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Siciak's extremal function via Bernstein and Markov constants for compact sets in $\mathbb{C}^N$

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Dedicated to Professor Józef Siciak
on the occasion of his 80th birthday

Abstract. The paper is concerned with the best constants in the Bernstein and Markov inequalities on a compact set $E \subset \mathbb{C}^N$. We give some basic properties of these constants and we prove that two extremal-like functions defined in terms of the Bernstein constants are plurisubharmonic and very close to the Siciak extremal function Φ_E. Moreover, we show that one of these extremal-like functions is equal to Φ_E if E is a nonpluripolar set with $\lim_{n\to\infty} M_n(E)^{1/n} = 1$ where

$$M_n(E) := \sup \| |\operatorname{grad} P| \|_E / \|P\|_E, \tag{0.1}$$

the supremum is taken over all polynomials P of N variables of total degree at most n and $\|\cdot\|_E$ is the uniform norm on E. The above condition is fulfilled e.g. for all regular (in the sense of the continuity of the pluricomplex Green function) compact sets in $\mathbb{C}^N$.

1. Introduction. Let $\mathcal{P}_\nu(\mathbb{C}^N)$ with $\nu = (\nu_1, \ldots, \nu_N) \in \mathbb{N}_0^N$ ($\mathbb{N}_0 = \{0, 1, 2, \ldots\}$) be a vector space of polynomials $P = P(z_1, \ldots, z_N)$ with complex coefficients of degree at most ν_i with respect to z_i ($i = 1, \ldots, N$).

For $\alpha, \nu \in \mathbb{N}_0^N$ we define the (α, ν) *Bernstein constant for a compact set* $E \subset \mathbb{C}^N$ *at a point* $w \in \mathbb{C}^N$ by setting

$$M_\nu^{(\alpha)}(w) = M_\nu^{(\alpha)}(E, w) := \sup\left\{ \frac{|D^\alpha P(w)|}{\|P\|_E} : P \in \mathcal{P}_\nu(\mathbb{C}^N),\ P_{|E} \not\equiv 0 \right\}$$

where $\|P\|_E := \max\{|P(z)| : z \in E\}$. The constant

$$M_\nu^{(\alpha)}(E) := \sup\left\{ \frac{\|D^\alpha P\|_E}{\|P\|_E} : P \in \mathcal{P}_\nu(\mathbb{C}^N),\ P_{|E} \not\equiv 0 \right\}$$

is called the (α, ν) *Markov constant for* E (see e.g. [Go], [To], [BC]). In the same manner we can define $M_n^{(\alpha)}(w) = M_n^{(\alpha)}(E, w)$ and $M_n^{(\alpha)}(E)$ for $n \in \mathbb{N}_0$

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by replacing in the definition of $M_\nu^{(\alpha)}(w)$ and $M_\nu^{(\alpha)}(E)$ the set $\mathcal{P}_\nu(\mathbb{C}^N)$ by the space $\mathcal{P}_n(\mathbb{C}^N)$ of all polynomials of total degree not greater than n. Since the Bernstein constants depend on a point $w \in \mathbb{C}^N$, the quantities $M_\nu^{(\alpha)}(E,w)$, $M_n^{(\alpha)}(E,w)$ will sometimes be called the *Bernstein functions*.

The constants $M_n^{(\alpha)}(E,w)$, $M_\nu^{(\alpha)}(E,w)$ and $M_n^{(\alpha)}(E)$, $M_\nu^{(\alpha)}(E)$ are directly connected with the Bernstein and Markov inequalities widely investigated owing to their relations to approximation and constructive theory of functions (e.g. [Pl2], [BoMi], [JoWa], [RaSch]). An important case is when

$$\mu_0(E) := \limsup_{n\to\infty} \frac{\log M_n(E)}{\log n} < \infty \tag{1.1}$$

where $M_n(E)$ is given by (0.1). A compact set E satisfying (1.1) is said to be a *Markov set* and $\mu_0(E)$ is called the *Markov exponent* of E (see [BaPl]).

