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Lelong classes on toric manifolds and a theorem of Siciak

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Abstract. We generalize a theorem of Siciak on the polynomial approximation of the Lelong class to the setting of toric manifolds with an ample line bundle. We also characterize Lelong classes by means of a growth condition on toric manifolds with an ample line bundle and construct an example of a nonample line bundle for which Siciak's theorem does not hold.

1. Introduction. There are many connections between pluripotential theory in $\mathbb{C}^n$ (both the classical theory and its weighted analog) and the study of complex polynomials. For example, the following theorem, due to Siciak ([Si]; see also [Kl, Theorem 5.1.6]), provides a polynomial approximation of the Lelong class $\mathcal{L}$ of plurisubharmonic (psh) functions in $\mathbb{C}^n$ of logarithmic growth. It also shows a relation between $\mathcal{L}$ and the class $\mathcal{H}_+^n$ of nonnegative psh functions in $\mathbb{C}^n$ which are absolutely homogeneous of order one. Recall that a nonnegative psh function is *absolutely homogeneous of order one* if $u(tz) = |t|u(z)$ for all $t \in \mathbb{C}$ and $z \in \mathbb{C}^n$.

Theorem 1.1. *Let $h : \mathbb{C}^n \to [0,\infty)$ and $u : \mathbb{C}^n \to [-\infty,\infty)$ be functions such that $h \not\equiv 0$ and $u \not\equiv -\infty$.*

(i) *If $h \in \mathcal{C}(\mathbb{C}^n) \cap \mathcal{H}_+^n$ and $h^{-1}(0) = \{0\}$, then $h(z) = \sup |Q(z)|^{1/\deg Q}$, $z \in \mathbb{C}^n$, where the* sup *is taken over all complex homogeneous polynomials Q such that $|Q|^{1/\deg Q} \le h$ in $\mathbb{C}^n$.*
(ii) *$h \in \mathcal{H}_+^n$ if and only if $h = (\limsup_{j\to\infty} |Q_j|^{1/j})^*$ for some sequence of complex homogeneous polynomials such that $\deg Q_j \le j$. In particular, if $h \in \mathcal{H}_+^n$, then $\log h \in \mathcal{L}$.*
(iii) *$u \in \mathcal{L}$ if and only if $e^u = (\limsup_{j\to\infty} |P_j|^{1/j})^*$ for some sequence of complex polynomials on $\mathbb{C}^n$ such that $\deg P_j \le j$.*

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When one onsiders $\mathbb{C}^n$ as the complement of the hyperplane at infinity in $\mathbb{CP}^n$, the class $\mathcal{L}(\mathbb{C}^n)$ is in one-to-one correspondence with the class $\mathrm{PSH}(\mathbb{CP}^n, \omega_{\mathrm{FS}})$ of functions quasi-plurisubharmonic with respect to the Fubini-Study form ω_{FS} (cf. [BS, Proposition 5 and Theorem 1], and the references given there). There is also a one-to-one correspondence with the class of positive singular metrics on the hyperplane bundle L over $\mathbb{CP}^n$. By Grauert's characterization of positive line bundles, all of these correspond with the class of plurisubharmonic functions on the total space of the universal bundle L' over $\mathbb{CP}^n$ (dual to the hyperplane bundle) which are nonnegative, not identically zero and absolutely homogeneous of order one in each fiber. Furthermore, homogeneous polynomials of degree $d \geq 1$ in L' can be thought of as sections of the dth tensor power dL (we use the additive notation for the tensor product operation) and $\mathbb{CP}^n = \operatorname{Proj} \bigoplus_{d=0}^{\infty} \Gamma(\mathbb{C}^n, dL)$.

