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The transfinite diameter of the real ball and simplex

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Abstract. We calculate the transfinite diameter for the real unit ball $B_d := \{x \in \mathbb{R}^d : |x| \leq 1\}$ and the real unit simplex $T_d := \{x \in \mathbb{R}^d_+ : \sum_{j=1}^d x_j \leq 1\}$.

1. Introduction. Suppose that $K \subset \mathbb{C}^d$ is compact. The transfinite diameter (and the associated notion of capacity) of K is a measure of the size of K, important in classical Complex Potential Theory (for $d = 1$) and Pluripotential Thoery (for $d > 1$). It is defined as follows. For $n \in \mathbb{Z}_+$ consider the monomials z^γ, $|\gamma| \leq n$, which we order as $\{e_1(z), \dots, e_{m_n}(z)\}$ where $e_i(z) = z^{\gamma_i}$, so that the ordering respects the degree, i.e. $|\gamma_j| < |\gamma_i|$ implies that $j < i$. Here $m_n := \binom{n+d}{n}$ is the dimension of the space of polynomials of degree at most n in d complex variables. Then for m_n points $z_j \in K$, $1 \leq j \leq m_n$, the *Vandermonde determinant* of degree n is defined to be

$$\mathrm{vdm}(z_1, \dots, z_{m_n}) := \det([e_i(z_j)])_{1 \leq i,j \leq m_n}).$$

The *transfinite diameter* of K is

$$\delta(K) := \lim_{n \to \infty} \Big(\max_{z_1, \dots, z_{m_n} \in K} |\mathrm{vdm}(z_1, \dots, z_{m_n})| \Big)^{1/\ell_n}$$

where $\ell_n := \frac{d}{d+1} n m_n$ is the degree of homogeneity of $\mathrm{vdm}(z_1, \dots, z_{m_n})$ considered as a polynomial on K^{m_n}.

That the limit exists is a result of Zakharyuta [Z] where the following remarkable formula is proved:

$$\delta(K) = \exp\bigg\{ \frac{1}{\mathrm{vol}(S_d)} \int_{S_d} \log(\tau(\theta))\, dV \bigg\} \tag{1.1}$$

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where

$$S_d := \Big\{\theta \in \mathbb{R}^d : \theta_j \geq 0 \text{ for } 1 \leq j \leq d, \text{ and } \sum_{j=1}^{d} \theta_j = 1\Big\} \tag{1.2}$$

is a symmetric simplex (of dimension $d-1$) in $\mathbb{R}^d$ and, for $\theta \in S_d$,

$$\tau(\theta) = \tau(\theta, K) := \limsup_{i\to\infty,\, \alpha_i/|\alpha_i|\to\theta} \inf\Big\{\Big\|e_i(z) + \sum_{j<i} c_j e_j(z)\Big\|_K^{1/|\alpha_i|}\Big\}$$

is the so-called *directional Chebyshev constant*. We note that, for $\theta \in \operatorname{int}(S_d)$, Zakharyuta showed that the lim sup in the definition of $\tau(\theta)$ may be replaced by an ordinary limit.

In certain cases explicit formulas for the transfinite diameter have been calculated. Jędrzejowski [J] has given the following formula for the unit complex euclidean ball $\mathbb{B}_d := \{z \in \mathbb{C}^d : |z| \leq 1\}$:

$$\delta(\mathbb{B}_d) = \exp\Big(-\frac{1}{2}\sum_{j=2}^{d}\frac{1}{j}\Big). \tag{1.3}$$

From the product formula of Schiffer and Siciak [SS] we also have, for the unit cube $Q_d := [-1,1]^d$,

$$\delta(Q_d) = \frac{1}{2}. \tag{1.4}$$

Rumely [R] has given a beautiful integral formula for the transfinite diameter using notions of pluripotential theory (see also [DR]). However, this appears to be difficult to explicitly evaluate, at least in the case of the real ball in which we are interested.

In this paper we will use Zakharyuta's formula (1.1) directly, by finding a formula for $\tau(\theta)$ and then computing its integral over the simplex, to give a compact formula (Theorem 2.5 below) for the transfinite diameter of the real euclidean unit ball $B_d := \{x \in \mathbb{R}^d : |x| \leq 1\}$ and the real unit simplex $T_d := \{x \in \mathbb{R}^d_+ : \sum_{j=1}^d x_j \leq 1\}$.