The Markov constants $(M_\nu^{(\alpha)}(E))_{\nu\in\mathbb{N}_0^N}$ are associated with the Chebyshev constant (if $\alpha = \nu$) and consequently, with the transfinite diameter of E (see [Za]). The Bernstein functions $(M_n^{(0)}(E,w))_{n\in\mathbb{N}_0}$ are strictly related to the Siciak extremal function, because

$$\Phi_E(w) := \sup\left\{\frac{|P(w)|}{\|P\|_E} : \mathcal{P}_n(\mathbb{C}^N),\ P_{|E} \not\equiv 0\right\}^{1/n} = \sup_{n\in\mathbb{N}} (M_n^{(0)}(E,w))^{1/n}$$

where $\mathbb{N} = \{1,2,\ldots\}$ and $D^0P := P$ (for basic properties of Φ_E see e.g. [Si1], [Si2]). We prove that also $(M_n^{(\alpha)}(E,w))_n$ and $(M_\nu^{(\alpha)}(E,w))_\nu$ with $\alpha \neq 0$ are very close to the Siciak extremal function (see Theorem 3.1 and Corollaries 3.4 and 3.5 below). It may be worth reminding the reader that $\log \Phi_E$ is equal to the pluricomplex Green function V_E of the set E with pole at infinity (for definition and background see [Kl]). If V_E is Hölder continuous with exponent s_E then E is a Markov set with $\mu_0(E) \le 1/s_E$.

The exact values of the Bernstein and Markov constants have been found for a few sets only. V. Markov made a very detailed investigation and discovered in 1892 a precise but intricate formula for $M_n^{(k)}([-1,1],w)$, $w \in [-1,1]$. He described these constants using the Zolotarev and Chebyshev polynomials (see e.g. [Sh]). Finally, he proved that

$$M_n^{(k)}([-1,1]) = T_n^{(k)}(1) = \frac{n^2[n^2-1]\ldots[n^2-(k-1)^2]}{1\cdot 3\cdot\ldots\cdot(2k-1)} \tag{1.2}$$

where $T_n(x) = \cos(n \arccos x)$ is the nth Chebyshev polynomial (for $k = 1$ this was proved by A. Markov in 1889). Moreover, thanks to the alternation theorem, we can show that $M_n^{(k)}([-1,1],w) = |T_n^{(k)}(w)|$ for $w \in \mathbb{R} \setminus (-1,1)$.

The exact values of $(M_n^{(k)}(E,w))_{k,n}$ are also known for $E = \{z \in \mathbb{C} : |z| \le r\}$ with $r > 0$, because by the Bernstein inequality, one can obtain $M_n^{(k)}(E,w) = n!|w|^{n-k}/((n-k)!r^n)$ for $|w| \geq r$ and thus $M_n^{(k)}(E) = n!/((n-k)!r^k)$. Due to a result of Baran [Ba], we can give an example in a multivariate space: if f is a fixed norm in $\mathbb{R}^N$ and $E = \{x \in \mathbb{R}^N : f(x) \leq 1\}$ then $M_n^{(\alpha)}(E) = n^2 f(\alpha)$ for any α with $|\alpha| = 1$.

The paper is organized as follows. In the second section we give some elementary properties and examples of the Bernstein and Markov constants. We show that the mapping $w \mapsto M_\nu^{(\alpha)}(E,w)$ (and $w \mapsto M_n^{(\alpha)}(E,w)$) is a plurisubharmonic continuous function in $\mathbb{C}^N$. In Section 3 we prove that the upper regularizations of two extremal-like functions defined by

$$\psi_E^{[\alpha]}(z) := \sup_{\nu \in \mathbb{N}_0^N \setminus \{0\}} (M_\nu^{(\alpha)}(E,z))^{1/|\nu|} \tag{1.3}$$

and

$$\varphi_E^{[\alpha]}(z) := \limsup_{n \to \infty} (M_n^{(\alpha)}(E,z))^{1/n} \tag{1.4}$$

are plurisubharmonic in $\mathbb{C}^N$ and very close to the Siciak extremal function Φ_E (Theorem 3.1). It is also shown that $\varphi_E^{[\alpha]} = \Phi_E$ for a large class of sets, e.g. for Markov sets and for all compacts with continuous pluricomplex Green function (Corollaries 3.4 and 3.5).

2. Basic properties of Bernstein and Markov constants. We start with inequalities that give an obvious bound on $\sup_\nu M_\nu^{(0)}(E,w)$ with respect to the Siciak extremal function. Namely, we have

$$\begin{aligned}\psi_E^{[0]}(w) &= \sup_{\nu \in \mathbb{N}_0^N \setminus \{0\}} (M_\nu^{(0)}(E,w))^{1/|\nu|} \leq \sup_{\nu \in \mathbb{N}_0^N \setminus \{0\}} (M_{|\nu|}^{(0)}(E,w))^{1/|\nu|} \\ &= \Phi_E(w)\end{aligned} \tag{2.1}$$

and

$$\sup_{\nu \in \mathbb{N}_0^N \setminus \{0\}} (M_\nu^{(0)}(E,w))^{1/|\nu|} \geq \sup_{k \in \mathbb{N}} (M_k^{(0)}(E,w))^{1/kN} = \Phi_E(w)^{1/N}. \tag{2.2}$$