The notion of homogeneous polynomials on the total space of a (dual) line bundle L' and the correspondence between them and holomorphic sections of tensor powers of L is also valid for ample line bundles over toric manifolds (see below for precise statements). This is the setting in which we will work in the present paper. First, we will extend Siciak's theorem to the setting of a toric manifold with an ample line bundle. Then, as an application, we will characterize Lelong classes on such toric manifolds by means of a growth condition (stated by [Be]). Finally, we will show that the theorem is not valid when the line bundle is globally generated but not ample. As a counterexample, we will construct a globally generated line bundle L over $X = \mathbb{CP}^1 \times \mathbb{CP}^1$ (using its properties as a toric manifold) and a continuous nonnegative plurisubharmonic function h on L', homogeneous in each fiber and vanishing only along the zero section, such that h cannot be obtained as $(\limsup_{j\to\infty} |Q_j|^{1/j})^*$ for any sequence $Q_j \in \Gamma(X, jL)$.

2. Line bundles over toric varieties. Let us begin by recalling some facts about toric varieties and line bundles over them. We omit many details, referring the interested reader to [CLS] to learn more. We will first describe the construction of an affine toric variety from a rational polyhedral cone (briefly, a cone). By a cone we understand a set $\sigma = \{c_1v_1 + \cdots + c_sv_s \in \mathbb{R}^n : c_1, \ldots, c_s \geq 0\}$, where $v_1, \ldots, v_s \in \mathbb{Z}^n \subset \mathbb{R}^n$, and we say that σ is generated by the vectors $v_1, \ldots, v_s$. Consider the dual cone $\check{\sigma} \subset M = \{m \in M : \langle m, v\rangle \geq 0 \ \forall v \in \sigma\}$, where $M \simeq \mathbb{Z}^n$ is thought of as the dual integral lattice. The affine semigroup $\check{\sigma} \cap M$ is finitely generated (by Gordan's Lemma; cf. [CLS, Proposition 1.2.17]). We define U_σ to be the maximal spectrum of the ring $\mathbb{C}[\check{\sigma} \cap M]$. It is an affine toric variety ([CLS, Theorem 1.2.18]). The characters $z^{m_1}, \ldots, z^{m_s}$ on the torus $T \simeq (\mathbb{C}^*)^n$ associated with the generators $m_1, \ldots, m_s$ of $\check{\sigma} \cap M$ introduce affine coordinates $x_1, \ldots, x_s$ on U_σ. Here $z^m := z_1^{m_1} \ldots z_n^{m_n}$.

A general toric variety X_Σ can be constructed from a fan Σ in $\mathbb{R}^n$, that is, from a certain collection of cones generated by vectors from $\mathbb{Z}^n$. (For the precise conditions characterizing a fan see e.g. [CLS, Definition 3.1.2]). Affine pieces are constructed from cones $\sigma \in \Sigma$, and then the construction of X_Σ requires gluing any two such pieces along a common face of σ and σ'. We also mention that torus-invariant hypersurfaces in X_Σ are determined by one-dimensional cones in Σ (e.g., via the so-called orbit-cone correspondence, [CLS, Theorem 3.2.6 and Proposition 3.2.7]).

Let $v_1, \dots, v_N$ denote the primitive generators of one-dimensional cones in Σ. A *character* on $\mathbb{Z}^N$ (i.e., an integer-valued function $\varphi : \mathbb{Z}^N \mapsto \mathbb{Z}$) is determined by its values $a_i = \varphi(e_i)$, where e_i, $i = 1, \dots, N$, are the vectors of the standard basis, and it defines a function $\varphi : \mathbb{R}^n \to \mathbb{R}$ by putting $\varphi(v_i) = -a_i$ and extending it as a linear function on each cone $\sigma \in \Sigma$. We will follow the convention from [CLS, Definition 6.1.17]: a function $\varphi : S \to \mathbb{R}$ defined on a convex set $S \subset \mathbb{R}^n$ is *convex* if $\varphi(tu+(1-t)tv) \geq t\varphi(u)+(1-t)\varphi(v)$ for all $u, v \in S$ and $t \in [0, 1]$. Note that this usage reverses the inequality occurring in the definition of convex functions considered in analysis. The function φ determines a torus-invariant divisor $D = \sum_{i=1}^N a_i D_i$ on X_Σ, where D_i is the hypersurface corresponding to the cone generated by v_i. Conversely, every such divisor $D = \sum_{i=1}^N a_i D_i$ gives rise to a character, which determines a function φ_D called the *support function* of D. If D is Cartier, then on each n-dimensional cone $\sigma \in \Sigma$ one has $\langle m_\sigma, v_i \rangle = -a_i$ for all $v_i \in \sigma$.

