ANNALES
POLONICI MATHEMATICI
106 (2012)

Plurisubharmonic functions on compact sets

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Abstract. Poletsky has introduced a notion of plurisubharmonicity for functions defined on compact sets in $\mathbb{C}^n$. We show that these functions can be completely characterized in terms of monotone convergence of plurisubharmonic functions defined on neighborhoods of the compact.

1. Introduction. The classical notion of plurisubharmonicity is well known and of undisputed importance for complex analysis and complex geometry. An upper semicontinuous function on an open set in $\mathbb{C}^n$ is said to be plurisubharmonic if its restrictions to complex lines are subharmonic. It is remarkable that an upper semicontinuous function is plurisubharmonic if and only if its compositions with analytic disks are subharmonic. Since this definition do not make any reference to the affine structure of $\mathbb{C}^n$, it can easily be extended to complex manifolds, or even complex spaces. There are however well-known examples of compact subsets of $\mathbb{C}^n$ that do not contain any analytic disks, but still exhibit very interesting behavior from a complex analytic point of view. Best known among these are probably the examples by Stolzenberg [St] and Wermer [We] of compact sets X with non-trivial polynomial hull $\widehat{X}$ such that $\widehat{X} \setminus X$ has no analytic structure. An appropriate notion of plurisubharmonicity on compact sets would make it possible to study such compact sets and shed some light on their pathological behavior.

In [Po4], Poletsky proposed a notion of plurisubharmonicity that does not require the existence of an analytic structure, but merely what could be called an "approximately analytic structure", in a sense that will be made precise later. Using this notion, Poletsky could answer questions about uniform algebras on Wermer type sets and he was also able to explain a pathological example regarding plurisubharmonic extension from [Si]. Recently,

2010 *Mathematics Subject Classification*: Primary 32U05, 31C10; Secondary 46A55.
Key words and phrases: plurisubharmonic functions on compacts, Jensen measures, monotone convergence.

DOI: 10.4064/ap106-0-11

Poletsky and Sigurdsson [PS] have focused on plurisubharmonic functions on compacts and studied them from the point of view of several different generalized Dirichlet problems.

Although it is beyond doubt that Poletsky's plurisubharmonic functions on compact sets are both useful and interesting, many aspects of their nature are still wrapped in mystery. The purpose of this paper is to study these functions and put them into the context of more familiar concepts. To this end our presentation of Poletsky's plurisubharmonicity will avoid most of the technical refinements and instead focus on the main ideas. At some points we will need results from [Po4], and in these cases for the reader's convenience we will present the full proofs in our somewhat stripped-down context.

The main result of this paper is the following theorem.

Main Theorem 1.1. *Let $X \subset \mathbb{C}^n$ be a compact set. Then u is plurisubharmonic on X if and only if it can be pointwise approximated on X by a decreasing sequence $\{u_j\}$ of functions continuous and plurisubharmonic on neighborhoods of X.*

This theorem has two natural interpretations. On one hand, it is an approximation theorem for plurisubharmonic functions on compact sets; on the other hand it relates the rather abstract notion of plurisubharmonicity on compact sets to the well-known notion of classical plurisubharmonicity.

Another of the main points of this paper is the strong connection between Poletsky's notion of plurisubharmonicity and Sibony's notion of B-regularity [Si]. In fact it has been shown by Nguyễn, Dung and Hung [NDH] that the Jensen measures in the sense of Sibony can be taken as the basis for an equivalent definition of plurisubharmonic functions on compact sets. However, in our opinion this fact has not been given proper emphasis, and we try to make up for this with an explicit treatment of the subject.

2. Notation. Throughout this paper, $\mathbb{D}$ denotes the unit disk in $\mathbb{C}$ and $\mathbb{T} := \partial\mathbb{D}$ its boundary. We will always denote an arbitrary compact set in $\mathbb{C}^n$ by X, and open and connected sets in $\mathbb{C}^n$ will be denoted by Ω. As usual we will denote the plurisubharmonic functions on Ω by $\mathcal{PSH}(\Omega)$. The set of plurisubharmonic functions on the compact set X will also be denoted by $\mathcal{PSH}(X)$, so that the meaning of $\mathcal{PSH}(E)$ will depend on whether E is compact or open. We will repeatedly make use of the set of functions that are the restrictions to X of continuous plurisubharmonic functions defined on neighborhoods of X. These will be denoted by $\mathcal{PSH}^o(X)$. The Dirac measure at the point z will be denoted by δ_z.

