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Relative tangent cone of analytic sets

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Abstract. We give a characterization of the relative tangent cone of an analytic curve and an analytic set with an improper isolated intersection. Moreover, we present an effective computation of the intersection multiplicity of a curve and a set with s-parametrization.

1. Introduction. We consider an analytic curve X and an analytic set Y in a neighbourhood Ω of a in $\mathbb{C}^m$ such that $X \cap Y = \{a\}$ and study their relative tangent cone, $C_a(X,Y)$. The relative tangent cone and the intersection multiplicity of analytic sets are additive, so we restrict our attention, without loss of generality, to an analytic curve with irreducible germ at a.

The main result of this paper is the formula $C_a(X,Y)+C_a(X)=C_a(X,Y)$ (see Theorem 2.2), where by $C_a(X)$ we mean the classical Whitney cone at a point (see [Whi 65]). This theorem, giving a strong geometric characterization of the relative tangent cone of an analytic curve and an analytic set, is an improvement of the result from [Cie 99].

In the last section we effectively calculate the intersection multiplicity of an analytic set with s-parametrization and an analytic curve.

2. Main result. Let X and Y be analytic sets in an open neighbourhood Ω of a point $a \in \mathbb{C}^m$ such that a is an isolated point of $X \cap Y$.

Definition 2.1. The *relative tangent cone* $C_a(X,Y)$ of the sets X, Y at a is defined to be the set of $\mathfrak{v} \in \mathbb{C}^m$ with the property that there exist sequences (x_ν) of points of X, (y_ν) of points of Y and (λ_ν) of complex numbers such that $x_\nu \to a$, $y_\nu \to a$ and $\lambda_\nu(x_\nu - y_\nu) \to \mathfrak{v}$ as $\nu \to \infty$.

The relative cone depends only on germs of analytic sets and is a closed cone with vertex at 0. If Ω is distinguished with respect to X and Y, then $C_a(X,Y) = C_0(Y-X)$ and $\dim(Y-X)_0 = \dim(X)_a + \dim(Y)_a$; moreover,

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if X has a p-dimensional germ at a and Y has a q-dimensional germ at a, then the relative cone $C_a(X,Y)$ is a $(p+q)$-dimensional algebraic cone.

In the definition of the Whitney tangent cone the scalars λ_ν may be taken to be positive real numbers (see [Whi 65, Sec. 7, Rem. 3D]). Moreover, if A is a locally analytic set in some neighbourhood of a and $\lambda_\nu \to +\infty$ is an arbitrary sequence of positive real numbers and $\mathfrak{v} \in C_a(A)$, then there exists a sequence (a_ν) of points of A such that $a_\nu \to a$ and $\lambda_\nu(a_\nu - a) \to \mathfrak{v}$. For the proof let $\Gamma : (-\varepsilon, \varepsilon) \to A$ be a $\mathcal{C}^1$-parametrization such that $\Gamma(0) = a$ and $\Gamma'(0) = \mathfrak{v}$. Then $\lambda_\nu(\Gamma(1/\lambda_\nu) - a)) \to \mathfrak{v}$ for $\nu \to \infty$. Due to the relation between the relative tangent cone and the Whitney cone, the scalars in the definition of the relative tangent cone may be taken in a form suitable for computation. For a detailed study of the relative tangent cone see [ATW 90], in which this object appeared for the first time. In fact the relative tangent cone is the limit of a join variety in the case that X and Y meet in one point (for definition and detailed study see [FOV 99]).

Without loss of generality we consider only analytic sets in a neighbourhood Ω of the origin. For the rest of the paper we assume that X is a pure 1-dimensional analytic set with irreducible germ at 0 (and for short will call it an *analytic curve*) with Puiseux parametrization

$$U \ni t \mapsto (t^p, \varphi(t)) \in X, \qquad \operatorname{ord}\varphi > p,$$

where $\operatorname{ord}\varphi = \min\{\operatorname{ord}\varphi_i : i = 2, \ldots, m\}$ (see [Łoj 91, II 6.2; Puiseux Theorem]).

The main goal of this paper is the following theorem.

