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Boundaries of Levi-flat hypersurfaces: special hyperbolic points

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Abstract. Let $S \subset \mathbb{C}^n$, $n \geq 3$, be a compact connected 2-codimensional submanifold having the following property: there exists a Levi-flat hypersurface whose boundary is S, possibly as a current. Our goal is to get examples of such S containing at least one special 1-hyperbolic point: a sphere with two horns, elementary models and their gluings. Some particular cases of S being a graph are also described.

1. Introduction. Let $S \subset \mathbb{C}^n$ be a compact connected 2-codimensional submanifold having the following property: there exists a Levi-flat hypersurface $M \subset \mathbb{C}^n \setminus S$ such that $dM = S$ (i.e. whose boundary is S, possibly as a current). The case $n = 2$ has been intensively studied since the beginning of the eighties, in particular by Bedford, Gaveau, Klingenberg, Shcherbina, Chirka, Tomassini, Słodkowski, Gromov, Eliashberg; it requires global conditions: S has to be contained in the boundary of a strictly pseudoconvex domain.

We consider the case $n \geq 3$; results on this case have been obtained since 2005 by Dolbeault, Tomassini and Zaitsev; local necessary conditions recalled in Section 2 have to be satisfied by S, singular CR points on S are supposed to be elliptic and the solution M is obtained in the sense of currents [DTZ05, DTZ10]. More recently a regular solution M has been obtained when S satisfies a supplementary global condition as in the case $n = 2$ [DTZ11], with singular CR points on S still supposed to be elliptic.

The problem we are interested in is to get examples of such S containing at least one special 1-hyperbolic point (Section 2.4). CR orbits near a special 1-hyperbolic point are large and, assuming they are compact, a careful examination has to be done (Sections 2.6, 2.7). As a topological preliminary, we need a generalization of a theorem of Bishop on the difference of the

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numbers of special elliptic and 1-hyperbolic points (Section 2.8); this result is a particular case of a theorem of Hon-Fei Lai [Lai72].

The first example considered is the sphere with two horns which has one special 1-hyperbolic point and three special elliptic points (Section 3.4). Then we consider elementary models and their gluings to obtain more complicated examples (Section 3.5). The results have been announced in [Dol08], and in a more precise way in [Dol11]; the first aim of this paper is to give complete proofs. Finally, we recall in detail and extend the results of [DTZ11] on regularity of the solution when S is a graph satisfying a supplementary global condition, as in the case $n = 2$, to the case of existence of special 1-hyperbolic points, and to gluing of elementary smooth models (Section 4).

2. Preliminaries: local and global properties of the boundary

2.1. Definitions. A smooth, connected, CR submanifold $M \subset \mathbb{C}^n$ is called *minimal* at a point p if there does not exist a submanifold N of M of lower dimension through p such that $HN = HM|_N$, where HN is the complex tangent bundle to N. By a theorem of Sussmann, all possible submanifolds N such that $HN = HM|_N$ contain, as germs at p, one of the minimal possible dimension, defining a so called *CR orbit* of p in M whose germ at p is uniquely determined.

A smooth compact connected oriented submanifold $S \subset \mathbb{C}^n$ of dimension $2n-2$ is said to be a *locally flat boundary* at a point p if it locally bounds a Levi-flat hypersurface near p. Assume that S is CR in a small enough neighborhood U of $p \in S$. If all CR orbits of S are 1-codimensional (which will appear as a necessary condition for our problem), the following two conditions are equivalent [DTZ05]:

(i) S is a locally flat boundary on U;
(ii) S is nowhere minimal on U.

2.2. Complex points of S (i.e. singular CR points on S) [DTZ05].

At such a point $p \in S$, T_pS is a complex hyperplane in $T_p\mathbb{C}^n$. In suitable local holomorphic coordinates $(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C}$ vanishing at p, with $w = z_n$ and $z = (z_1, \ldots, z_{n-1})$, S is locally given by the equation

$$w = \varphi(z) = Q(z) + O(|z|^3),$$
$$Q(z) = \sum_{1 \le i,j \le n-1} (a_{ij} z_i z_j + b_{ij} z_i \overline{z}_j + c_{ij} \overline{z}_i \overline{z}_j). \tag{1}$$

S is called *flat* at a complex point $p \in S$ if $\sum b_{ij} z_i \overline{z}_j \in \lambda\mathbb{R}$, $\lambda \in \mathbb{C}$. We also say that p *is flat.*

Let $S \subset \mathbb{C}^n$ be a locally flat boundary with a complex point p. Then p is flat.