A bounded holomorphic mapping $f : \mathbb{D} \to \mathbb{C}^n$ will be called an *analytic disk*. It follows from Fatou's theorem that such a mapping has radial boundary values almost everywhere and so we will consider analytic disks as mappings from $\overline{\mathbb{D}}$ to $\mathbb{C}^n$. By σ, we will mean the normalized arc length measure on $\mathbb{T}$. As is common in the literature, we will also allow ourselves to confuse an analytic disk with its image.

3. Holomorphic measures and plurisubharmonic functions. One way to look at plurisubharmonicity is to say that an upper semicontinuous function on an open set Ω is plurisubharmonic on Ω if it satisfies the mean value inequality with respect to integration on each complex line in Ω. A natural generalization of plurisubharmonicity is to allow for a wider range of measures to integrate against. This is the idea of Poletsky.

We start by noting that every analytic disk f induces a measure on $\mathbb{C}^n$ by pushing forward the arc-length measure on the unit disk. To be more precise:

DEFINITION 3.1. For $E \subset \mathbb{C}^n$ and f an analytic disk in $\mathbb{C}^n$, let

$$\mu_f(E) := f_*\sigma(E) = \int_{\mathbb{T}\cap f^{-1}(E)} d\sigma.$$

Next we consider the weak-* closure of such measures.

DEFINITION 3.2. Let $L = \{f_j\}$ be a uniformly bounded sequence of analytic disks. We say that L *is weak-* convergent* if the measures μ_{f_j} weak-* converge on $\mathbb{C}^n$, that is, there is a measure μ_L such that

$$\int \phi\, d\mu_L = \lim_{j\to\infty} \int \phi\, d\mu_{f_j}, \quad \forall \phi \in C(\mathbb{C}^n).$$

Given such a sequence, we will denote its limit measure by μ_L and let $\lim_j f_j(0) = z_L$.

To confine our attention to measures on specific sets, we introduce the notion of the cluster of a sequence of analytic disks.

DEFINITION 3.3. Let $L = \{f_j\}$ be a sequence of analytic disks. The *cluster* of L, denoted $\mathcal{K}(L)$, is the set of all $z \in \mathbb{C}^n$ such that the set

$$\{j \in \mathbb{N} : f_j(\mathbb{D}) \cap B(z,r) \neq \emptyset\}$$

is infinite for every $r > 0$.

REMARK 3.4. Since the complement of the cluster is easily seen to be open, the cluster itself will always be a closed set.

We are now ready to define what we mean by a holomorphic measure.

DEFINITION 3.5. Suppose that L is a uniformly bounded sequence of analytic disks. If $\mathcal{K}(L) \subset X$ and L is weak-* convergent, we say that μ_L is a *holomorphic measure* at the point $z := z_L$. We denote of all such measures by $\mathcal{M}_z(X)$.

Remark 3.6. It follows from the definition of weak-* convergence that all holomorphic measures are probability measures.

Remark 3.7. Given $z \in X$, let $f_j(\zeta) = z$. Then $\lim \mu_{f_j} = \delta_z$, which means that $\delta_z \in \mathcal{M}_z(X)$.

The above definition of holomorphic measure is worth some extra comments. Although this is the original definition given by Poletsky [Po4], one may ask if it is too general. To achieve the full power of Poletsky theory, it is often necessary to work with analytic disks that are holomorphic not only in the interior of $\mathbb{D}$ but in a neighborhood of $\overline{\mathbb{D}}$. To make this precise, we introduce the following definition.

Definition 3.8. Let $\Omega \subset \mathbb{C}^n$ be a bounded open set. We define $\mathcal{H}_z(\Omega)$ to be the class of mappings f from $\overline{\mathbb{D}}$ to Ω such that $f(0) = z$ and f is holomorphic in a neighborhood of $\overline{\mathbb{D}}$.

The following proposition shows that we end up with the same class of holomorphic measures if we demand the analytic disks in Definition 3.5 to be holomorphic in a neighborhood of $\overline{\mathbb{D}}$.

Proposition 3.9. *If $\mu \in \mathcal{M}_z(X)$, then there is a domain $\Omega \supset X$ and a sequence $L = \{f_j\} \subset \mathcal{H}_{z_j}(\Omega)$ of analytic disks such that $\mathcal{K}(L) \subset X$ and $\{\mu_{f_j}\}$ converges weak-* to μ.*

To prove the proposition, we need the following compactness result.

