarrow E \in \mathcal{M}.$$

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Mirosław Baran
Institute of Mathematics
Jagiellonian University
Łojasiewicza 6
30-348 Kraków, Poland
E-mail: miroslaw.baran@im.uj.edu.pl

Beata Milówka, Paweł Ozorka
Institute of Mathematical and Natural Sciences
State Higher Vocational School in Tarnów
Mickiewicza 8
33-100 Tarnów, Poland
E-mail: bmilowka@wp.pl
pozorka@op.pl

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