2. The ball. We begin with some standard facts about univariate orthogonal polynomials.

2.1. Univariate Gegenbauer polynomials. Let $C_n^\lambda(x)$, $n \in \mathbb{Z}_+$, $\lambda > -1/2$, denote the Gegenbauer polynomials of degree n and parameter λ, i.e., the classical univariate polynomials orthogonal on $[-1,1]$ with respect to the inner product

$$\langle f, g\rangle_\lambda := \int_{-1}^{1} f(x)g(x)w_\lambda(x)\,dx \tag{2.1}$$

where

$$w_\lambda(x) := (1-x^2)^{\lambda-1/2}.$$

It is known that (see e.g. [AS])

$$C_n^\lambda(x) = k_n x^n + \text{lower degree terms}$$

where

$$k_n := \frac{2^n \Gamma(\lambda+n)}{n!\Gamma(\lambda)} \tag{2.2}$$

and that

$$\int_{-1}^{1} (C_n^\lambda(x))^2 w_\lambda(x)\,dx = \frac{\pi 2^{1-2\lambda}\Gamma(n+2\lambda)}{n!(n+\lambda)(\Gamma(\lambda))^2}.$$

If we let $\widehat{C}_n^\lambda(x) := k_n^{-1} C_n^\lambda(x)$ denote the associated *monic* orthogonal polynomials, it follows easily that

$$\widehat{C}_n^\lambda(x) = x^n + \text{lower degree terms}$$

and that

$$h(n,\lambda) := \int_{-1}^{1} (\widehat{C}_n^\lambda(x))^2 w_\lambda(x)\,dx = \frac{\pi 2^{1-2(n+\lambda)}\Gamma(n+1)\Gamma(n+2\lambda)}{(\Gamma(n+\lambda))^2(n+\lambda)}. \tag{2.3}$$

We will make use of the following nth root asymptotics of $h(n,\lambda)$.

LEMMA 2.1. *Suppose that* $a_n, b_n \ge 0$ *are such that* $\lim_{n\to\infty} a_n/n = a$ *and* $\lim_{n\to\infty} b_n/n = b$ *with* $a+b>0$. *Then*

$$\lim_{n\to\infty} h(a_n,b_n)^{1/n} = \frac{a^a(a+2b)^{a+2b}}{2^{2(a+b)}(a+b)^{2(a+b)}}.$$

Proof. From (2.3) we have

$$h(a_n,b_n) = \frac{\pi 2^{1-2(a_n+b_n)}\Gamma(a_n+1)\Gamma(a_n+2b_n)}{(\Gamma(a_n+b_n))^2(a_n+b_n)}.$$

Clearly, the terms $\pi^{1/n}$ and $(a_n+b_n)^{1/n}$ both tend to 1 and hence can be ignored. Further,

$$(2^{1-2(a_n+b_n)})^{1/n} = 2^{1/n-2(a_n/n+b_n/n)} \to 2^{-2(a+b)}$$

and hence we are left with showing that

$$\lim_{n\to\infty}\left(\frac{\Gamma(a_n+1)\Gamma(a_n+2b_n)}{(\Gamma(a_n+b_n))^2}\right)^{1/n} = \frac{a^a(a+2b)^{a+2b}}{(a+b)^{2(a+b)}}.$$

But this is an easy consequence of Stirling's formula, and we are done. ∎

2.2. A family of orthogonal polynomials on B_d. For $\mu > -1/2$ and $x = (x_1, \dots, x_d) \in B_d$ let

$$W_\mu(x) = (1 - |x|^2)^{\mu - 1/2}$$

be a generalized Gegenbauer weight. Now, suppose that $\alpha \in \mathbb{Z}_+^d$ is a multi-index. We define (cf. Prop. 2.3.2 of [DX]) the polynomials

$$\begin{aligned} P_\alpha(x) = \widehat{C}_{\alpha_1}^{\lambda_1}(x_1) &\times \left((1-x_1^2)^{\alpha_2/2}\widehat{C}_{\alpha_2}^{\lambda_2}\left(\frac{x_2}{\sqrt{1-x_1^2}}\right)\right) \\ &\times \left((1-x_1^2-x_2^2)^{\alpha_3/2}\widehat{C}_{\alpha_3}^{\lambda_3}\left(\frac{x_3}{\sqrt{1-x_1^2-x_2^2}}\right)\right) \times \cdots \\ &\times \left((1-x_1^2-\cdots-x_{d-1}^2)^{\alpha_d/2}\widehat{C}_{\alpha_d}^{\lambda_d}\left(\frac{x_d}{\sqrt{1-x_1^2-\cdots-x_{d-1}^2}}\right)\right) \end{aligned}$$