Lemma 3.10 ([PS]). *Suppose that $\{X_j\}$ is a decreasing sequence of compact sets in $\mathbb{C}^n$ such that $X = \bigcap X_j$. Suppose also that $\mu_j \in \mathcal{M}_{z_j}(X_j)$. Then there is a subsequence $\mu_{j(k)}$ converging to a measure $\mu \in \mathcal{M}_z(X)$ with $z = \lim z_{j(k)}$.*

Proof. Let B be the closure of a ball such that $X_1 \subset B$. It follows from the Riesz representation theorem that the probability measures on B can be identified with the dual space of $C(B)$ equipped with the uniform norm, and so the Banach–Alaoglu theorem implies that the space of probability measures on B is compact when equipped with the weak-* topology of $C(B)$. This means that there is a subsequence $\mu_{j(k)}$ converging to a probability measure μ on B. It remains to show that $\mu \in \mathcal{M}_z(X)$.

For notational comfort, we begin by renaming our subsequence μ_k. By definition, there exist sequences $\{f_{k\ell}\}$ of analytic disks such that $\lim f_{k\ell}(0) = z_k$, $\mu_{f_{k\ell}}$ weak-* converges to μ_k and $\mathcal{K}(\{f_{k\ell}\}) \subset X_k$. Since $C(B)$ is separable, a diagonal argument yields a sequence $L := \{f_{k\ell_k}\} \subset \{f_{k\ell}\}$ such that $\mu_L = \mu$. Obviously $\mathcal{K}(\{f_{k\ell_k}\}) \subset X$ and $\lim f_{k\ell_k}(0) = z$, and we are done. ■

Proof of Proposition 3.9. By the definition of $\mathcal{M}_z(X)$, there is a uniformly bounded sequence $L' = \{f_j\}$ of analytic disks such that μ_{f_j} weak-*

converges to μ. Pick a domain Ω such that $f_j(\overline{\mathbb{D}}) \Subset \Omega$ for all $j \in \mathbb{N}$. Define

$$f_{jk}(\zeta) = f_j((1 - 1/k)\zeta).$$

We note that all the functions f_{jk} are in $\mathcal{H}_{z_j}(\Omega)$.

By Lemma 3.10, there is a weak-* convergent subsequence L of $\{f_{jj}\}$ such that $\mu_L \in \mathcal{M}_z(X)$. By definition, this measure must be identical to μ. Of course $\mathcal{K}(L) \subset \mathcal{K}(L') \subset X$. This finishes the proof. ∎

With this technical question settled, we can define Poletsky's notion of plurisubharmonicity on compact sets.

DEFINITION 3.11. Let u be an upper semicontinuous function on X. We say that u is *plurisubharmonic on* X if for all $z \in X$,

$$u(z) \leq \int u \, d\mu, \quad \forall \mu \in \mathcal{M}_z(X).$$

We denote the set of all plurisubharmonic functions on X by $\mathcal{PSH}(X)$.

The following elementary examples show that the class $\mathcal{PSH}(X)$ shares some basic properties with classical plurisubharmonic functions defined on open sets.

EXAMPLE 3.12. Let $u, v \in \mathcal{PSH}(X)$. Then it follows from the properties of the integral that $su + tv \in \mathcal{PSH}(X)$ for all $s, t \geq 0$.

EXAMPLE 3.13. Let $u_1, u_2 \in \mathcal{PSH}(X)$. Then for $z \in X$ and $\mu \in \mathcal{M}_z(X)$,

$$\int \max(u_1, u_2) \, d\mu \geq \int u_i \, d\mu \geq u_i(z) \quad \text{for } i = 1, 2.$$

Hence $\max(u_1, u_2) \in \mathcal{PSH}(X)$.

EXAMPLE 3.14. Let $u \in \mathcal{PSH}(X)$ and let ϕ be a convex, strictly increasing function on the range of u. Then $\phi \circ u \in \mathcal{PSH}(X)$ by Jensen's inequality.

EXAMPLE 3.15. Let u be a plurisubharmonic function defined in a neighborhood of X. Suppose that $z \in X$ and $\mu \in \mathcal{M}_z(X)$. This means that there are analytic disks f_j such that

$$\int \phi \, d\mu = \lim_{j \to \infty} \int_{\mathbb{T}} \phi \circ f_j \, d\sigma$$

for all $\phi \in C(\mathbb{C}^n)$. Since u can be approximated by a decreasing sequence u_k of smooth plurisubharmonic function defined in a slightly smaller neighborhood of X, the functions $u_k \circ f_j$ are subharmonic on $\mathbb{D}$. It follows that

$$\int u \, d\mu = \lim_{j,k \to \infty} \int_{\mathbb{T}} u_k \circ f_j \, d\sigma \geq \lim_{j,k \to \infty} u_k \circ f_j(0) = u(z),$$

which shows that $u \in \mathcal{PSH}(X)$.