THEOREM 2.2. *If* $X \cap Y = \{0\}$, *then* $C_0(X,Y) + C_0(X) = C_0(X,Y)$.

Proof. The Whitney cone $C_0(X)$ is a complex line (see: [Łoj 91] or [Chi 89]). If this line is transverse to Y, which means that $C_0(X) \cap C_0(Y) = \{0\}$, then by [ATW 90, Property 2.9] we have $C_0(X,Y) = C_0(X) + C_0(Y)$. In the opposite case, we have $C_0(X) \subset C_0(Y)$. Then, after a suitable biholomorphic change of coordinates, $C_0(X) = \mathbb{C}_1 := \{x \in \mathbb{C}^m : x_2 = \cdots = x_m = 0\}$. Fix $\mathfrak{v} = (v_1, \ldots, v_m) \in C_0(X,Y)$ and $(c, 0, \ldots, 0) \in \mathbb{C}_1 = C_0(X)$. By the definition of the relative tangent cone there are sequences (t_ν) of complex numbers and $(y_{1,\nu}, \ldots, y_{m,\nu})$ of points of Y such that $t_\nu \to 0$; for $i \in \{1, \ldots, m\}$ we have $y_{i,\nu} \to 0$ and $\nu^p(t_\nu^p - y_\nu^1) \to v_1$; and for $i \in \{2, \ldots, m\}$ we have

$$\nu^p(\varphi_i(t_\nu) - y_{i,\nu}) \to v_i.$$

Without loss of generality we may assume that (νt_ν) is convergent in $\widehat{\mathbb{C}}$, so the application of [Cie 99, Lemma 2.1] to the sequence (t_ν) yields a sequence (h_ν) with the following properties:

(i) $h_\nu \to 0$,
(ii) $\nu^d((t_\nu + h_\nu)^d - t_\nu^d) \to c$,

(iii) for any holomorphic function $\varphi : \Omega \to \mathbb{C}$ defined in an open neighbourhood Ω of $0 \in \mathbb{C}$ with $\operatorname{ord} \varphi > d$ we have

$$\nu^d(\varphi(t_\nu + h_\nu) - \varphi(t_\nu)) \to 0.$$

Substituting $t_\nu + h_\nu$ for t_ν in the Puiseux parametrization of the curve we move the points a little along the curve. Then for the first coordinate we obtain

$$\nu^p((t_\nu + h_\nu)^p - y_{1,\nu}) = \nu^p((t_\nu + h_\nu)^p - t_\nu^p) + \nu^p(t_\nu^p - y_{1,\nu})$$

and observe that the left-hand side converges to the first coordinate of some vector in $C_a(X, Y)$, whereas the summands converge respectively to c and v_1. Similarly for $i \in \{2, \ldots, m\}$ we have

$$\nu^p(\varphi_i(t_\nu + h_\nu) - y_{i,\nu}) = \nu^p(\varphi_i(t_\nu + h_\nu) - \varphi_i(t_\nu)) + \nu^p(\varphi_i(t_\nu) - y_{i,\nu})$$

and observe that the left-hand side converges to the ith coordinate of the vector in $C_0(X, Y)$, whereas the first term on the right converges to 0 and the second converges to v_i. Let $\mathfrak{u}$ denote the vector of the left-hand side limits, so $\mathfrak{u} \in C_0(X, Y)$ and $\mathfrak{u} = \mathfrak{v} + (c, 0, \ldots, 0)$. Since $C_0(X) = \mathbb{C}_1$ we conclude that $C_0(X, Y) + C_0(X) = C_0(X, Y)$ and the theorem follows. ∎

The following corollary is an elegant geometric description of the relative tangent cone of a curve and a set:

Corollary 2.3. *Let X be an analytic curve and let Y be an analytic set, such that $X \cap Y = \{0\}$. If the tangent cone of X is the axis $\mathbb{C}_1$, then there exists an algebraic cone $\mathbb{S} \subset \mathbb{C}^{m-1}$ such that $C_0(X, Y) = \mathbb{C} \times \mathbb{S}$.*

Note that $\dim \mathbb{S} = \dim Y$. Moreover, if Y is an analytic curve, then the relative tangent cone $C_0(X, Y)$ is a set-theoretic finite union of complex planes (for an effective formula see [Kra 01]).