From now on, we assume that E is a nonpluripolar compact set. Consider the linear functional $L_w^{(\alpha)} : P \mapsto D^\alpha P(w)$ defined on the finite-dimensional vector space $\mathcal{P}_\nu(\mathbb{C}^N)$ with the norm $\|\cdot\|_E$. Since $L_w^{(\alpha)}$ is bounded and $\|L_w^{(\alpha)}\| = M_\nu^{(\alpha)}(E,w)$, there exists a polynomial $Q \in \mathcal{P}_\nu(\mathbb{C}^N)$ such that

$$M_\nu^{(\alpha)}(E,w) = D^\alpha Q(w) \quad \text{and} \quad \|Q\|_E = 1. \tag{2.3}$$

The set of all such polynomials will be denoted by $\mathcal{M}_\nu^{(\alpha)}(w) = \mathcal{M}_\nu^{(\alpha)}(E,w)$ and its elements will be called *extremal polynomials* for $M_\nu^{(\alpha)}(E,w)$. Analogously, we define the set $\mathcal{M}_n^{(\alpha)}(w) = \mathcal{M}_n^{(\alpha)}(E,w)$ of extremal polynomials for $M_n^{(\alpha)}(E,w)$.

Observe now that (2.1) becomes an equality if E is the Cartesian product of N subsets of $\mathbb{C}$:

Proposition 2.1. *If $E = E_1 \times \cdots \times E_N$ is a nonpluripolar compact set in $\mathbb{C}^N$ then for all $w = (w_1, \ldots, w_N) \in \mathbb{C}^N$ and $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}_0^N$,*

$$M_\nu^{(\alpha)}(E, w) = M_{\nu_1}^{(\alpha_1)}(E_1, w_1) \cdot \ldots \cdot M_{\nu_N}^{(\alpha_N)}(E_N, w_N), \tag{2.4}$$

$$M_\nu^{(\alpha)}(E) = M_{\nu_1}^{(\alpha_1)}(E_1) \cdot \ldots \cdot M_{\nu_N}^{(\alpha_N)}(E_N). \tag{2.5}$$

Furthermore,

$$\psi_E^{[0]}(w) = \sup_{\nu \in \mathbb{N}_0^N \setminus \{0\}} (M_\nu^{(0)}(E, w))^{1/|\nu|} = \Phi_E(w). \tag{2.6}$$

Proof. For a fixed $P \in \mathcal{P}_\nu(\mathbb{C}^N)$ we have

$$\begin{aligned} |D^\alpha P(w)| &= \left| \frac{\partial^{\alpha_1} D^{(0,\alpha_2,\ldots,\alpha_N)} P}{\partial z_1^{\alpha_1}}(w) \right| \\ &\le M_{\nu_1}^{(\alpha_1)}(E_1, w_1) \max_{z_1 \in E_1} |D^{(0,\alpha_2,\ldots,\alpha_N)} P(z_1, w_2, \ldots, w_N)| \\ &\le \cdots \le M_{\nu_1}^{(\alpha_1)}(E_1, w_1) \ldots M_{\nu_N}^{(\alpha_N)}(E_N, w_N) \|P\|_E. \end{aligned}$$

Consequently, (2.4) will follow once we take $P = Q_1 \cdot \ldots \cdot Q_N$ where $Q_j \in \mathcal{M}_{\nu_j}^{(\alpha_j)}(E_j, w_j)$, $j = 1, \ldots, N$, because $M_\nu^{(\alpha)}(E, w) \ge D^\alpha P(w)/\|P\|_E = Q_1^{(\alpha_1)}(w_1) \ldots Q_N^{(\alpha_N)}(w_N)$. From (2.4) and since

$$M_\nu^{(\alpha)}(E) = \sup_{w \in E} M_\nu^{(\alpha)}(E, w),$$

we can easily deduce (2.5).

By a result of Siciak (see [Si2, 3.17]), we have

$$\Phi_E(w) = \max\{\Phi_{E_1}(w_1), \ldots, \Phi_{E_N}(w_N)\}.$$

There is no loss of generality in assuming that $\Phi_E(w) = \Phi_{E_1}(w_1)$. It follows that

$$\sup_{\nu \in \mathbb{N}_0^N \setminus \{0\}} (M_\nu^{(0)}(E, w))^{1/|\nu|} \ge \sup_{\nu_1 \in \mathbb{N}} (M_{\nu_1}^{(0)}(E_1, w_1))^{1/\nu_1} = \Phi_{E_1}(w_1) = \Phi_E(w),$$