Example 3.16. Let X be a compact set with interior points and suppose that $u \in \mathcal{PSH}(X)$. Then u is plurisubharmonic in the classical sense on the interior of X. This follows from the fact that every complex line in the interior of X determines a holomorphic measure $\mu \in \mathcal{M}_z(X)$.

4. Approximation of plurisubharmonic functions. The key tool in proving our main theorem on approximation of plurisubharmonic functions on compact sets will be an Edwards type duality theorem for plurisubharmonic functions and holomorphic measures. A very similar theorem was proved by Poletsky [Po4], whose proof we will also follow.

Theorem 4.1. *Suppose that $\phi \in C(X)$ for a compact set $X \subset \mathbb{C}^n$. Define*

$$u(z) = \inf\Big\{\int \phi\, d\mu : \mu \in \mathcal{M}_z(X)\Big\},$$
$$v(z) = \sup\{\psi(z) : \psi \in \mathcal{PSH}^o(X),\ \psi \le \phi\}.$$

Then $u = v$ on X and there is an increasing sequence $u_j \in \mathcal{PSH}^o(X)$ converging pointwise to this function.

Before we prove the theorem, we define a notion of holomorphic measure on open sets. This definition should be compared to the holomorphic current Φ_1 (the Poisson functional) of [Po3, Example 3.1].

Definition 4.2. Suppose that $\Omega \Subset \mathbb{C}^n$ is an open set. We define $\mathcal{M}_z(\Omega)$ to be the weak-* closure of the set $\{\mu_f : f \in \mathcal{H}_z(\Omega)\}$.

Remark 4.3. Since $\mathcal{M}_z(\Omega)$ is a closed subset of the set of probability measures, and since the set of probability measures is weak-* compact, the same holds true for $\mathcal{M}_z(\Omega)$.

Remark 4.4. Suppose that $\mu \in \mathcal{M}_z(\Omega)$. Then μ is the weak-* limit of some sequence μ_{f_j} for $f_j \in \mathcal{H}_z(\Omega)$. The sequence $L = \{f_j\}$ is uniformly bounded and $\mathcal{K}(L) \subset \overline{\Omega}$. Hence $\mathcal{M}_z(\Omega) \subset \mathcal{M}_z(\overline{\Omega})$.

We are now ready to prove the duality theorem.

Proof of Theorem 4.1. Since $\delta_z \in \mathcal{M}_z(X)$, it follows that $u \le \phi$. If $\psi \in \mathcal{PSH}^o(X)$, $\psi \le \phi$ and $\mu \in \mathcal{M}_z(X)$, then

$$\psi(z) \le \int \psi\, d\mu \le \int \phi\, d\mu.$$

Taking the supremum of all such ψ we get

$$v(z) = \sup\{\psi(z) : \psi \in \mathcal{PSH}^o(X),\ \psi \le \phi\} \le \int \phi\, d\mu.$$

Now we take the infimum over all $\mu \in \mathcal{M}_z(X)$ to get $v \le u \le \phi$ on X.

The idea is now to approximate u with functions smaller than v, to see that u and v have to be equal. Let $\{\Omega_j\}$ be a sequence of bounded open domains such that $X \subset \Omega_{j+1} \Subset \Omega_j$ and $\bigcap_{j=1}^{\infty} \Omega_j = X$. By the Tietze extension theorem there exists a continuous extension $\widetilde{\varphi}$ of φ onto Ω_1. Since $\mathcal{M}_z(\Omega_j)$ is weak-* compact (see Remark 4.3), the point-evaluation functional defined by $\mu \mapsto \int_{\Omega_j} \widetilde{\varphi}\, d\mu$, being continuous in the weak-* topology, must attain its minimum on $\mathcal{M}_z(\Omega_j)$, so there exists $\mu_j^z \in \mathcal{M}_z(\Omega_j)$ such that

$$\int \widetilde{\phi}\, d\mu_j^z = \inf\Big\{\int \widetilde{\phi}\, d\mu_f : f \in \mathcal{H}_z(\Omega_j)\Big\}.$$