3. Multiplicity of the intersection of analytic sets. The effective formulas for the intersection multiplicity of two analytic curves are presented in [Kra 01] and [KN 03]. Now we present an effective computation of the intersection multiplicity of an analytic curve and a pure dimensional analytic set.

Let Ω be an open neighborhood of $0 \in \mathbb{C}^m$. Consider an analytic curve $X \subset \Omega$, irreducible at the origin and with a Puiseux parametrization. Let $Y \subset \Omega$ be an irreducible k-dimensional set for which there exists a proper, finite holomorphic mapping $\Psi : D \ni \tau \mapsto \psi(\tau) \in Y$, defined on a k-dimensional manifold D, such that Ψ is an s-sheeted analytic cover over the regular part of Y. Following [TW 89], the mapping Ψ will be called an *s-parametrization* of Y. Moreover, we assume that $X \cap Y = \{0\}$ and $\Psi^{-1}(0) = \{0\}$. By Corollary 2.3 there exists a k-dimensional algebraic cone

$\mathbb{S} \subset \mathbb{C}^m$ such that

$$C_0(X,Y) = \mathbb{C} \times \mathbb{S} = \mathbb{C}_1 + (\{0\} \times \mathbb{S}) \subset \mathbb{C} \times \mathbb{C}^{m-1} = \mathbb{C}^m.$$

For the computation of the intersection multiplicity we state a simpler version of [TW 89, Theorem 4.2] more suitable for our purpose.

THEOREM 3.1. *In the setting introduced above, for any holomorphic mapping* $f : \Omega \to \mathbb{C}^k$, *if* $\dim f^{-1}(0) = m - k$ *and* $f^{-1}(0) \cap Y = \{a\}$, *then*

$$\deg(Z_f \cdot Y; a) = \frac{1}{s} \sum_{b \in \Psi^{-1}(a)} \deg(Z_{f\circ\Psi}; b).$$

By $\deg(Z_{f\circ\Psi}; b)$ we mean the degree (Lelong number) at $b \in \Psi^{-1}(a)$ of the cycle of zeros $Z_{f\circ\Psi}$, and $\deg(Z_f \cdot Y; a)$ is the degree of the intersection cycle $Z_f \cdot Y$ at a. For a linear surjection $l : \mathbb{C}^{m-1} \to \mathbb{C}^k$ we denote

$$f_l : \mathbb{C}^m \times \mathbb{C}^m \ni (x, y) = ((x_1, x'), (y_1, y')) \mapsto (x_1 - y_1, l(x' - y')) \in \mathbb{C}^{k+1}.$$

Using the above notation and the multiplicity of the holomorphic map we have an effective formula:

THEOREM 3.2. *If* $\Psi^{-1}(0) = \{0\}$, $f_l \circ (\Phi \times \Psi)$ *has an isolated zero at the origin and* $\ker l \cap \mathbb{S} = \{0\}$, *then*