By making the change of coordinates $(z,w)\mapsto(z,\lambda^{-1}w)$, we get $\sum b_{ij}z_i\overline{z}_j \in \mathbb{R}$ for all z. By a change of coordinates $(z,w)\mapsto(z,w+\sum a'_{ij}z_iz_j)$ we can choose the holomorphic term in (1) to be the conjugate of the antiholomorphic one and so make the whole form Q real-valued.

We say that S is in a *flat normal form* at p if the coordinates (z,w) as in (1) are chosen such that $Q(z)\in\mathbb{R}$ for all $z\in\mathbb{C}^{n-1}$.

Properties of Q. Assume that S is in a flat normal form; then the quadratic form Q is real-valued. If Q is positive definite or negative definite, the point $p\in S$ is said to be *elliptic*; if $p\in S$ is not elliptic, and if Q is nondegenerate, p is said to be *hyperbolic*. From Section 2.4 on, we will only consider particular cases of the quadratic form Q.

2.3. Elliptic points

PROPOSITION 2.1 ([DTZ05, DTZ10]). *Assume that $S\subset\mathbb{C}^n$ $(n\geq 3)$ is nowhere minimal at all its CR points and has an elliptic flat complex point p. Then there exists a neighborhood V of p such that $V\setminus\{p\}$ is foliated by compact real $(2n-3)$-dimensional CR orbits diffeomorphic to the sphere $\mathbb{S}^{2n-3}$ and there exists a smooth function ν having the CR orbits as level surfaces.*

Sketch of proof (see [DTZ10]). In the case of a quadric S_0 $(w=Q(z))$, the CR orbits are defined by $w_0=Q(z)$, where w_0 is constant. Using (1), we approximate the tangent space to S by the tangent space to S_0 at a point with the same coordinate z; the same is done for the tangent spaces to the CR orbits on S and S_0; then we construct the global CR orbit on S. ■

2.4. Special flat complex points. From [Bis65], for $n=2$, in suitable local holomorphic coordinates centered at 0, we have $Q(z)=z\overline{z}+\lambda\operatorname{Re}z^2$, $\lambda\geq 0$, under the notation of [BK91]; for $0\leq\lambda<1$, p is said to be *elliptic*, and for $\lambda>1$, it is said to be *hyperbolic*. The parabolic case $\lambda=1$, not generic, will be omitted [BK91]. When $n\geq 3$, Bishop's reduction cannot be generalized.

We say that the flat complex point $p\in S$ is *special* if in certain holomorphic coordinates centered at 0,

$$Q(z)=\sum_{j=1}^{n-1}(z_j\overline{z}_j+\lambda_j\operatorname{Re}z_j^2),\qquad \lambda_j\geq 0. \tag{2}$$

Let $z_j=x_j+iy_j$, x_j,y_j real, $j=1,\dots,n-1$. Then

$$Q(z)=\sum_{l=1}^{n-1}((1+\lambda_l)x_l^2+(1-\lambda_l)y_l^2). \tag{3}$$

A flat point $p\in S$ is said to be *special elliptic* if $0\leq\lambda_j<1$ for any j.

A flat point $p \in S$ is said to be *special k-hyperbolic* if $\lambda_j > 1$ for $j \in J \subset \{1, \dots, n-1\}$, and $0 \leq \lambda_j < 1$ for $j \in \{1, \dots, n-1\} \setminus J \neq \emptyset$, where k denotes the number of elements of J.

Special elliptic (*resp. special k-hyperbolic*) *points are elliptic* (*resp. hyperbolic*).