We know that $\mu_j^z \in \mathcal{M}_z(\overline{\Omega}_j)$ by Remark 4.4. Also $\int \widetilde{\phi}\, d\mu_j^z$ increases with j (since $\Omega_{j+1} \Subset \Omega_j$). By letting $X_j = \overline{\Omega}_j$ we are in the situation of Lemma 3.10, and hence $\{\mu_j^z\}$ has a subsequence $\mu_k^z = \mu_{j(k)}^z$ which weak-* converges to $\mu^z \in \mathcal{M}_z(X)$. Define

$$u_k(z) = \int \widetilde{\phi}\, d\mu_k^z - 1/k.$$

By the Poletsky minimum principle [Po1, Theorem 1], $u_k \in \mathcal{PSH}(\Omega_k)$ and by the observation above $u_{k+1} > u_k$. We also have

$$\lim_{k\to\infty} u_k = \int \phi\, d\mu^z.$$

Since $\mu^z \in \mathcal{M}_z(X)$, it follows that $\lim_{k\to\infty} u_k(z) \ge u(z)$. By definition, we know that $u_k \in \mathcal{PSH}^o(X)$, and $u_k \le \varphi$ since $\delta_z \in \mathcal{M}_z(\Omega_k)$. Hence $u_k \le v$. As we have already noted that $v \le u$, this gives $v = u$. ∎

Poletsky noted that this theorem immediately gives an approximation theorem for continuous plurisubharmonic functions.

Corollary 4.5. *Suppose that $u \in \mathcal{PSH}(X) \cap C(X)$. Then there is a sequence $u_j \in \mathcal{PSH}^o(X)$ such that $u_j \nearrow u$.*

Proof. Since $\delta_z \in \mathcal{M}_z(X)$,

$$u(z) = \inf\Big\{\int u\, d\mu : \mu \in \mathcal{M}_z(X)\Big\},$$

and by Theorem 4.1, this function can be approximated by a monotone sequence $u_k \in \mathcal{PSH}^o(X)$. ∎

Corollary 4.6. *$\mathcal{PSH}(X) \cap C(X)$ is the uniform closure of $\mathcal{PSH}^o(X)$.*

Proof. This follows directly from the previous corollary and Dini's theorem. ∎

The previous corollary only applies to continuous functions on X, but as announced in the introduction, all plurisubharmonic functions on X can be characterized by monotone convergence.

Theorem 4.7 (Main Theorem). *A function u belongs to $\mathcal{PSH}(X)$ if and only if there is a sequence $u_j \in \mathcal{PSH}^o(X)$ such that $u_j \searrow u$ on X.*

Proof. First assume that u can be approximated as in the statement. Then, being the decreasing limit of continuous functions, u is upper semicontinuous. Since all u_j belong to $\mathcal{PSH}(X)$, for $z \in X$ and $\mu \in \mathcal{M}_z(X)$ we have

$$u(z) = \lim_{j\to\infty} u_j(z) \le \lim_{j\to\infty} \int u_j \, d\mu = \int u \, d\mu,$$

where in the last inequality we have used the fact that $\{u_j\}$ is monotone.

Conversely, suppose that $u \in \mathcal{PSH}(X)$. We begin by showing that for every $f \in C(X)$ such that $u < f$ on X, we can find $v \in \mathcal{PSH}^o(X)$ such that $u < v \le f$. Let

$$F(z) := \sup\{\varphi(z) : \varphi \in \mathcal{PSH}^o(X),\ \varphi \le f\} = \inf\Big\{\int f \, d\mu : \mu \in \mathcal{M}_z(X)\Big\}.$$

As in the proof of Theorem 4.1, for each $z \in X$ we can find $\mu_z \in \mathcal{M}_z(X)$ such that $F(z) = \int f \, d\mu_z$. Since

$$F(z) = \int f \, d\mu_z > \int u \, d\mu_z \ge u(z),$$

we have $u < F$. By the construction of F, for every $z \in X$ there exists $v_z \in \mathcal{PSH}^o(X)$ such that $v_z \le F$ and $u(z) < v_z(z) \le F(z)$. As $u - v_z$ is upper semicontinuous, the set $U_z := \{w \in X : u(w) - v_z(w) < 0\}$ is open in X. By the compactness of X there are finitely many points $z_1, \dots, z_k$ with corresponding functions $v_{z_1}, \dots, v_{z_k}$ and open sets $U_{z_1}, \dots, U_{z_k}$ such that $u < v_{z_j}$ in U_{z_j} and $X = \bigcup_{j=1}^k U_{z_j}$. The function $v = \max\{v_{z_1}, \dots, v_{z_k}\}$ belongs to $\mathcal{PSH}^o(X)$ and $u < v \le f$.