which gives (2.6) when combined with (2.1), and the proof is complete. ∎

Example 2.2. As a consequence of Proposition 2.1 we can obtain some exact formulas for $M_\nu^{(\alpha)}(E, w)$ and $M_\nu^{(\alpha)}(E)$ for certain sets. To see an example, let E be a polydisc of polyradius $r = (r_1, \ldots, r_N) \in (0, \infty)^N$, i.e. $E = P(a, r) = \{z \in \mathbb{C}^N : |z_1 - a_1| \le r_1, \ldots, |z_N - a_N| \le r_N\}$. For $w \in \mathbb{C}^N$

such that $|w_1 - a_1| \geq r_1, \ldots, |w_N - a_N| \geq r_N$ we have

$$M_\nu^{(\alpha)}(P(a,r),w) = \frac{\nu!}{(\nu-\alpha)!} \cdot \frac{|w_1 - a_1|^{\nu_1-\alpha_1} \ldots |w_N - a_N|^{\nu_N-\alpha_N}}{r^\nu},$$

$$M_\nu^{(\alpha)}(P(a,r)) = \frac{\nu!}{(\nu-\alpha)!\, r^\alpha}$$

where $r^\nu = r_1^{\nu_1} \ldots r_N^{\nu_N}$. As another example, we can take $I = [a_1, b_1] \times \cdots \times [a_N, b_N] \subset \mathbb{R}^N \subset \mathbb{R}^N + i\mathbb{R}^N = \mathbb{C}^N$. In this case we get

$$M_\nu^{(\alpha)}(I) = \frac{2^{|\alpha|}}{(b-a)^\alpha} T_{\nu_1}^{\alpha_1}(1) \cdot \ldots \cdot T_{\nu_N}^{\alpha_N}(1)$$

with $a = (a_1, \ldots, a_N)$, $b = (b_1, \ldots, b_N)$.

PROPOSITION 2.3. *Let E be a nonpluripolar compact set in $\mathbb{C}^N$. Then for every $\alpha, \nu \in \mathbb{N}_0^N$, $n \in \mathbb{N}_0$ and $w \in \mathbb{C}^N$, $r \in (0,\infty)^N$ we have*

$$M_\nu^{(\alpha)}(E,w) \leq \frac{\alpha!}{r^\alpha} \|\Phi_E\|_{P(w,r)}^{|\nu|}, \tag{2.7}$$

$$M_n^{(\alpha)}(E,w) \leq \frac{\alpha!}{r^\alpha} \|\Phi_E\|_{P(w,r)}^{n}. \tag{2.8}$$

Moreover, if $\nu \geq \alpha$, $n \geq |\alpha|$ then

$$M_\nu^{(\alpha)}(E) \geq \frac{\nu!}{(\nu-\alpha)!} \cdot \frac{1}{(\operatorname{diam} E)^{|\alpha|}}, \tag{2.9}$$

$$M_n^{(\alpha)}(E) \geq \frac{1}{(\operatorname{diam} E)^{|\alpha|}} \left[\frac{n}{|\alpha|}\right]^{|\alpha|} > \frac{1}{(\operatorname{diam} E)^{|\alpha|}} \left(\frac{n}{|\alpha|} - 1\right)^{|\alpha|} \tag{2.10}$$

where $\operatorname{diam} E := \max\{\|z-w\|_2 : z, w \in E\}$, $\|\cdot\|_2$ *is the Euclidean norm and $[k]$ is the greatest integer less than or equal to k.*

Proof. By Cauchy's integral formula and the Bernstein–Walsh–Siciak inequality, we get the following inequalities for a fixed polynomial P:

$$|D^\alpha P(w)| \leq \frac{\alpha!}{r^\alpha} \|P\|_{P(w,r)} \leq \frac{\alpha!}{r^\alpha} \|\Phi_E\|_{P(w,r)}^{\deg P} \|P\|_E,$$

which establishes both (2.7) and (2.8).

In order to prove inequality (2.9), we take $u, w \in E$ such that $|(u-w)^\nu| = \max\{|(s-t)^\nu| : s, t \in E\} > 0$. Put $P(z) = (z-u)^\nu$. We have

$$\begin{aligned} M_\nu^{(\alpha)}(E) \geq M_\nu^{(\alpha)}(E,w) &\geq \frac{|D^\alpha P(w)|}{\|P\|_E} \\ &\geq \frac{\nu!}{(\nu-\alpha)!} \cdot \frac{(w-u)^{\nu-\alpha}}{(w-u)^\nu} \geq \frac{\nu!}{(\nu-\alpha)!} \cdot \frac{1}{(\operatorname{diam} E)^{|\alpha|}}. \end{aligned}$$