2.5. Special hyperbolic points. For S given by (1), let S_0 be the quadric of equation $w = Q(z)$.

LEMMA 2.2. *Suppose that S_0 is flat at 0 and that 0 is a special k-hyperbolic point. Then, in a neighborhood of 0, and with the above local coordinates, S_0 is CR and nowhere minimal outside 0, and the CR orbits of S_0 are $(2n-3)$-dimensional submanifolds given by $w = \text{const} \neq 0$.*

Proof. The submanifolds $w = \text{const} \neq 0$ have the same complex tangent space as S_0 and are of minimal dimension among submanifolds having this property, so they are CR orbits of codimension 1, and from the end of Section 2.1, S_0 is nowhere minimal outside 0.

The section $w = 0$ of S_0 is a real quadratic cone Σ_0' in $\mathbb{R}^{2n}$ whose vertex is 0 and, outside 0, it is a CR orbit Σ_0 in a neighborhood of 0. We will call Σ_0' a *singular CR orbit.* ∎

2.6. Foliation by CR orbits in a neighborhood of a special 1-hyperbolic point. We first imitate and transpose the beginning of the proof of Proposition 2.1, i.e. of 2.4.2 in [DTZ05, DTZ10].

2.6.1. *Local 2-codimensional submanifolds.* In order to use simple notation we will assume $n = 3$.

In $\mathbb{C}^3$, consider the 4-dimensional submanifold S locally defined by the equation

$$w = \varphi(z) = Q(z) + O(|z|^3), \tag{1}$$

and the 4-dimensional submanifold S_0 of equation

$$w = Q(z) \tag{4}$$

with

$$Q(z) = (\lambda_1 + 1)x_1^2 - (\lambda_1 - 1)y_1^2 + (1 + \lambda_2)x_2^2 + (1 - \lambda_2)y_2^2$$

having a special 1-hyperbolic point at 0 ($\lambda_1 > 1$, $0 \leq \lambda_2 < 1$), and the cone Σ_0' whose equation is $Q = 0$. On S_0, a CR orbit is a 3-dimensional submanifold $\mathcal{K}_{w_0}$ whose equation is $w_0 = Q(z)$. If $w_0 > 0$, then $\mathcal{K}_{w_0}$ does not meet the line $L = \{x_1 = x_2 = y_2 = 0\}$; if $w_0 < 0$, then $\mathcal{K}_{w_0}$ cuts L at two points.

LEMMA 2.3. *$\Sigma_0 = \Sigma_0' \setminus 0$ has two connected components in a neighborhood of 0.*

Proof. The equation of $\Sigma'_0 \cap \{y_1 = 0\}$ is $(\lambda_1 + 1)x_1^2 + (1 + \lambda_2)x_2^2 + (1 - \lambda_2)y_2^2 = 0$ whose only zero, in a neighborhood of 0, is 0. The two connected components are obtained for $y_1 > 0$ and $y_1 < 0$ respectively. ■

2.6.2. *CR orbits.* By differentiating (1), we get for the tangent spaces the asymptotics

$$T_{(z,\varphi(z))}S = T_{(z,Q(z))}S_0 + O(|z|^2), \qquad z \in \mathbb{C}^2. \tag{5}$$

Here both $T_{(z,\varphi(z))}S$ and $T_{(z,Q(z))}S_0$ depend continuously on z near the origin.

Consider

(i) the hyperboloid $H_- = \{Q = -1\}$, (then $Q(z/(-Q(z))^{1/2}) = -1$), and the projection

$$\pi_- : \mathbb{C}^3 \setminus \{z = 0\} \to H_-, \qquad (z, w) \mapsto z/(-Q(z))^{1/2},$$

(ii) for every $z \in H_-$, a real orthonormal basis $e_1(z), \ldots, e_6(z)$ of $\mathbb{C}^3 \cong \mathbb{R}^6$ such that

$$e_1(z), e_2(z) \in H_z H_-, \qquad e_3(z) \in T_z H_-,$$

where HH_- is the complex tangent bundle to H_-.