It is now easy to prove that u can be approximated as desired. Indeed, since u is upper semicontinuous, it can be approximated by a decreasing sequence f_j of continuous functions. We can find $v_1 \in \mathcal{PSH}^o(X)$ such that $u < v_1 \le f_1$. Assuming that we have found a decreasing sequence $\{v_1, \dots, v_k\}$ such that $v_j \in \mathcal{PSH}^o(X)$ and $u < v_j < f_j$ for $j = 1, \dots, k$, we find $v_{k+1} \in \mathcal{PSH}^o(X)$ such that $u < v_{k+1}$ and $v_{k+1} \le \min\{f_{k+1}, v_k\}$. Now the conclusion follows by induction. ∎

5. A Choquet-theoretic definition of holomorphic measures. In [Si], Sibony used a Choquet-theoretic approach to study the Dirichlet problem for plurisubharmonic functions. For this he studied the class of Jensen measures with respect to a certain class of plurisubharmonic functions, and by analysing these measures he was able to draw far-reaching conclusions regarding the solvability of the Dirichlet problem. Specifically he studied sets whose only Jensen measures are the Dirac measures, so called B-regular sets, and put these in connection with Catlin's Property (P) [Ca] and the $\bar\partial$-Neumann problem.

In [Po4], Poletsky hinted at a connection between his holomorphic measures and Sibony's Jensen measures, but without making any explicit statements. Later Nguyễn, Dung and Hung showed in an appendix to their paper [NDH] that the two classes of measures actually coincide. Thus, Poletsky's results on holomorphic measures give an approximation result for Jensen measures which should be compared to that of Bu and Schachermayer [BS] on classical Jensen measures.

Since we believe that this result has got neither the attention nor the presentation it deserves, we dedicate the last section of the paper to it. We emphasize that even though the result of this section is already known, the method of proof is new.

Definition 5.1. A probability measure μ on X is said to be a *Jensen measure* at $z \in X$ if

$$u(z) \le \int u\, d\mu, \quad \forall u \in \mathcal{PSH}^o(X). \tag{5.1}$$

The set of all Jensen measures at z is denoted by $\mathcal{J}_z(X)$.

Remark 5.2. This set is always non-empty, since $\delta_z \in \mathcal{J}_z(X)$.

Remark 5.3. In the terminology of Choquet theory the measures in $\mathcal{J}_z(X)$ are said to be *representation measures* or $\mathcal{PSH}^o$*-measures* for z. A good reference for Choquet theory is the monograph [Ga] by Gamelin.

For clarity, we point out that there is a certain freedom of choice in defining $\mathcal{J}_z(X)$. We have chosen to use the set $\mathcal{PSH}^o(X)$, but this is more or less a matter of taste. We could for example require the inequality (5.1) to hold for merely upper semicontinuous plurisubharmonic functions (as in [NDH]), or for those functions on X that can be uniformly approximated by functions in $\mathcal{PSH}^o(X)$ (as in [Si]). In both cases it is easy to see that we end up with the same Jensen measures.

For our purposes it is important to notice that $\mathcal{PSH}^o(X)$ is a convex cone of continuous functions containing the constants and separating points. This means that we can apply the techniques of Choquet theory; specifically, we automatically get an analogue to Theorem 4.1.

Theorem 5.4 (Edwards' Theorem [Ed]). *Let ϕ be a lower semicontinuous function on X. Then*

$$\sup\{\psi(z) : \psi \in \mathcal{PSH}^o(X),\ \psi \le \phi\} = \inf\Big\{\int \phi\, d\nu : \nu \in \mathcal{J}_z(X)\Big\}.$$

As the suprema in Theorems 5.4 and 4.1 are the same, this theorem provides a link between Jensen measures and holomorphic measures. Together with the Hahn–Banach separation theorem, this will enable us to prove that $\mathcal{J}_z(X) = \mathcal{M}_z(X)$. To be able to use the Hahn–Banach theorem, we have to prove the convexity of the set of holomorphic measures.

The following lemma is a variant of a theorem of Bu and Schachermayer [BS], but the proof we present is due to Poletsky [Po2].

Lemma 5.5. *The set $\mathcal{M}_z(X)$ is convex.*

Proof. Without loss of generality, we may assume that $z=0$. We want to show that $\lambda\mu+(1-\lambda)\nu\in\mathcal{M}_z(X)$ for all $\mu,\nu\in\mathcal{M}_z(X)$ and $\lambda\in[0,1]$. By the definition of $\mathcal{M}_z(X)$, the measures μ and ν are weak-* limits of some sequences, $\{\mu_{f_j}\}$ and $\{\nu_{g_j}\}$ respectively, where $\mathcal{K}(\{f_j\}),\mathcal{K}(\{g_j\})\subset X$ and the disks $\{f_j\}$ and $\{g_j\}$ all lie in $B(0,M)$ for some large $M>0$.