To deal with (2.10), consider $\nu = [n/|\alpha|]\alpha \geq \alpha$. From (2.9) it follows that

$$M_\nu^{(\alpha)}(E) \geq \frac{\nu!}{(\nu-\alpha)!}\cdot\frac{1}{(\operatorname{diam} E)^{|\alpha|}}$$
$$\geq \frac{[([n/|\alpha|]-1)\alpha_1+1]^{\alpha_1}\dots[([n/|\alpha|]-1)\alpha_N+1]^{\alpha_N}}{(\operatorname{diam} E)^{|\alpha|}} \geq \left[\frac{n}{|\alpha|}\right]^{|\alpha|}\frac{1}{(\operatorname{diam} E)^{|\alpha|}}.$$

Since $M_n^{(\alpha)}(E) \geq M_{[n/|\alpha|]|\alpha|}^{(\alpha)}(E) \geq M_\nu^{(\alpha)}(E)$, the above inequalities yield (2.10). ∎

THEOREM 2.4. *Let E be a nonpluripolar compact set in $\mathbb{C}^N$. For every $\alpha, \nu \in \mathbb{N}_0^N$ the mapping $\mathbb{C}^N \ni w \mapsto M_\nu^{(\alpha)}(E,w) \in [0,\infty)$ is a plurisubharmonic continuous function in $\mathbb{C}^N$. The same holds for $M_n^{(\alpha)}(E,\cdot)$, $n \in \mathbb{N}$.*

Proof. We will only prove that $M_\nu^{(\alpha)}$ is plurisubharmonic and continuous. The case of $M_n^{(\alpha)}$ is similar.

Observe first that $u_P := |D^\alpha P|/\|P\|_E$ is a plurisubharmonic function in $\mathbb{C}^N$ for every $P \in \mathcal{P}_\nu(\mathbb{C}^N)$. Inequality (2.7) shows that the family $\{u_P : P \in \mathcal{P}_\nu(\mathbb{C}^N),\ P_{|E} \not\equiv 0\}$ is locally uniformly bounded from above. Hence, by [Kl, Th. 2.9.14], the upper regularization of $M_\nu^{(\alpha)}$ is plurisubharmonic.

Now, it is sufficient to show that $M_\nu^{(\alpha)}$ is continuous. Since $M_\nu^{(\alpha)}$ is a supremum of continuous functions, it is lower semicontinuous. To prove the upper semicontinuity, take an arbitrary $w_0 \in \mathbb{C}^N$ and a sequence $(w_l)_{l\in\mathbb{N}}$ such that $w_l \to w_0$ and $\limsup_{w\to w_0} M_\nu^{(\alpha)}(w) = \lim_{l\to\infty} M_\nu^{(\alpha)}(w_l)$. Consider a sequence of extremal polynomials $Q_l \in \mathcal{M}_\nu^{(\alpha)}(w_l)$ for $l = 1, 2, \dots$. In particular, $\|Q_l\|_E = 1$ for every l. As $\mathcal{P}_\nu(\mathbb{C}^N)$ is finite-dimensional, the norm $\|\cdot\|_E$ is equivalent to the sum of the moduli of the coefficients. It is therefore possible to choose a convergent subsequence $(Q_{l_m})_m$ that tends to a polynomial, say Q, such that for every $\beta \in \mathbb{N}_0^N$ the sequence of derivatives $(D^\beta Q_{l_m})_m$ tends to $D^\beta Q$ uniformly on compact sets. Clearly, $Q \in \mathcal{P}_\nu(\mathbb{C}^N)$ and an elementary verification shows that $\|Q\|_E = 1$. Moreover, the Schwarz lemma leads to $\lim_{m\to\infty} D^\alpha Q_{l_m}(w_{l_m}) = D^\alpha Q(w_0)$.

Summarizing, we have

$$M_\nu^{(\alpha)}(w_0) \geq |D^\alpha Q(w_0)| = D^\alpha Q(w_0) = \lim_{m\to\infty} D^\alpha Q_{l_m}(w_{l_m}) = \lim_{m\to\infty} M_\nu^{(\alpha)}(w_{l_m})$$
$$= \limsup_{w\to w_0} M_\nu^{(\alpha)}(w) \geq \liminf_{w\to w_0} M_\nu^{(\alpha)}(w) = M_\nu^{(\alpha)}(w_0),$$

the last inequality being a consequence of the lower semicontinuity of $M_\nu^{(\alpha)}(w)$. This completes the proof. ∎

We have the following obvious consequence of Theorem 2.4.