where

$$\begin{aligned} \lambda_1 &= \mu + \alpha_2 + \alpha_3 + \cdots + \alpha_d + (d-1)/2, \\ \lambda_2 &= \mu + \alpha_3 + \alpha_4 + \cdots + \alpha_d + (d-2)/2, \\ \lambda_3 &= \mu + \alpha_4 + \alpha_5 + \cdots + \alpha_d + (d-3)/2, \\ &\;\vdots \\ \lambda_d &= \mu, \end{aligned}$$

that is,

$$\lambda_j := \mu + (d-j)/2 + \sum_{k=j+1}^{d} \alpha_k.$$

Note that since the weight $w_\lambda(x)$ is symmetric, the polynomials $\widehat{C}_n^\lambda(x)$ are even for n even and odd for n odd. Hence the P_α, as defined above, are indeed d-variate polynomials.

LEMMA 2.2. *The polynomials $P_\alpha(x)$ are orthogonal with respect to the inner product*

$$\langle f, g\rangle_\mu := \int_{B_d} f(x)g(x)W_\mu(x)\,dx. \tag{2.4}$$

Proof. This is given in Proposition 2.3.2 of [DX]. ∎

LEMMA 2.3. *The polynomials $P_\alpha(x)$ are* monic, *i.e., of the form*

$$P_\alpha(x) = x^\alpha + \textit{lower order terms}$$

where the ordering of the monomials is graded-lexicographic with $x_1 \prec \cdots \prec x_d$. Moreover,

$$\int_{B_d} (P_\alpha(x))^2 W_\mu(x)\,dx = \prod_{j=1}^{d} h(\alpha_j, \lambda_j).$$

Proof. The mononoticity is obvious from the construction. The norm is easily calculated by writing

$$\int_{B_d} f(x)W_\mu(x)\,dx = \int_{B_{d-1}} \int_{-\sqrt{1-|x'|^2}}^{\sqrt{1-|x'|^2}} f(x',x_d)W_\mu(x',x_d)\,dx_d\,dx',$$

where $x' := (x_1,\ldots,x_{d-1})$, substituting $y_d := x_d/\sqrt{1-|x'|^2}$, and proceeding by induction. ∎

2.2.1. *The measures $W_\mu\,dx$ are Bernstein–Markov.* A measure ν on a compact set $K \subset \mathbb{C}^d$ is said to be a *Bernstein–Markov measure* when there exist constants $C(n)$ with the property that

$$\lim_{n\to\infty} C(n)^{1/n} = 1$$

such that for all polynomials p of degree n,

$$\|p\|_K \le C(n)\|p\|_{L_2(K;\nu)}.$$

For many purposes, including the calculation of transfinite diameters, this means that the uniform norm can be substituted by the L_2 norm with respect to the measure ν.

It turns out that, for the measures $W_\mu(x)\,dx$, $\mu \ge 0$, there is a stronger statement. Specifically, from Lemma 1 of [B2], it follows that, for $\alpha = 0$,

$$\|p\|_{B_d} \le \sqrt{\frac{2}{\omega_d}\left(\binom{n+d}{d} + \binom{n+d-1}{d}\right)}\,\|p\|_{L_2(B_d;W_0dx)} \tag{2.5}$$

for all polynomials p of degree n. Here ω_d is the surface area of the d-dimensional unit sphere in $\mathbb{R}^{d+1}$. In other words, for this measure, $C(n) = O(n^{d/2})$ is of polynomial growth. Similarly, the considerations of [B2] show that $C(n)$ is also of polynomial growth for all $\mu \ge 0$.

2.3. The directional Chebyshev constants. Since $W_\mu(x)\,dx$, $\mu \ge 0$, are Bernstein–Markov measures, we may use any of the associated 2-norms,

$$\|f\|_\mu := \langle f, f\rangle_\mu^{1/2},$$

to compute the directional Chebyshev constants (cf. [B1, p. 320]). Specifically consider $\theta \in S_d$, the unit simplex in $\mathbb{R}^d$ (see (1.2)). Then for $n = |\alpha| = \alpha_1 + \cdots + \alpha_d$ we have

$$\tau(\theta) = \lim_{\alpha/n\to\theta} \|P_\alpha\|_\mu^{1/n} = \lim_{\alpha/n\to\theta} \Big\{\prod_{j=1}^d h(\alpha_j,\lambda_j)\Big\}^{1/2n}$$

by Lemma 2.3.