Choose $0<r_j<1$ and consider the mappings

$$h_j(\zeta)=f_j(\zeta)+g_j(r_j/\zeta),$$

defined on the annuli $R_j=\{\zeta: r_j<\zeta<1\}$.

For every $\zeta\in\mathbb{D}$, either $|\zeta|$ or $r_j/|\zeta|$ is less than $\sqrt{r_j}$. Hence choosing r_j small enough will guarantee that $h_j(R_j)$ lies in an arbitrarily small neighborhood of $f_j(\mathbb{D})\cup g_j(\mathbb{D})$. Suppose from now on that $r_j\to 0$.

Now let ψ_j be the conformal mapping from $\mathbb{D}$ to $\{\zeta:\log r_j<\operatorname{Re}\zeta<0\}$ defined by the formula

$$\psi_j(\zeta):=\frac{i\log r_j}{\pi}\log\biggl(e^{-\lambda i\pi}\frac{\zeta-e^{\lambda i\pi}}{\zeta-e^{-\lambda i\pi}}\biggr)+\log r_j.$$

Note that $\psi_j(0)=(1-\lambda)\log r_j$, $\psi_j(1)=0$ and $\psi_j(-1)=\log r_j$. Hence the mapping $\omega_j=e^{\psi_j}$ maps the arc $\gamma_1:=\{e^{i\theta}:|\theta|<\lambda\pi\}$ onto $\mathbb{T}$ and the arc $\gamma_2:=\mathbb{T}\setminus\gamma_1$ onto $r_j\mathbb{T}$. Now $p_j:=h_j\circ\omega_j$ is an analytic disk in $\mathbb{C}^n$, and we claim that the corresponding measures μ_{p_j} weak-* converge to $\lambda\mu+(1-\lambda)\nu$.

To prove this, first observe that $\mathcal{K}(\{p_j\})\subset\mathcal{K}(\{f_j\})\cup\mathcal{K}(\{g_j\})\subset X$. Next, let $\phi\in C(\mathbb{C}^n)$. Then

$$\int\phi\,d\mu_{p_j}=\int_{\gamma_1}\phi\bigl(f_j(\omega_j)+g_j(r_j/\omega_j)\bigr)\,d\sigma+\int_{\gamma_2}\phi\bigl(f_j(\omega_j)+g_j(r_j/\omega_j)\bigr)\,d\sigma.\tag{5.2}$$

We first study the integral over γ_1. Since ω_j maps γ_1 onto $\mathbb{T}$, and by the construction of r_j, the Schwarz lemma shows that for $\zeta\in\gamma_1$, $|g_j(r_j/\omega_j(\zeta))|\to 0$ as $j\to\infty$. By the continuity of ϕ it follows that

$$\int_{\gamma_1}\phi(f_j(\omega_j)+g_j(r_j/\omega_j))\,d\sigma=\int_{\gamma_1}\phi(f_j(\omega_j))\,d\sigma+\delta_j\tag{5.3}$$

for some $\delta_j\to 0$ as $j\to\infty$.

Now let u_j be the harmonic function on $\mathbb{D}$ with boundary values $u_j(\zeta):=\phi(f_j(\zeta))$. Since every u_j is bounded by

$$K:=\sup_{z\in B(0,M)}\phi(z),$$

by the Schwarz lemma we have

$$|u_j(\zeta)-u_j(0)|<K|\zeta|,\quad\forall j\in\mathbb{N}.\tag{5.4}$$

Since $u_j \circ \omega_j$ is also a harmonic function it follows that

$$u_j(\omega_j(0)) = \int_{\mathbb{T}} u_j(\omega_j)\, d\sigma = \int_{\gamma_1} \phi(f_j(\omega_j))\, d\sigma + \int_{\gamma_2} u_j(\omega_j)\, d\sigma. \tag{5.5}$$

Using (5.4) we know $|u_j(\omega_j(\zeta)) - u_j(0)| < Kr_j$ for $\zeta \in \gamma_2 \cup \{0\}$. Putting this together with (5.5) we see that

$$u_j(0) = \int_{\gamma_1} \phi(f_j(\omega_j))\, d\sigma + (1-\lambda)u_j(0) + \delta'_j,$$

where $\delta'_j \to 0$. Rearranging and combining (5.3) and the definition of u we arrive at

$$\int_{\gamma_1} \phi(f_j(\omega_j) + g_j(r_j/\omega_j))\, d\sigma = \lambda \int_{\mathbb{T}} \phi(f_j(\zeta))\, d\sigma + \delta''_j$$

for some sequence $\delta''_j \to 0$.