COROLLARY 2.5. *The Markov constant* $M_\nu^{(\alpha)}(E)$ *is attained at some point* $w_\nu \in E$ *and some polynomial* $Q \in \mathcal{P}_\nu(\mathbb{C}^N)$, *i.e.* $\|Q\|_E = 1$ *and* $M_\nu^{(\alpha)}(E) = D^\alpha Q(w_\nu)$. *The same holds for* $M_n^{(\alpha)}(E)$, $n \in \mathbb{N}$.

Note that, unlike Markov constants, the Markov exponent defined by (1.1) may not be achieved, which is shown in [BaBCMi] on sets in dimension $N \geq 2$, and in [Go] on the real line.

3. The main result. It is clear that the constants defined with $\mathcal{P}_n(\mathbb{C}^N)$ are more closely related to the Siciak extremal function Φ_E than the ones with $\mathcal{P}_\nu(\mathbb{C}^N)$. However, both functions $(M_\nu^{(\alpha)}(E,w))^{1/|\nu|}$ and $(M_n^{(\alpha)}(E,w))^{1/n}$ are asymptotically (as $\nu, n \to \infty$) very close to Φ_E. We formulate this result in terms of the functions $\psi_E^{[\alpha]}$ and $\varphi_E^{[\alpha]}$ defined by (1.3) and (1.4), respectively. As usual, the upper regularization of f will be denoted by f^*, i.e. $f^*(z) = \limsup_{w\to z} f(w)$. Let

$$m_\psi^{(\alpha)}(E) := \sup_{\nu\in\mathbb{N}_0^N\setminus\{0\}} (M_\nu^{(\alpha)}(E))^{1/|\nu|}, \qquad m_\varphi^{(\alpha)}(E) := \limsup_{n\to\infty} (M_n^{(\alpha)}(E))^{1/n}.$$

By (2.7)–(2.10), we get $m_\psi^{(\alpha)}(E), m_\varphi^{(\alpha)}(E) \in [1,\infty)$.

THEOREM 3.1. *If* $E \subset \mathbb{C}^N$ *is a nonpluripolar compact set and* $\alpha \in \mathbb{N}_0^N$ *then* $(\psi_E^{[\alpha]})^*$ *and* $(\varphi_E^{[\alpha]})^*$ *are plurisubharmonic functions in* $\mathbb{C}^N$ *and for every* $w \in \mathbb{C}^N$ *we have*

$$\Phi_E(w)^{1/N} \leq \psi_E^{[\alpha]}(w) \leq m_\psi^{(\alpha)}(E)\Phi_E(w), \tag{3.1}$$

$$\Phi_E(w) \leq \varphi_E^{[\alpha]}(w) \leq m_\varphi^{(\alpha)}(E)\Phi_E(w). \tag{3.2}$$

Proof. Let us first prove that $(\psi_E^{[\alpha]})^*$, $(\varphi_E^{[\alpha]})^* \in \mathrm{PSH}(\mathbb{C}^N)$. Since $\log(|D^\alpha P(z)|/\|P\|_E) \in \mathrm{PSH}(\mathbb{C}^N)$ for every polynomial P, we have $(|D^\alpha P(z)|/\|P\|_E)^{1/|\nu|} \in \mathrm{PSH}(\mathbb{C}^N)$. From (2.7) and [Kl, Th. 2.9.14], we get the plurisubharmonicity of $(\psi_E^{[\alpha]})^*$. Inequality (2.8) and the fact that the upper regularization of the upper limit of a sequence of plurisubharmonic functions locally bounded above is plurisubharmonic (see [JaJa, Th. 3.4.17]) lets us prove that $(\varphi_E^{[\alpha]})^* \in \mathrm{PSH}(\mathbb{C}^N)$.

The right inequalities of (3.1) and (3.2) are consequences of the fact that

$$\frac{|D^\alpha P(w)|}{\|P\|_E} \leq \Phi_E(w)^{\deg P-|\alpha|} M_\nu^{(\alpha)}(E) \leq \Phi_E(w)^{|\nu|} M_\nu^{(\alpha)}(E) \quad \text{for } P \in \mathcal{P}_\nu,$$

$$\frac{|D^\alpha P(w)|}{\|P\|_E} \leq \Phi_E(w)^{\deg P-|\alpha|} M_n^{(\alpha)}(E) \leq \Phi_E(w)^{n} M_n^{(\alpha)}(E) \quad \text{for } P \in \mathcal{P}_n.$$

For $\alpha = 0$ the first inequality of (3.2) is obvious and that of (3.1) follows from (2.2). Therefore, we now assume that $|\alpha| \geq 1$.