The support function φ_D of a Cartier divisor D on X_Σ is called *strictly convex* if it is convex and for every n-dimensional cone $\sigma \in \Sigma$ it satisfies $\varphi_D(u + v) > \varphi_D(u) + \varphi_D(v)$ for all u, v not belonging to the same $\sigma \in \Sigma$. By [CLS, Propositions 4.3.3 and 4.3.8] (see also [Ro, Proposition 2]), the line bundle associated with D has global sections $\Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) = \bigoplus_{m\in P_D\cap\mathbb{Z}^n} \mathbb{C} \cdot \chi^m$, where $P_D \subset \mathbb{R}^n$ is the polyhedron defined by $P_D = \{m \in \mathbb{R}^n : \langle m, v_i \rangle \geq -a_i,\ i = 1, \dots, N\}$. By Lemma 6.1.9 in [CLS], $P_D = \{m \in \mathbb{R}^n : \varphi(x) \leq \langle m, v \rangle\ \forall v \in \bigcup_{\sigma\in\Sigma} \sigma\}$. By [CLS, Theorem 6.1.10], φ_D is convex if and only if D is basepoint-free (i.e., $\mathcal{O}_{X_\Sigma}(D)$ is generated by global sections). This in turn is equivalent to $\varphi_D(u) = \min_{m\in P_D} \langle m, u \rangle$ for all $u \in \mathbb{R}^n$. Finally by [CLS, Theorem 6.1.15], D is ample if and only if φ_D is strictly convex.

In this section we assume that X is a compact toric variety with a line bundle $L = \mathcal{O}_X(D)$ coming from a Cartier divisor D with a strictly convex support function (hence L is ample). Recall Grauert's ampleness criterion: a holomorphic line bundle L over a compact complex manifold X is ample if and only if the zero section $Z(L')$ of the dual bundle L' has a strongly pseudoconvex neighborhood. It follows that $Z(L')$ can be blown down to a finite set of points.

Consider the cone $C_\varphi = \{(x,\alpha) \in \mathbb{R}^n \times \mathbb{R} : \alpha \geq \varphi(x)\}$ in $\mathbb{R}^{n+1}$. It is equal to $\mathrm{Cone}\{(0,1),(v_i,-a_i) : i = 1,\ldots,N\}$. By [Ro, Proposition 1] (cf. also [CLS, Exercise 6.1.13]), the function φ is convex if and only if C_φ is a convex cone.

Now, for $\sigma \in \Sigma$ let

$$\begin{aligned}\tilde{\sigma} &= \{(u,\alpha) : u \in \sigma,\ \alpha \geq \varphi(u)\}\\ &= \mathrm{Cone}\{(0,1),(v_i,-a_i) : v_i \in \sigma(1),\ i = 1,\ldots,N\}.\end{aligned}$$

(Here $\sigma(1)$ denotes the collection of all 1-dimensional cones in the fan σ.) By [CLS, Proposition 7.3.1], the affine variety $U_{\tilde{\sigma}}$ is isomorphic to $U_\sigma \times \mathbb{C}$, which is a trivializing neighborhood for the line bundle $\mathcal{O}_{X_\Sigma}(D)$. The projection induces a map $U_{\tilde{\sigma}} \to \mathbb{C}$, which is $\chi^{(-m_\sigma,1)}$. This suggests another way of looking at the sections of $\mathcal{O}_{X_\Sigma}(D)$. Specifically, the following characterization appears in [Ro, discussion between Lemma 5 and Proposition 3]:

Let A_φ be the affine toric variety (with the action of the torus T^{n+1}) associated with the cone C_φ, that is, $A_\varphi = \mathrm{Spec}\ \mathbb{C}[\mathbb{Z}^{n+1} \cap C_\varphi^*]$. This is the affine variety obtained by blowing down the zero section of the line bundle $\mathcal{O}_{X_\Sigma}(-D)$. The affine coordinates on A_φ are given by the characters χ^m such that $m \in P_D \cap \mathbb{Z}^n$. This yields the identification

$$\Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) = \{f : A_\varphi \to \mathbb{C} : f(\lambda \cdot z) = \lambda f(z),\ \lambda \in \mathbb{C}^*,\ z \in A_\varphi\},$$

so the sections of L can be thought of as linear forms on A_φ. This kind of identification can be extended to $\Gamma(X, dL)$. In fact, the following holds:

Proposition 2.1 ([Ba]; [Co, Section 5]; cf. also [CLS, Appendix A to the Chapter 7]). *Let X and L be as above. Consider the graded ring $R_D = \bigoplus_{d\in\mathbb{N}} \Gamma(X, dL)$. Then there exists a ring isomorphism between R_D and $\mathbb{C}[\mathbb{Z}^{n+1} \cap C_\varphi^*]$, preserving the grading.*

The grading in this proposition is given by $\deg t_0^d \chi^m(t) = d$, $d \geq 1$. A closely related result, proved using the above isomorphism, is that if one considers R_D as a $\mathbb{C}$-algebra with multiplication of complex functions, then R_D is finitely generated [El]. We can now formulate our first result:

Theorem 2.2. *Let X be a toric manifold and L be a line bundle over X defined by a Cartier divisor D with a strictly convex support function. Let $H : L' \to [0,\infty)$, $H \not\equiv 0$, be a continuous plurisubharmonic function which is homogeneous in each fiber, with $H^{-1}(0) = Z_{L'}$, where $Z_{L'}$ is the zero section of L'. Then $H(z) = \sup |Q(z)|^{1/\deg Q}$, where Q is a linear combination of the variables $t_0^d \chi^m(t)$ (a homogeneous polynomial in the coordinates of A_φ), with $\deg Q = d$ and $m \in dP_D$, such that $|Q|^{1/\deg Q} \leq H$.*

Proof. There is a strictly convex neighborhood Ω of the zero section and there is $r > 0$ such that $\{H < r\} \subset \Omega$. By rescaling H, we can assume that

$r = 1$. Let $a \in L'$ be such that $H(a) = 1$ and let $g(a)$ denote the above supremum evaluated at a. It is enough to show that $g(a) \geq 1$.

After blowing down the zero section, we get a holomorphically convex bounded neighborhood $\tilde{\Omega}$ of a point in A_φ, hence a polynomially convex set. Hence for any $\lambda \in (0,1)$ we have $\hat{K_\lambda} \subset \tilde{\Omega}$, where $K_\lambda = H^{-1}([0,\lambda])$. Now recall that the polynomial ring $\mathbb{C}[\mathbb{Z}^{n+1} \cap C_\varphi^*]$ is the subring of $\mathbb{C}[t_0, t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ spanned by the Laurent monomials $t_0^d \chi^m(t)$ with $m \in dP_D$, and the sections of dL correspond exactly to homogeneous polynomials which are sums of $t_0^d \chi^m(t)$. Recall also that the fiber L_z^j can be identified with the space of j-linear functions $L'_z \times \cdots \times L'_z \to \mathbb{C}$. In particular, every section in L^j is a homogeneous function of degree j on every fiber of L'.

Fix a $\lambda \in (0,1)$. There exists a $\mu \in (\lambda, 1)$ and a homogeneous polynomial Q (viewed as a homogeneous function on A_φ) such that $Q(\mu a) \geq 1$ and $\|Q\|_{K_\lambda} \leq 1$. By homogeneity, $\lambda |Q|^{1/\deg Q}$, so $\lambda \leq |\lambda^{\deg Q} Q(\mu a)|^{1/\deg Q} \leq g(\mu a) = \mu g(a)$. Letting $\lambda \to 1$ implies that $\mu \to 1$ and $1 \leq g(a)$. ■

We also have the following:

THEOREM 2.3. *Under the assumptions of Theorem* 2.2,

$$H = (\limsup_{j \to \infty} |Q_j|^{1/j})^*$$

for some sequence Q_j of homogeneous polynomials on A_φ with $\deg Q_j \leq j$.