Suppose now that $\theta \in S_d$ with $\theta > 0$ (i.e., each component is positive). Then under the hypothesis that $\alpha/n \to \theta$ we have

$$\lim_{n\to\infty} \alpha_j/n = \theta_j, \qquad 1 \le j \le d, \tag{2.6}$$

$$\lim_{n\to\infty} \lambda_j/n = \sum_{k=j+1}^{d} \theta_k, \quad 1 \le j \le d. \tag{2.7}$$

Hence, by Lemma 2.1,

$$\lim_{n\to\infty} h(\alpha_j, \lambda_j)^{1/n} = \frac{\theta_j^{\theta_j} (\theta_j + 2\sum_{k=j+1}^{d} \theta_k)^{\theta_j + 2\sum_{k=j+1}^{d} \theta_k}}{2^{2(\sum_{k=j}^{d} \theta_k)} (\sum_{k=j}^{d} \theta_k)^{2(\sum_{k=j}^{d} \theta_k)}} =: H_j(\theta). \tag{2.8}$$

We thus may conclude

LEMMA 2.4. *For $\theta \in S_d$ with $\theta > 0$,*

$$\tau(\theta) = \Big\{\prod_{j=1}^{d} H_j(\theta)\Big\}^{1/2}.$$

2.4. Zakharyuta's formula for the transfinite diameter. We will use Zakharyuta's formula for the transfinite diameter, given in the introduction:

$$\delta(B_d) = \exp\bigg\{\frac{1}{\mathrm{vol}(S_d)} \int_{S_d} \log(\tau(\theta))\, dV\bigg\}. \tag{2.9}$$

THEOREM 2.5. *The transfinite diameter of the unit ball B_d is:*

(a) *for d even,*

$$\delta(B_d) = \frac{1}{2}\exp\bigg(-\frac{1}{4}\,\frac{2d+1}{d}\sum_{j=1}^{d}\frac{1}{j} + \frac{1}{2} + \frac{1}{2}\log(2) + \frac{1}{4d}\sum_{j=1}^{d}\frac{(-1)^j}{j}\bigg),$$

(b) *for d odd,*

$$\delta(B_d) = \frac{1}{2}\exp\bigg(-\frac{1}{4}\,\frac{2d+1}{d}\sum_{j=1}^{d}\frac{1}{j} + \frac{1}{2} + \frac{d-1}{2d}\log(2) - \frac{1}{4d}\sum_{j=1}^{d}\frac{(-1)^j}{j}\bigg).$$

REMARK. For $d = 1$ the formula reduces to the classical $\delta([-1,1]) = 1/2$ and for $d = 2$ we obtain $\delta(B_2) = 1/\sqrt{2e}$, in agreement with the result of [B1].

Proof of Theorem 2.5. We use the Zakharyuta formula together with the formula for $\tau(\theta)$ given in Lemma 2.1 to obtain

$$\delta(B_d) = \exp\bigg(\frac{(d-1)!}{2}(C_d + F_d - D_d - E_d)\bigg)$$

where

$$C_d = \sum_{j=1}^{d} \int_{S_d} \log(\theta_j^{\theta_j})\, dV = -\frac{1}{(d-1)!} \sum_{j=2}^{d} \frac{1}{j},$$

$$D_d = \sum_{j=1}^{d} \int_{S_d} \log(2^{2\sum_{k=j}^{d} \theta_k})\, dV = \frac{d+1}{(d-1)!} \log(2),$$

$$E_d = \sum_{j=1}^{d} \int_{S_d} \log\Big(\Big(\sum_{k=j}^{d} \theta_k\Big)^{2(\sum_{k=j}^{d} \theta_k)}\Big)\, dV = -\frac{1}{2(d-2)!},$$

$$F_d = \sum_{j=1}^{d} \int_{S_d} \log\Big(\Big(\theta_j + 2\sum_{k=j+1}^{d} \theta_k\Big)^{\theta_j + 2\sum_{k=j+1}^{d} \theta_k}\Big)\, dV$$

$$= \frac{1}{2}\frac{1}{(d-2)!}\Big\{2\log(2) - 1 + \int_1^2 (x-1)^{d-2} x \log(x)\, dx\Big\} - \frac{1}{2}\frac{1}{d!}\sum_{j=2}^{d}\frac{1}{j}.$$

The values for the integrals are given in a sequence of lemmas below. Note that $\mathrm{vol}(S_d) = 1/(d-1)!$; putting them together and simplifying gives the result. ∎

We can express integrals over the simplex S_d as univariate integrals by means of B-splines.