Locally such a basis can be chosen continuously depending on z. For every $(z, w) \in \mathbb{C}^3 \setminus \{z = 0\}$, consider the basis $e_1(\pi_-(z, w)), \ldots, e_6(\pi_-(z, w))$. The unit vectors $e_1(\pi_-(z, w_0)), e_2(\pi_-(z, w_0)), e_3(\pi_-(z, w_0))$ are tangent to the CR orbit $\mathcal{K}_{w_0}$ at (z, w_0) for $w_0 < 0$. Then, from (5), we have

$$H_{(z,\varphi(z))}S = H_{(z,Q(z))}S_0 + O(|z|^2), \qquad z \neq 0,\ z \to 0. \tag{6}$$

As in [DTZ10], in a neighborhood of 0, denote by $E(q)$, $q \in S \setminus \{0\}$, $w < 0$, the tangent space to the local CR orbit $\mathcal{K}$ on S through q, and by $E_0(q_0)$, $q_0 \in S_0 \setminus \{0\}$, $w < 0$, the analogous object for S_0. We have

$$E(z, \varphi(z)) = E_0(z, Q(z)) + O(|z|^2), \qquad z \neq 0,\ z \to 0. \tag{7}$$

Given $\underline{q} \in S$, by integration of $E(q)$, $q \in S$, we get, locally, the CR orbit (leaf) on S through $\underline{q}$; given $\underline{q}_0 \in S_0$, by integration of $E_0(q_0)$, $q_0 \in S_0$, we get, locally, the CR orbit (leaf) on S_0 through $\underline{q}_0$ (theorem of Sussmann). On S_0, a leaf is the 3-dimensional submanifold $\mathcal{K}_{\underline{q}_0} = \mathcal{K}_{w_0} = \mathcal{K}_0$ whose equation is $w_0 = Q(z)$, with $\underline{q} = (z_0, w_0 = Q(z_0))$. Moreover, $d\pi_-$ projects each $E_0(q)$, $q \in S_0, w < 0$, bijectively onto $T_{\pi(q)}H_-$, so $\pi_-|_{\mathcal{K}_0}$ is a diffeomorphism onto H_-; this implies, from (7), that, in a suitable neighborhood of the origin, the restriction of π_- to each local CR orbit of S is a local diffeomorphism.

We have $\varphi(z) = Q(z) + \Phi(z)$ with $\Phi(z) = O(|z|^3)$.

2.6.3. *Behavior of local CR orbits.* We follow the construction of $E(z, \varphi(z))$; compare with $E_0(z, Q(z))$. We know the integral manifold, the orbit of $E_0(z, Q(z))$; and we deduce an evaluation of the integral manifold $\mathcal{K}$ of $E(z, \varphi(z))$.

LEMMA 2.4. *Under the above hypotheses, the local orbit Σ corresponding to Σ_0 has two connected components in a neighborhood of* 0.

Proof. Using the real coordinates, as for Lemma 2.3, consider $\Sigma' \cap \{y_1 = 0\}$. Locally, the connected components are obtained for $y_1 > 0$ and $y_1 < 0$ respectively, from formula (1). ∎

We will call $\Sigma' = \overline{\Sigma}$ a *singular CR orbit* and a *singular leaf of the foliation.* We intend to prove that:

1) $\mathcal{K}$ does not cross the singular leaf through 0;
2) the only separatrix is the singular leaf through 0.

From the orbit $\mathcal{K}_0$, we will construct the differential equation defining it, and using (7), we will construct the differential equation defining $\mathcal{K}$.

In $\mathbb{C}^3$, we use the notation $x = x_1$, $y = y_1$, $u = x_2$, $v = y_2$; it suffices to consider the particular case $Q = 3x^2 - y^2 + u^2 + v^2$. On S_0, the orbit $\mathcal{K}_0$ issuing from the point $(c, 0, 0, 0)$ is defined by $3x^2 - y^2 + u^2 + v^2 = 3c^2$, i.e., for $x \geq 0$, $x = \frac{1}{\sqrt{3}}(y^2 - u^2 - v^2 + 3c^2)^{1/2} = A(y, u, v)$; the local coordinates on the orbit are (y, u, v). The orbit $\mathcal{K}_0$ satisfies the differential equation $dx = dA$. From (7), the orbit $\mathcal{K}$ issuing from $(c, 0, 0, 0)$ satisfies $dx = dA + \Psi$ with $\Psi(y, u, v; c) = O(|z|^2)$; hence $\Psi = d\Phi$, so $x = A + \Phi$ with $\Phi = O(|z|^3)$. More explicitly, $\mathcal{K}$ is defined by

$$x = x_{\mathcal{K},c} = \frac{1}{\sqrt{3}}(y^2 - u^2 - v^2 + 3c^2)^{1/2} + \Phi(y, u, v; c), \qquad \Phi(y, u, v; c) = O(|z|^3).$$