$$i(X, Y; 0) = \frac{1}{s}\mu_0(f_l \circ (\Phi \times \Psi)).$$

Proof. Let $T := X \times Y, \pi : \mathbb{C}^m \times \mathbb{C}^m \ni (x,y) \mapsto x - y \in \mathbb{C}^m$ and $\Delta := \ker \pi$. By the theory developed in [ATW 90], to compute the multiplicity of the isolated intersection of X and Y we should calculate the multiplicity of the isolated intersection of T with the subspace Δ at the point $0 \in \mathbb{C}^m \times \mathbb{C}^m$ ([ATW 90, Def. 5.1]). Observe that $\ker f_l$ is a linear subspace of $\mathbb{C}^m \times \mathbb{C}^m$ of codimension $k + 1$ and by [ATW 90, Lemma 2.4] we have $C_0(T, \Delta) = \pi^{-1}(\mathbb{C} \times \mathbb{S}) = \{(x, y) \in \mathbb{C}^m \times \mathbb{C}^m \mid x' - y' \in \mathbb{S}\}$. Thus $\ker f_l \cap C_0(T, \Delta) = \Delta$ and since it is easy to see that the origin is an isolated point of $\ker f_l \cap T$, by [ATW 90, Theorem 4.4] we conclude that $i(X \cdot Y; 0) = i(T \cdot \ker f_l; 0)$. The mapping $\Phi \times \Psi : U \times D \to T = X \times Y$ is an s-parametrization of T and $\ker f_l = Z_{f_l}$ is the cycle of zeros of f_l. By Theorem 3.1 we have $\deg(T \cdot Z_{f_l}; 0) = \frac{1}{s}\deg(Z_{f_l\circ(\Phi\times\Psi)}; 0)$ and $\deg(T \cdot Z_{f_l}; 0) = \deg(T \cdot \ker f_l; 0) = i(X, Y; 0)$, so $\deg(Z_{f_l\circ(\Phi\times\Psi)}; 0) = \mu_0(f_l \circ (\Phi \times \Psi))$, which completes the proof. ∎

EXAMPLE 3.3. Consider the analytic curve

$$X = \{(t^3, t^5, 0, 0) \in \mathbb{C}^4 \mid t \in \mathbb{C}\}$$

and the pure dimensional analytic set

$$Y = \{(\tau^2, \tau^2\varrho, \tau\varrho^2, \varrho^3) \in \mathbb{C}^4 \mid \tau, \varrho \in \mathbb{C}\}$$

of dimension 2. The curve X has the Puiseux parametrization, Y has the 3-parametrization

$$\Psi : \mathbb{C}^2 \ni (\tau, \varrho) \mapsto (\tau^3, \tau^2\varrho, \tau\varrho^2, \varrho^3) \in Y$$

and $X \cap Y = \{0\}$. Moreover, the axis $\mathbb{C}_1$ is the Whitney cone $C_0(X)$ and lies in $C_0(Y)$. The relative tangent cone $C_0(X, Y)$ has the form $\mathbb{C} \times \mathbb{S}$ for some algebraic 2-dimensional cone $\mathbb{S}$. Observe that $(0, 1, 1) \notin \mathbb{S}$. By Theorem 3.1, $i(X, Y; 0) = \frac{1}{3}\mu_0(f_l)$ where

$$f_l : \mathbb{C}^3 \ni (t, \tau, \varrho) \mapsto (t^3 - \tau^3, l(t^5 - \tau^2\varrho, \tau\varrho^2, \varrho^3)) \in \mathbb{C}^3$$

and $l : \mathbb{C}^3 \to \mathbb{C}^2$ is a linear surjection such that $\ker l \cap \mathbb{S} = \{0\}$. Consider $l : \mathbb{C}^3 \ni (x_2, x_3, x_4) \mapsto (x_2, x_3 - x_4) \in \mathbb{C}^2$, so $\ker l = \{(0, t, t) : t \in \mathbb{C}\}$ and $\ker l \cap \mathbb{S} = \{0\}$.

We now compute the multiplicity $\mu_0(f_l)$ using Theorem 4.3 from [TW 89]. Denote

$$g : \mathbb{C}^3 \ni (t, \tau, \varrho) \mapsto t^5 - \tau^2\varrho \in \mathbb{C},$$
$$h : \mathbb{C}^3 \ni (t, \tau, \varrho) \mapsto (t^3 - \tau^3, \tau\varrho^2 - \varrho^3) \in \mathbb{C}^2.$$

To compute the multiplicity of $f_l = (g, h)$ at 0, we observe that $Z_h = 2A_1 + A_2$ where A_1 and A_2 are the sets of three lines with respective equations $t^3 - \tau^3 = 0$, $\varrho = 0$ and $t^3 - \tau^3 = 0$, $\tau - \varrho = 0$. Now, we have $\mu_0(f_l) = 3\cdot 2\cdot 5 + 3\cdot 3 = 39$, so $i(X, Y; 0) = 13$.

Note that in our method we do not need to know the form of the relative tangent cone. It is enough to know a suitable linear subspace which allows one to choose a linear surjection.

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