Lemma 2.6. *Suppose that* $a_1 \le \cdots \le a_d$ *with* $d \ge 2$ *and that* $f \in L_1[a_1, a_d]$. *Then*

$$\int_{S_d} f\Big(\sum_{j=1}^{d} \theta_j a_j\Big)\, dV = \frac{1}{(a_d - a_1)(d-2)!} \int_{-\infty}^{\infty} f(x) B(x \,|\, a_1, \ldots a_d)\, dx$$

where

$$B(x \,|\, a_1, \ldots, a_d) := (a_d - a_1)(\cdot - x)_+^{d-2}[a_1, \ldots, a_d]$$

is the B-spline of degree $d-2$ *with knots* $a_1, \ldots, a_d$. *Here*

$$y_+ := \begin{cases} y, & y \ge 0, \\ 0, & y < 0, \end{cases}$$

and $f[a_1, \ldots, a_d]$ *denotes the divided difference of the function* f *at the points* a_i.

Proof. This is a standard formula of spline theory, based on the fact that B-splines are the Peano kernel for divided differences; see, for example, [deB, p. 88]. ∎

We will need the following fact.

Lemma 2.7. *For $1 < j \le d$ and $d \ge 2$, we have*

$$B(x \mid \underbrace{0,\ldots,0}_{j-1},\underbrace{1,\ldots,1}_{d-j+1}) = \begin{cases} \binom{d-2}{j-2} x^{d-j}(1-x)^{j-2}, & x\in[0,1], \\ 0, & \text{otherwise.}\end{cases}$$

Proof. This is again a standard fact that follows easily from the recurrence formula for B-splines (see [deB, p. 89]). ■

Lemma 2.8. *For $1 \le j \le d$, we have*

$$\int_{S_d} \theta_j \log(\theta_j)\, dV = \frac{1}{(d-2)!}\int_0^1 (1-x)^{d-2} x\log(x)\,dx = -\frac{1}{d!}\sum_{j=2}^{d}\frac{1}{j}.$$

Hence

$$C_d = \sum_{j=1}^{d}\int_{S_d}\theta_j\log(\theta_j)\,dV = -\frac{1}{(d-1)!}\sum_{j=2}^{d}\frac{1}{j}.$$

Proof. By symmetry we need only consider $j = d$. Then, using the integral formula of Lemma 2.6, we have

$$\int_{S_d}\theta_d\log(\theta_d)\,dV = \frac{1}{(1-0)(d-2)!}\int_{-\infty}^{\infty} x\log(x)B(x\mid \underbrace{0,\ldots,0}_{d-1},1)\,dx$$
$$= \frac{1}{(d-2)!}\int_0^1 x\log(x)(1-x)^{d-2}\,dx$$

since, by Lemma 2.7 with $j = d-1$,

$$B(x\mid \underbrace{0,\ldots,0}_{d-1},1) = \begin{cases}(1-x)^{d-2}, & x\in[0,1],\\ 0, & \text{otherwise.}\end{cases}$$

By Lemma 2.9 below with $m = d-2$ we obtain

$$\frac{1}{(d-2)!}\biggl\{-\frac{1}{(d-1)d}\sum_{j=2}^{d}\frac{1}{j}\biggr\}$$

and the result follows. ■

Lemma 2.9. *For $m \in \mathbb{Z}_+$ we have*

$$\int_0^1 (1-x)^m x\log(x)\,dx = -\frac{1}{(m+1)(m+2)}\sum_{j=2}^{m+2}\frac{1}{j}.$$