The cone Σ_0' whose equation is $Q = 0$ is a separatrix for the orbits $\mathcal{K}_0$. The corresponding object $\Sigma' = \{\varphi(z) = 0\}$ for S has the singular point 0 and for $x > 0, y > 0$, $u > 0, v > 0$ it is defined by the differential equation $dx = d(A + \Phi)$ with $c = 0$, i.e. the local equation of Σ' is

$$x = x_{\mathcal{K},0} = \frac{1}{\sqrt{3}}(y^2 - u^2 - v^2)^{1/2} + \Phi(y, u, v; 0), \qquad \Phi(y, u, v; 0) = O(|z|^3).$$

For given (y, u, v), $x_{\mathcal{K},c} - x_{\mathcal{K},0} = x_{\mathcal{K}_0,c} - x_{\mathcal{K}_0,0} + \Phi(y, u, v; c) - \Phi(y, u, v; 0)$. But $x_{\mathcal{K}_0,c} - x_{\mathcal{K}_0,0} = O(1)$ and $\Phi(y, u, v; c) - \Phi(y, u, v; 0) = O(|z|^3)$.

As a consequence, for $x > 0$, $y > 0$, $u > 0$, $v > 0$, locally, Σ' is the unique separatrix for the orbits $\mathcal{K}$. The same holds for $x < 0$.

What has been done for the hyperboloid $H_- = \{Q = -1\}$ can be repeated for the hyperboloid $H_+ = \{Q = 1\}$. As at the beginning of Section 2.6.2, we consider

(i) the hyperboloid $H_+ = \{Q = 1\}$ and the projection

$$\pi_+ : \mathbb{C}^3 \setminus \{z = 0\} \to H_+, \qquad (z, w) \mapsto z/(Q(z))^{1/2},$$

(ii) for every $z \in H_+$, a real orthonormal basis $e_1(z), \ldots, e_6(z)$ of $\mathbb{C}^3 \cong \mathbb{R}^6$ such that

$$e_1(z), e_2(z) \in H_z H_+, \qquad e_3(z) \in T_z H_+,$$

where HH_+ is the complex tangent bundle to H_+.

LEMMA 2.5. *Given φ, there exists $R > 0$ such that, in $B(0,R) \cap \{x > 0,\, y > 0,\, u > 0,\, v > 0\} \subset \mathbb{C}^2$, the CR orbits $\mathcal{K}$ have Σ' as a unique separatrix.*

Proof. When c tends to zero, $x_{\mathcal{K},c} - x_{\mathcal{K},0} = x_{\mathcal{K}_0,c} - x_{\mathcal{K}_0,0} + \Phi(y,u,v;c) - \Phi(y,u,v;0) = O(|z|)$ and $\Phi(y,u,v;c) - \Phi(y,u,v;0) = O(|z|^3)$. For $\varphi(z) = Q(z) + \Phi(z)$ with $\Phi(z) = O(|z|^3)$ given, in (7), $E(z,\varphi(z)) - E_0(z,Q(z)) = O(|z|^2)$ and $\Phi(y,u,v;c) - \Phi(y,u,v;0) = O(|z|^3)$ are also given. Then there exists R such that, for $|z| < R$, $x_{\mathcal{K},c} - x_{\mathcal{K},0} > 0$. ∎

2.7. CR orbits near a subvariety containing a special 1-hyperbolic point. In this section we will impose conditions on S and give a local property in a neighborhood of a compact $(2n-3)$-subvariety of S.