Proof. Let $H : L' \to [0,\infty)$, $H \not\equiv 0$, be a continuous plurisubharmonic function which is homogeneous in each fiber, with $H^{-1}(0) = Z_{L'}$, where $Z_{L'}$ is the zero section of L'. The set $\Omega = \{z : H(z) < 1\}$ is a balanced pseudoconvex neighbourhood of $Z_{L'}$. We let Φ denote the map which blows down the zero section of L', and let $\tilde{\Omega}$ denote the set that Ω maps to.

Since $\tilde{\Omega}$ is holomorphically convex, there exists a holomorphic function $\tilde{F}$ on $\tilde{\Omega}$ which does not extend beyond $\tilde{\Omega}$. Observe that $\tilde{\Omega}$ is a balanced neighborhood of 0 in A_φ so $\tilde{F}$ has a homogeneous expansion. That is, $\tilde{F}(z) = \sum_{j=0}^{\infty} Q_j(z)$ for $z \in \tilde{\Omega}$, where Q_j denotes a homogeneous polynomial with $\deg Q_j \leq j$. Define $v(z) = (\limsup_{j\to\infty} |Q_j(z)|^{1/j})^*$ for $z \in A_\varphi$. By Cauchy's convergence criterion for numerical series, $(\limsup_{j\to\infty} |Q_j(z)|^{1/j})^* \leq 1$ for $z \in \tilde{\Omega}$. Consequently $\tilde{\Omega} = \{z \in A_\varphi : v(z) < 1\}$. By the homogeneity of v and H we have $v \equiv H$ on A_φ. ■

It is quite straightforward to show that the existence of a plurisubharmonic function $H : L' \to [0,\infty)$ as in Theorem 2.2 is equivalent to the existence of a positive singular hermitian metric on L, i.e., to L being pseudoeffective. Namely ([BS], Theorem 1 and the references there), given such an H and a system of trivializations $\theta_i : L'|_{U_i} \to U_i \times \mathbb{C}$, the functions $h_i(x) = \log(H \circ \theta_i^{-1}(x,t)/|t|)$, $x \in U_i$, $t \neq 0$, give a positive singular metric

h^L on L, and the same formula defines H as in Theorem 2.2 if the collection $\{h_i\}$ gives a positive singular metric. As an example of a positive singular metric one can take $\log|s|$, where s is a holomorphic section of L. From [GZ] we have the following relation between positive singular metrics and quasi-plurisubharmonic functions: Consider the metric with weight λ (or fix any smooth metric on the line bundle L over X), and set $\omega = dd^c\lambda$. The class $\mathrm{PSH}(X,\omega)$ of functions $u \in L^1(X, \mathbb{R}\cup\{-\infty\})$ such that u is upper semicontinuous and satisfies $dd^c u \geq -\omega$ can be obtained by taking all positive singular metrics ψ on L and setting $u = \psi - \lambda$. Conversely, if $u \in \mathrm{PSH}(X,\omega)$, then $u + \lambda$ defines a positive singular metric on L. We call $\mathrm{PSH}(X,\omega)$ the *Lelong class* on X.

In analogy to the case of $\mathbb{CP}^n$, a growth condition for plurisubharmonic functions on a complex torus was introduced in [Be, Section 4]. It generalizes the growth condition for the Lelong class in $\mathbb{C}^n$ with a view to studying quasi-plurisubharmonic functions on toric manifolds. Moreover, it is equivalent to the statement that an ω-quasi-psh function v on a toric manifold X corresponds uniquely to a plurisubharmonic function u on the torus T^n satisfying

$$\begin{aligned} u(z_1,\ldots,z_n) &\leq \psi(z_1,\ldots,z_n) + \mathcal{O}(1) \\ &:= \sup_{m\in P_D} \langle m, (\log|z_1|,\ldots,\log|z_n|)\rangle + \mathcal{O}(1). \end{aligned}$$

(Note that ψ is the composition of the support function of P_D with the map $z \mapsto (\log|z_1|,\ldots,\log|z_n|)$.) Accordingly, $u - \psi + \lambda$ is a positive singular metric.