Proof. This is a special case of formula 1 of §4.253 of [GR], however, for completeness, we provide an elementary proof. Let A_m denote the integral in question. Then integrating

$$A_{m+1} = \int_0^1 (1-x)^{m+1} x \log(x)\,dx$$

by parts with $u = (1-x)^{m+1}$ and $v' = x\log(x)$ we easily obtain the recurrence

$$A_{m+1} = \frac{m+1}{m+3} A_m - \frac{1}{(m+2)(m+3)^2}$$

from which it is easy to verify the stated formula by induction. ■

LEMMA 2.10. *For $d \ge 2$ we have*

$$D_d = \sum_{j=1}^{d} \int_{S_d} \log(2^{2\sum_{k=j}^{d}\theta_k})\,dV = \frac{d+1}{(d-1)!}\log(2).$$

Proof. The $j=1$ term is slightly different. In fact, for $j=1$, $\sum_{k=j}^{d}\theta_k = 1$ and the integrand is just $2\log(2)$, so the integral equals

$$\int_{S_d} \log(2^{2\sum_{k=1}^{d}\theta_k})\,dV = 2\log(2)\operatorname{vol}(S_d) = \frac{2\log(2)}{(d-1)!}. \tag{2.10}$$

For the other terms we compute

$$\begin{aligned}
&\sum_{j=2}^{d} \int_{S_d} \log(2^{2\sum_{k=j}^{d}\theta_k})\,dV \\
&\quad = 2\log(2)\sum_{j=2}^{d}\int_{S_d}\Big(\sum_{k=j}^{d}\theta_k\Big)\,dV \\
&\quad = 2\log(2)\sum_{j=2}^{d}\frac{1}{(1-0)(d-2)!}\int_0^1 xB(x \mid \underbrace{0,\ldots,0}_{j-1},\underbrace{1,\ldots,1}_{d-j+1})\,dx \\
&\quad = \frac{2\log(2)}{(d-2)!}\sum_{j=2}^{d}\int_0^1 x\binom{d-2}{j-2}x^{d-j}(1-x)^{j-2}\,dx \qquad \text{(by Lemma 2.7)} \\
&\quad = \frac{2\log(2)}{(d-2)!}\int_0^1 x\sum_{j=2}^{d}\binom{d-2}{j-2}x^{d-j}(1-x)^{j-2}\,dx \\
&\quad = \frac{2\log(2)}{(d-2)!}\int_0^1 x\sum_{j=0}^{d-2}\binom{d-2}{j}x^{(d-2)-j}(1-x)^{j}\,dx \\
&\quad = \frac{2\log(2)}{(d-2)!}\int_0^1 x\,dx = \frac{\log(2)}{(d-2)!}.
\end{aligned}$$

Combining this with (2.10) gives

$$\sum_{j=1}^{d}\int\limits_{S_d}\log(2^{2\sum_{k=j}^{d}\theta_k})\,dV=\frac{2\log(2)}{(d-1)!}+\frac{\log(2)}{(d-2)!}$$

and the result follows.

As pointed out by an anonymous reviewer, this integral can also be computed by completely elementary means. First observe that by symmetry,

$$\int\limits_{S_d}\theta_j\,dV=\int\limits_{S_d}\theta_k\,dV=\frac{1}{d}\operatorname{vol}(S_d)=\frac{1}{d!}.$$

Then

$$\begin{aligned}D_d&=\sum_{j=1}^{d}\int\limits_{S_d}\log(2^{2\sum_{k=j}^{d}\theta_k})\,dV=2\log(2)\sum_{j=1}^{d}\sum_{k=j}^{d}\frac{1}{d!}\\&=\frac{2}{d!}\log(2)\sum_{j=1}^{d}(d-j+1)=\frac{2}{d!}\log(2)\frac{d(d+1)}{2}=\frac{d+1}{(d-1)!}\log(2).\ \blacksquare\end{aligned}$$

Lemma 2.11. *For $d\ge 2$ we have*

$$E_d=\sum_{j=1}^{d}\int\limits_{S_d}2\Big(\sum_{k=j}^{d}\theta_k\Big)\log\Big(\sum_{k=j}^{d}\theta_k\Big)\,dV=-\frac{1}{2(d-2)!}.$$