Assume that $S \subset \mathbb{C}^n$ $(n \geq 3)$ is a locally closed $(2n-2)$-submanifold, nowhere minimal at all its CR points, which has a unique 1-hyperbolic flat complex point p, and such that:

(i) if Σ is the orbit whose closure Σ' contains p, then Σ' is compact.

Let $q \in S$, $q \neq p$; then, in a neighborhood U of q not containing p, S is CR, CR-dim $S = n-2$, S is nonminimal and Σ is 1-codimensional. We show that the CR orbits constitute a foliation on S whose separatrix is Σ'. This is true in U since $\Sigma \cap U$ is a leaf. Moreover, let U_0 be the ball $B(0,R)$ centered at $p = 0$ as in Lemma 2.5; if $U \cap U_0 \neq \emptyset$, the leaves in U glue with the leaves in U_0 on $U \cap U_0$. Since Σ' is compact, there exist a finite number of points $q_j \in \Sigma'$, $j = 0, 1, \ldots, J$, and open neighborhoods U_j, as above, such that $(U_j)_{j=0}^J$ is an open covering of Σ'. Moreover the leaves in U_j glue respectively with the leaves in U_k if $U_j \cap U_k \neq \emptyset$.

PROPOSITION 2.6. *Assume that $S \subset \mathbb{C}^n$ $(n \geq 3)$ is a locally closed $(2n-2)$-submanifold, nowhere minimal at all its CR points, which has a unique special 1-hyperbolic flat complex point p, and such that:*

(i) *if Σ is the orbit whose closure Σ' contains p, then Σ' is compact;*
(ii) *Σ has two connected components σ_1, σ_2 whose closures are homeomorphic to spheres of dimension $2n-3$.*

Then there exists a neighborhood V of Σ' such that $V \setminus \Sigma'$ is foliated by compact real $(2n-3)$-dimensional CR orbits whose equation in a neighborhood of p is (3), and, the $w(= x_n)$-axis being assumed to be vertical, each orbit is diffeomorphic to either

- *the sphere $\mathbb{S}^{2n-3}$ above Σ', or*
- *the union of two spheres $\mathbb{S}^{2n-3}$ under Σ',*

and there exists a smooth function ν having the CR orbits as level surfaces.

Proof. This follows from the above and the following remark:

When x_n tends to 0, the orbits tend to Σ', and because of the geometry of the orbits near p, they are diffeomorphic to a sphere above Σ', and to the union of two spheres under Σ'. The existence of ν is proved as in Proposition 2.1, namely, consider a smooth curve $\gamma : [0,\varepsilon) \to S$ such that $\gamma(0) = q$, where q is a point of Σ close to p, and γ is a diffeomorphism onto its image $\Gamma = \gamma([0,\varepsilon))$. Let $\nu = \gamma^{-1}$ on the image of γ. Then, close enough to q, every CR orbit cuts Γ at a unique point $q(t)$, $t \in [0,\varepsilon)$. Hence there is a unique extension of ν from $\gamma([0,\varepsilon))$ to $V \setminus p$ where V is a neighborhood of Σ' having CR orbits as its level surfaces. As ν is smooth away from p, it is smooth on the orbit Σ and, if we set $\nu(p) = \nu(q) = 0$, ν is smooth on a neighborhood of $\Sigma \cup \{p\} = \Sigma'$. ∎

2.8. Geometry of the complex points of S. The results of Section 2.8 are particular cases of theorems of Lai [Lai72], which I learnt from F. Forstnerič in July 2011.

In [BK91] E. Bedford and W. Klingenberg cite the following theorem of E. Bishop [Bis65, Section 4, p. 15]: *On a 2-sphere embedded in $\mathbb{C}^2$, the difference between the numbers of elliptic points and of hyperbolic points is the Euler–Poincaré characteristic, i.e.* 2. For the proof, Bishop uses a theorem of [CS51, Section 4].

We extend this result to $n \geq 3$ and give proofs which are essentially similar to the proofs of the general case [Lai72, Lai74] but simpler.