By a similar estimation of the integral over γ_2 in (5.2), we conclude that

$$\lim_{j\to\infty} \int \phi\, d\mu_{p_j} = \lambda \int \phi\, d\mu + (1-\lambda) \int \phi\, d\nu. \ \blacksquare$$

With the question of convexity settled, we can finally prove the equality of the Jensen measures and holomorphic measures. This was originally proved by Nguyễn, Dung and Hung [NDH], who used another proof.

THEOREM 5.6. *For every $z \in X$, $\mathcal{J}_z(X) = \mathcal{M}_z(X)$.*

Proof. Suppose that $\mu \in \mathcal{M}_z(X)$ and pick $u \in \mathcal{PSH}^o(X)$. Then u can be extended to a plurisubharmonic function on a neighborhood of X and Example 3.15 yields

$$u(z) \le \int u\, d\mu, \tag{5.6}$$

which shows that $\mu \in \mathcal{J}_z(X)$.

Conversely, suppose that there is a $\mu \in \mathcal{J}_z(X) \setminus \mathcal{M}_z(X)$. Since we have shown that $\mathcal{M}_z(X)$ is convex, and by Lemma 3.10 it is weak-* compact, the Hahn–Banach separation theorem yields a $\phi \in C(X)$ such that

$$\int \phi\, d\mu < \inf\Big\{\int \phi\, d\nu : \nu \in \mathcal{M}_z(X)\Big\} = \sup\Big\{\psi(z) : \psi \in \mathcal{PSH}^o(X),\ \psi \le \phi\Big\}.$$

By Theorem 5.4, this supremum equals

$$\inf\Big\{\int \phi\, d\nu : \nu \in \mathcal{J}_z(X)\Big\} \le \int \phi\, d\mu,$$

and this is a contradiction. $\blacksquare$

Acknowledgements. We would like to thank Professors E. Poletsky and P. Åhag for inspiring discussions on the theme of this paper. The first named author was partially supported by NCN grant 2011/01/B/ST1/00879.

References

[BS] S. Q. Bu and W. Schachermayer, *Approximation of Jensen measures by image measures under holomorphic functions and applications*, Trans. Amer. Math. Soc. 331 (1992), 585–608.

[Ca] D. Catlin, *Global regularity of the $\bar{\partial}$-Neumann problem*, in: Complex Analysis of Several Variables (Madison, 1982), Proc. Sympos. Pure Math. 41, Amer. Math. Soc., Providence, RI, 1984, 39–49.

[Ed] D. A. Edwards, *Choquet boundary theory for certain spaces of lower semicontinuous functions*, in: Function Algebras (New Orleans, LA, 1965), F. Birtel (ed.), Scott-Foresman, Chicago, IL, 1966, 300–309.

[Ga] T. W. Gamelin, *Uniform Algebras and Jensen Measures*, London Math. Soc. Lecture Note Ser. 32, Cambridge Univ. Press, Cambridge, 1978.

[NDH] Q. D. Nguyễn, N. T. Dung and D. H. Hung, *B-regularity of certain domains in* $\mathbb{C}^n$, Ann. Polon. Math. 86 (2005), 137–152.

[Po1] E. A. Poletsky, *Plurisubharmonic functions as solutions of variational problems*, in: Proc. Sympos. Pure Math. 52, Part 1, Amer. Math. Soc., Providence, RI, 1991, 163–171.

[Po2] E. A. Poletsky, *Sequences of analytic disks*, arXiv:math/9207202v1.

[Po3] E. A. Poletsky, *Holomorphic currents*, Indiana Univ. Math. J. 42 (1993), 85–144.

[Po4] E. A. Poletsky, *Analytic geometry on compacta in* $\mathbb{C}^n$, Math. Z. 222 (1996), 407–424.

[PS] E. A. Poletsky and R. Sigurdsson, *Dirichlet problems for plurisubharmonic functions on compact sets*, Math. Z. 271 (2012), 877–892.

[Si] N. Sibony, *Une classe de domaines pseudoconvexes*, Duke Math. J. 55 (1987), 299–319.

[St] G. Stolzenberg, *A hull with no analytic structure*, J. Math. Mech. 12 (1963), 103–111.

[We] J. Wermer, *Polynomially convex hulls and analyticity*, Ark. Mat. 20 (1982), 29–35.

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Received 30.11.2011
and in final form 16.5.2012 (2625)