We will now apply our approximation results from the previous a section to prove that this condition provides a unique characterization of ω-quasi-psh functions.

THEOREM 2.4. *Let $X = X_\Sigma$ be a n-dimensional toric manifold with a complete fan Σ and let D be a very ample divisor on X. Let u be a plurisubharmonic function on the torus T^n. Then the function*

$$v = u(z_1,\ldots,z_n) - \psi(\log|z_1|,\ldots,\log|z_n|)$$

extends to an ω-psh function on X if and only if

$$\begin{aligned} u(z_1,\ldots,z_n) &\leq \psi(\log|z_1|,\ldots,\log|z_n|) + \mathcal{O}(1) \\ &:= \sup_{m\in P_D\cap M} \langle m, (\log|z_1|,\ldots,\log|z_n|)\rangle + \mathcal{O}(1). \end{aligned}$$

Proof. If u is a psh function satisfying the growth condtion, then v and its extension to X satisfy $dd^c v \geq dd^c(-\psi) = -[D]$. Conversely, by Theorem 2.3, $u - \psi + \lambda = (\limsup_{j\to\infty}(1/j)\log|Q_j|)^*$ for some sequence $Q_j \in \Gamma(X, \mathcal{O}_X(jD))$, $j = 1,2,\ldots$. We have $(1/j)\log|Q_j| - \lambda \leq \mathcal{O}(1)$ for all j, and the inequality persists under the upper limit and upper semicontinuous regularization, so $u - \psi \leq \mathcal{O}(1)$. ∎

3. A counterexample. The problem of approximation of arbitrary plurisubharmonic functions by functions of the form $\log|f|$, with f holomorphic, dates back to Lelong ([Le]) and Bremermann ([Br1], [Br2]). In the cited articles of the latter author, the problem of approximating psh functions by so-called Hartogs functions (a class which includes $\log|f|$) was solved positively in a pseudoconvex domain in $\mathbb{C}^n$. Also, counterexamples were given that such approximation may be impossible when the domain is not pseudoconvex. Having this problem in mind, one may ask whether in Siciak's theorem the assumption that the line bundle is ample can be relaxed. Now we will answer this question in the negative. More specifically, we will present a globally generated (basepoint-free) nonample line bundle L on $X = \mathbb{P}^1 \times \mathbb{P}^1$ and a plurisubharmonic function $H : L' \to [0, \infty)$, $H \not\equiv 0$, which is homogeneous in each fiber of L' and which is not of the form $H(z) = \sup |Q(z)|^{1/\deg Q}$ for any collection of sections Q with $|Q|^{1/\deg Q} \le H$. We need the following ingredients:

1) The fan Σ for $X = \mathbb{CP}^1 \times \mathbb{CP}^1$ and affine coordinate patches (cf. [CLS, Example 3.1.12]): The fan Σ consists of the following cones in $\mathbb{R}^2$: the 0-dimensional cone $\{(0,0)\}$; four one-dimensional cones generated by the vectors $\pm e_i$, $i = 1, 2$, where e_1, e_2 are the vectors of the canonical basis in $\mathbb{R}^2$; four two-dimensional cones $\sigma_I, \sigma_{II}, \sigma_{III}, \sigma_{IV}$ equal to the corresponding quadrants in $\mathbb{R}^2$. Taking dual cones one gets affine coordinate neighborhoods in $\mathbb{P}^1 \times \mathbb{P}^1$: $U_I = \operatorname{Spec} \mathbb{C}[x, y]$; $U_{II} = \operatorname{Spec} \mathbb{C}[x^{-1}, y]$; $U_{III} = \operatorname{Spec} \mathbb{C}[x^{-1}, y^{-1}]$; $U_{IV} = \operatorname{Spec} \mathbb{C}[x, y^{-1}]$.