Proof. The $j=1$ term is 0 since $\sum_{k=1}^{d}\theta_k=1$. Hence we compute

$$\begin{aligned}&\sum_{j=2}^{d}\int\limits_{S_d}2\Big(\sum_{k=j}^{d}\theta_k\Big)\log\Big(\sum_{k=j}^{d}\theta_k\Big)\,dV\\&\quad=2\sum_{j=2}^{d}\int\limits_{S_d}\Big(\sum_{k=j}^{d}\theta_k\Big)\log\Big(\sum_{k=j}^{d}\theta_k\Big)\,dV\\&\quad=2\sum_{j=2}^{d}\frac{1}{(1-0)(d-2)!}\int\limits_{0}^{1}x\log(x)B(x\mid\underbrace{0,\dots,0}_{j-1},\underbrace{1,\dots,1}_{d-j+1})\,dx\\&\quad=\frac{2}{(d-2)!}\int\limits_{0}^{1}x\log(x)\sum_{j=2}^{d}B(x\mid\underbrace{0,\dots,0}_{j-1},\underbrace{1,\dots,1}_{d-j+1})\,dx\\&\quad=\frac{2}{(d-2)!}\int\limits_{0}^{1}x\log(x)\,dx\qquad\text{(as in Lemma 2.10)}\\&\quad=\frac{2}{(d-2)!}\cdot\Big(-\frac{1}{4}\Big)\end{aligned}$$

and the result follows. ■

LEMMA 2.12. *For $d \geq 2$, we have*

$$F_d = \sum_{j=1}^{d} \int_{S_d} \Big(\theta_j + 2 \sum_{k=j+1}^{d} \theta_k\Big) \log\Big(\theta_j + 2 \sum_{k=j+1}^{d} \theta_k\Big)\, dV$$
$$= \frac{1}{2} \frac{1}{(d-2)!} \Big\{ 2\log(2) - 1 + \int_1^2 (x-1)^{d-2} x \log(x)\, dx \Big\} - \frac{1}{2} \frac{1}{d!} \sum_{j=2}^{d} \frac{1}{j}.$$

The univariate integral above is evaluated in Lemma 2.13.

Proof of Lemma 2.12. Let

$$I_j := \int_{S_d} \Big(\theta_j + 2 \sum_{k=j+1}^{d} \theta_k\Big) \log\Big(\theta_j + 2 \sum_{k=j+1}^{d} \theta_k\Big)\, dV.$$

We must compute $\sum_{j=1}^{d} I_j$. First note that, by Lemma 2.6,

$$I_1 = \int_{S_d} \Big(\theta_1 + 2 \sum_{k=2}^{d} \theta_k\Big) \log\Big(\theta_2 + 2 \sum_{k=2}^{d} \theta_k\Big)\, dV \tag{2.11}$$
$$= \frac{1}{2-1} \frac{1}{(d-2)!} \int_{-\infty}^{\infty} x \log(x) B(x \mid 1, \underbrace{2, \ldots, 2}_{d-1})\, dx$$
$$= \frac{1}{(d-2)!} \int_{-\infty}^{\infty} x \log(x) B(x \mid 1, \underbrace{2, \ldots, 2}_{d-1})\, dx,$$

while

$$I_d = \int_{S_d} \theta_d \log(\theta_d)\, dV = \frac{1}{1-0} \frac{1}{(d-2)!} \int_{-\infty}^{\infty} x \log(x) B(x \mid \underbrace{0, \ldots, 0}_{d-1}, 1)\, dx$$
$$= \frac{1}{(d-2)!} \int_{-\infty}^{\infty} x \log(x) B(x \mid \underbrace{0, \ldots, 0}_{d-1}, 1)\, dx.$$

Further, for $2 \leq j \leq d-1$,

$$I_j = \int_{S_d} \Big(\theta_j + 2 \sum_{k=j+1}^{d} \theta_k\Big) \log\Big(\theta_j + 2 \sum_{k=j+1}^{d} \theta_k\Big)\, dV$$
$$= \frac{1}{2-0} \frac{1}{(d-2)!} \int_{-\infty}^{\infty} x \log(x) B(x \mid \underbrace{0, \ldots, 0}_{j-1}, 1, \underbrace{2, \ldots, 2}_{d-j})\, dx$$
$$= \frac{1}{2} \frac{1}{(d-2)!} \int_{-\infty}^{\infty} x \log(x) B(x \mid \underbrace{0, \ldots, 0}_{j-1}, 1, \underbrace{2, \ldots, 2}_{d-j})\, dx.$$

Hence

$$\frac{1}{2}I_1+\sum_{j=2}^{d-1}I_j+\frac{1}{2}I_d$$
$$=\frac{1}{2}\,\frac{1}{(d-2)!}\int_{-\infty}^{\infty}x\log(x)\Big\{\sum_{j=1}^{d}B(x\,|\,\underbrace{0,\dots,0}_{j-1},1,\underbrace{2,\dots,2}_{d-j})\Big\}\,dx.$$