We proceed to show the first inequality of (3.1). To this end, fix $w \in \mathbb{C}^N$, $\delta \in (0,1)$ and consider $Q_n \in \mathcal{M}_n^{(0)}(E,w)$. We can find $\varepsilon_n > 0$ satisfying

$$\frac{|Q_n(w)| - \varepsilon_n}{\|Q_n\|_E + \varepsilon_n} = \frac{Q_n(w) - \varepsilon_n}{1 + \varepsilon_n} > (1-\delta) Q_n(w).$$

Put $M := \max\{\|w\|_1, \max_{z\in E} \|z\|_1\}$ with $\|z\|_1 = |z_1| + \dots + |z_N|$ and

$$W_n(z) := Q_n(z) + \frac{\varepsilon_n}{M} \sum z_j \quad \text{for } z = (z_1, \dots, z_N) \in \mathbb{C}^N,$$

where the sum is taken over $j \in \{1,\dots,N\}$ such that $\frac{\partial Q_n}{\partial z_j}(w) = 0$. In this way we get $\frac{\partial W_n}{\partial z_j}(w) \neq 0$ for all $j = 1,\dots,N$. We can assume, by decreasing ε_n if necessary, that $W_n(w) \neq 0$. It is easily seen that

$$\frac{|W_n(w)|}{\|W_n\|_E} > (1-\delta) Q_n(w).$$

Moreover, for a fixed $k \in \mathbb{N}$, $k > |\alpha|$, with the notation

$$S_{n,k} = \left| \frac{(k-1)\dots(k-|\alpha|+1)}{W_n^{|\alpha|}(w)} \left(\frac{\partial W_n(w)}{\partial z_1}\right)^{\alpha_1} \cdot \ldots \cdot \left(\frac{\partial W_n(w)}{\partial z_N}\right)^{\alpha_N} + \dots + \frac{1}{W_n(w)} \frac{\partial^\alpha W_n(w)}{\partial z^\alpha} \right|,$$

we have $|D^\alpha(W_n^k)(w)| = k|W_n^k(w)| S_{n,k}$. Observe that for every fixed n,

$$\frac{S_{n,k}}{k^{|\alpha|-1}} \to \frac{1}{|W_n(w)|^{|\alpha|}} \left|\frac{\partial W_n(w)}{\partial z_1}\right|^{\alpha_1} \cdot \ldots \cdot \left|\frac{\partial W_n(w)}{\partial z_N}\right|^{\alpha_N} > 0 \quad \text{as } k \to \infty,$$

and thus $(kS_{n,k})^{1/k} = k^{|\alpha|/k} (S_{n,k}/k^{|\alpha|-1})^{1/k} \to 1$ as $k \to \infty$. By the above, we can find a sequence $(k_n)_n$ such that $k_n > |\alpha|$, $(k_n S_{n,k_n})^{1/k_n} > 1-\delta$ and $k_n > k_{n-1}$ for any $n > 1$.

In this way we get

$$\begin{aligned}
\sup_{\nu\in\mathbb{N}_0^N\setminus\{0\}} (M_\nu^{(\alpha)}(E,w))^{\frac{1}{|\nu|}} &\geq \sup_{\nu_n=(nk_n,\dots,nk_n),\, n\in\mathbb{N}} (M_{\nu_n}^{(\alpha)}(E,w))^{\frac{1}{nNk_n}} \\
&\geq \sup_{n\in\mathbb{N}} \left(\frac{|D^\alpha(W_n^{k_n})(w)|}{\|W_n^{k_n}\|_E}\right)^{\frac{1}{nNk_n}} = \sup_{n\in\mathbb{N}} \left(\frac{k_n |W_n(w)|^{k_n} S_{n,k_n}}{\|W_n\|_E^{k_n}}\right)^{\frac{1}{nNk_n}} \\
&\geq \sup_{n\in\mathbb{N}} (Q_n(w))^{\frac{1}{nN}} (1-\delta)^{\frac{2}{nN}} \geq (1-\delta) \Big(\sup_{n\in\mathbb{N}} M_n^{(0)}(E,w)\Big)^{\frac{1}{nN}} \\
&= (1-\delta) \Phi_E(w)^{1/N}.
\end{aligned}$$

Letting δ tend to zero we obtain the first inequality of (3.1).