2) A divisor on $\mathbb{P}^1 \times \mathbb{P}^1$: let $D_i^{\pm}$ be the divisors corresponding to the ray generators $\pm e_i$, $i = 1, 2$. We will take $D = D_1^+ - D_1^- + D_2^+ + D_2^-$.

3) The support function associated with D: If $D = \sum a_\rho D_\rho$, then $\varphi_D(u_\rho) = -a_\rho$ on the ray generator u_ρ and φ_D extends as a linear function to each cone $\sigma \in \Sigma$ ([CLS, Definition 4.2.11 and Theorem 4.2.12]). The values of φ on the ray generators are: $\varphi(1,0) = -1$, $\varphi(0,1) = -1$, $\varphi(-1,0) = 1$, $\varphi(0,-1) = -1$. Thus the support function of D is $\varphi_D(x,y) = -x - y$ for $y \ge 0$ and $\varphi_D(x,y) = -x + y$ for $y \le 0$ and all real x.

Proposition 3.1. *The divisor D in $\mathbb{CP}^1 \times \mathbb{CP}^1$ is basepoint-free, but not ample.*

Proof. According to [CLS, Theorem 6.1.10(a), (g)], a torus-invariant divisor D is basepoint-free if and only if φ_D is convex, and according to [CLS, Theorem 6.1.15], a divisor D is ample if and only if φ_D is strictly convex. The function $\varphi_D(x,y)$ is convex, but not strictly convex, since $\varphi_D(1,0) + \varphi_D(-1,0) = \varphi_D(0,0)$. ∎

4) The polytope associated with D (cf. [CLS, Lemma 6.1.9]): This is a degenerate polytope in $\mathbb{R}^2$: $m_1 = 0$, $-1 \le m_2 \le 1$.

The cone C_φ^* (which equals the cone over $P_D \times \{1\}$) is generated by the vectors $(-1,-1,1),(-1,1,1)$.

5) Sections of the line bundle $L = \mathcal{O}(D)$ ([CLS, Propositions 4.3.3 and 4.3.8]): $\Gamma(X, \mathcal{O}(D)) = \{az_1^{-1}z_2^{-1}t + bz_1^{-1}z_2t + cz_1^{-1}t\} : a,b,c \in \mathbb{C}\}$. A similar representation is valid for $\Gamma(X, \mathcal{O}(nD))$, $n \geq 1$.

6) The total space of the line bundle $L' = \mathcal{O}(-D)$ ([CLS, §7.3]) minus the zero section: the affine coordinate neighborhoods are determined by the cones $\tilde{\sigma} = \mathrm{Cone}\,\{(0,0,1),(u_\rho,a_\rho) : \rho \in \sigma(1)\}$ and $\sigma \in \Sigma$. We thus have $\tilde{U}_I = \mathrm{Spec}\,\mathbb{C}[z_1^{-1}z_2^{-1}t, z_1, z_2]$; $\tilde{U}_{II} = \mathrm{Spec}\,\mathbb{C}[z_1^{-1}z_2^{-1}t, z_1^{-1}, z_2]$; $\tilde{U}_{III} = \mathrm{Spec}\,\mathbb{C}[z_1^{-1}z_2t, z_1^{-1}, z_2^{-1}]$; $\tilde{U}_{IV} = \mathrm{Spec}\,\mathbb{C}[z_1^{-1}z_2t, z_1, z_2^{-1}]$.

7) Let us now define a function on the total space on L' as follows: $H(\zeta_1,\zeta_2,\tau) := |t|$, i.e., H equals $|\zeta_1\zeta_2\tau|$ on $\sigma_1 \cup \sigma_4$, $|\zeta_1\tau/\zeta_2|$ on σ_2 and $|\zeta_2\tau/\zeta_1|$ on σ_3. Then H is plurisubharmonic, homogeneous in each fiber, and satisfies $H^{-1}(0) = Z_{L'}$, but it cannot be approximated by combinations of sections of L.

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