However, just as Lemma 2.10, it can be shown (cf. [deB]) that

$$\sum_{j=1}^{d}B(x\,|\,\underbrace{0,\dots,0}_{j-1},1,\underbrace{2,\dots,2}_{d-j})\equiv 1$$

on its support, i.e., on $(0,2)$ in this case. Thus

$$\frac{1}{2}I_1+\sum_{j=2}^{d-1}I_j+\frac{1}{2}I_d=\frac{1}{2}\,\frac{1}{(d-2)!}\int_0^2 x\log(x)\,dx=\frac{1}{2}\,\frac{1}{(d-2)!}(2\log(2)-1).$$

It follows that

$$\sum_{j=1}^{d}I_j=\frac{1}{2}\,\frac{1}{(d-2)!}(2\log(2)-1)+\frac{1}{2}(I_1+I_d).$$

But, by (2.11),

$$I_1=\frac{1}{(d-2)!}\int_{-\infty}^{\infty}x\log(x)B(x\,|\,1,\underbrace{2,\dots,2}_{d-1})\,dx$$
$$=\frac{1}{(d-2)!}\int_1^2(x-1)^{d-2}x\log(x)\,dx$$

and from Lemma 2.8,

$$I_d=-\frac{1}{d!}\sum_{j=2}^{d}\frac{1}{j}.$$

Putting these together yields the result. ■

Lemma 2.13. *For $m\in\mathbb{Z}_+$ we have*

$$\int_1^2(x-1)^m x\log(x)\,dx=a_m\log(2)+b_m$$

where

$$a_m:=\begin{cases}\dfrac{2}{m+1}, & m\ \text{even},\\[2mm] \dfrac{2}{m+2}, & m\ \text{odd},\end{cases}$$

and

$$b_m := -\frac{(-1)^m}{(m+1)(m+2)}\Big\{(-1)^m - \sum_{k=1}^{m+2}(-1)^k/k\Big\}.$$

Proof. Let A_m denote the integral in question. Just as before we may integrate by parts (with $u = (x-1)^m$ and $v' = x\log(x)$) to show that

$$A_{m+1} = -\frac{m+1}{m+3}A_m + \frac{4}{m+3}\log(2) - \frac{2m+5}{(m+2)(m+3)^2}$$

and from this one can easily verify the given formula by induction. ∎

3. The simplex. By the mapping result of Bloom and Calvi [BC] we immediately obtain

THEOREM 3.1. *The transfinite diameter of the simplex T_d is given by*

$$\delta(T_d) = (\delta(B_d))^2.$$

Proof. Just note that T_d is the image of B_d under the mapping

$$(x_1, \ldots, x_d) \mapsto (x_1^2, \ldots, x_d^2).\ ∎$$

4. Concluding remarks. Using our formulas for $\delta(B_d)$ and the formula (1.3) for $\delta(\mathbb{B}_d)$ we easily find that

$$\lim_{d\to\infty}\frac{\delta(\mathbb{B}_d)}{\delta(B_d)} = \sqrt{2}$$

showing that the real and complex balls have finitely comparable transfinite diameters.

Further, for the *inscribed* cube $\frac{1}{\sqrt{d}}Q_d \subset B_d$ we have

$$\lim_{d\to\infty}\frac{\delta(B_d)}{\delta\big(\frac{1}{\sqrt{d}}Q_d\big)} = \sqrt{2\exp(1-\gamma)}$$

where γ is Euler's constant, while for the *superscribed* cube,

$$\lim_{d\to\infty}\frac{\delta(B_d)}{\delta(Q_d)} = 0.$$

Since $\frac{1}{\sqrt{d}}Q_d \subset B_d \subset Q_d$ it is natural to ask why $\delta(B_d)$ is comparable to the inscribed cube and not to the superscribed cube. However, even in the euclidean case,

$$\mathrm{vol}(B_d) = \frac{\pi^{d/2}}{\Gamma(1+d/2)} \sim \frac{1}{\sqrt{\pi d}}\bigg(\sqrt{\frac{2\pi e}{d}}\bigg)^d,$$

by Stirling's formula, showing that the volume of B_d behaves like that of a cube with side length proportional to $1/\sqrt{d}$, i.e., like that of the inscribed cube.

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