Finally, we take a sequence of polynomials $L_n \in \mathcal{P}_n(\mathbb{C}^N)$ such that

$$\Phi_E(w) = \lim_{n\to\infty} \left(\frac{|L_n(w)|}{\|L_n\|_E}\right)^{1/n}$$

(see [Si1]). The left inequality of (3.2) can be shown in much the same way as that of (3.1) but we need to consider polynomials L_n instead of Q_n. The proof of the theorem is complete. ■

Theorem 3.1 underlines the key role of the Markov constants $M_\nu^{(\alpha)}(E)$, $M_n^{(\alpha)}(E)$ in the estimate of the growth of the Bernstein functions $M_\nu^{(\alpha)}(E,w)$, $M_n^{(\alpha)}(E,w)$ and the extremal functions $\psi_E^{[\alpha]}$, $\varphi_E^{[\alpha]}$ in the whole space. Furthermore, the sets satisfying the condition

$$\limsup_{n\to\infty}\,(M_n^{(\alpha)}(E))^{1/n} = 1 \tag{3.3}$$

gain a particular meaning, reflected in the following obvious implication of Theorem 3.1.

COROLLARY 3.2. *Under the same assumptions as in Theorem* 3.1*,*

$$m_\varphi^{(\alpha)}(E) = 1 \Rightarrow \varphi_E^{[\alpha]} \equiv \Phi_E \text{ in } \mathbb{C}^N.$$

The next observation follows immediately from inequality (2.10) and the fact that $M_n^{(\alpha)}(E) \le (\max_{|\beta|=1} M_n^{(\beta)}(E))^{|\alpha|}$.

REMARK 3.3. The following conditions are equivalent:

(i) $m_\varphi^{(\alpha)}(E) = 1$ for all $\alpha \in \mathbb{N}_0^N$,
(ii) $\max_{|\beta|=1} m_\varphi^{(\beta)}(E) = 1$,
(iii) $\lim_{n\to\infty}(M_n^{(\alpha)}(E))^{1/n} = 1$ for all $\alpha \in \mathbb{N}_0^N$,
(iv) $\lim_{n\to\infty}(M_n(E))^{1/n} = 1$

where $M_n(E)$ is defined by (0.1).

Let us emphasize that the compacts with property (3.3) form a wide class of sets in $\mathbb{C}^N$. We have the following consequence of the definition of Markov sets (see (1.1)).

COROLLARY 3.4. *If $E \subset \mathbb{C}^N$ is a Markov set then condition* (iv) *is satisfied and thus $\Phi_E \equiv \varphi_E^{[\alpha]}$ in $\mathbb{C}^N$ for any $\alpha \in \mathbb{N}_0^N$.*

Recall that the Hölder continuity of the pluricomplex Green function V_E implies that E is a Markov set, as noted in the introduction. It seems that also the converse holds but a proof is an open problem. However, the Hölder continuity is not a necessary condition for the assertion of Corollary 3.2, because (ii) is satisfied whenever V_E (or equivalently Φ_E) is merely continuous. The sets with continuous pluricomplex Green function are often called regular sets.

COROLLARY 3.5. *If $E \subset \mathbb{C}^N$ is a regular compact set then condition* (ii) *is satisfied and thus $\Phi_E \equiv \varphi_E^{[\alpha]}$ in $\mathbb{C}^N$ for any $\alpha \in \mathbb{N}_0^N$.*

The first proof of (ii) for regular compact sets was given in [To] in the univariate case. This proof can be easily adapted to the general case of sets in $\mathbb{C}^N$, as proved in [CaLe] and [BC].

The question about a relationship between the assumptions in Corollaries 3.4 and 3.5, i.e. between property (1.1) and the regularity of E, is an interesting and nontrivial problem. A partial answer is only given in the real one-dimensional case (see [Pl1], [BCEg]).

To end the paper we exhibit a set without property (3.3). The example seems to be known at least in the univariate case. For the convenience of the reader, we sketch the proof.

EXAMPLE 3.6. If E is a polynomially convex compact set with an isolated point then for no α is condition (3.3) satisfied.

Proof. We can assume that $E = F \cup \{z_0\}$ and $\operatorname{dist}(z_0, F) > 0$. The set F is polynomially convex. For $Q \in \mathcal{M}^{(0)}_{n-|\alpha|}(F, z_0)$ put $P(z) = (z - z_0)^\alpha Q(z)$. It follows that

$$M_n^{(\alpha)}(E) \geq \frac{|D^\alpha P(z_0)|}{\|P\|_E} \geq \frac{\alpha! Q(z_0)}{(\operatorname{diam} E)^{|\alpha|}\|Q\|_F} = \frac{\alpha!}{(\operatorname{diam} E)^{|\alpha|}} M^{(0)}_{n-|\alpha|}(F, z_0)$$

and thus

$$\limsup_{n\to\infty} (M_n^{(\alpha)}(E))^{1/n} \geq \Phi_F(z_0) > 1. \ \blacksquare$$

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