Let S be a smooth compact connected oriented submanifold of dimension $2n-2$. Let G be the manifold of oriented real linear $(2n-2)$-subspaces of $\mathbb{C}^n$. The submanifold S of $\mathbb{C}^n$ has a given orientation which defines an orientation $o(p)$ of the tangent space to S at any point $p \in S$. By mapping each point of S to its oriented tangent space, we get a smooth Gauss map

$$t : S \to G.$$

Denote by $-t(p)$ the tangent space to S at p with the opposite orientation $-o(p)$.

Properties of G

(a) $\dim G = 2(2n-2)$.

Proof. G is a two-fold covering of the Grassmannian $M_{m,k}$ of linear k-subspaces of $\mathbb{R}^m$ [Ste99, Part I, Section 7.9], for $m = 2n$ and $k = 2n-2$; they have the same dimension. We have

$$M_{m,k} \cong O_m/O_k \times O_{m-k}.$$

But $\dim O_k = \frac{1}{2}k(k-1)$, hence

$$\dim M_{m,k} = \tfrac{1}{2}(m(m-1) - k(k-1) - (m-k)(m-k-1)) = k(m-k).$$

(b) G has the complex structure of a smooth quadric of complex dimension $2n-2$ in $\mathbb{C}P^{2n-1}$ [Lai74], [Pol08].

(c) There exists a canonical isomorphism $h : G \to \mathbb{C}P^{n-1} \times \mathbb{C}P^{n-1}$.

(d) Homology of G (cf. [Pol08]): Let S_1, S_2 be generators of $H_{2n-2}(G, \mathbb{Z})$; we assume that S_1 and S_2 are fundamental cycles of complex projective subspaces of complex dimension $n-1$ of the complex quadric G. We also denote S_1, S_2 the ordered two factors $\mathbb{C}P^{n-1}$, so that $h : G \to S_1 \times S_2$.

Proposition 2.7. *For $n \geq 2$, in general, S has isolated complex points.*

Proof. Let $\pi \in G$ be a complex hyperplane of $\mathbb{C}^n$ whose orientation is induced by its complex structure; the set of such π is $H = G^{\mathbb{C}}_{n-1,n} = \mathbb{C}P^{n-1*} \subset G$, as a real submanifold. If p is a complex point of S, then $t(p) \in H$ or $-t(p) \in H$. The set of complex points of S is the inverse image under t of the intersections $t(S) \cap H$ and $-t(S) \cap H$ in G. Since $\dim t(S) = 2n-2$, $\dim H = 2(n-1)$, $\dim G = 2(2n-2)$, it follows that the intersection is 0-dimensional in general. ∎

Denoting also by S the fundamental cycle of the submanifold S and by t_* the homomorphism defined by t, we have

$$t_*(S) \sim u_1 S_1 + u_2 S_2$$

where $\sim$ means "homologous to".

Lemma 2.8 (proved for $n = 2$ in [CS51]). *With the above notation, we have $u_1 = u_2$ and $u_1 + u_2 = \chi(S)$, the Euler–Poincaré characteristic of S.*

The proof for $n = 2$ works for any $n \geq 3$, namely:

Let G' be the manifold of oriented real linear 2-subspaces of $\mathbb{C}^n$. Let $\alpha : G \to G'$ map each oriented $2(n-1)$-subspace R to its normal 2-subspace R' oriented so that R, R' determine the orientation of $\mathbb{C}^n$. Then α is a canonical isomorphism. Let $n : S \to G'$ be the map defined by taking oriented normal planes. Then $n = \alpha t$ and $t = \alpha^{-1} n$, hence we have the mapping $h\alpha h^{-1} : S_1 \times S_2 \to S_1 \times S_2$. Let $(x, y) \in S_1 \times S_2$. Then

$$h\alpha h^{-1}(x, y) = (x, -y). \tag{†}$$

Over G, there is a bundle V of spheres with fiber over a real oriented linear $(2n-2)$-subspace of $\mathbb{C}^n$ through 0 being the unit sphere $\mathbb{S}^{2n-3}$ of this subspace. Let Ω be the characteristic class of V, and let Ω_t, Ω_n denote the characteristic classes of the tangent and normal bundles of S. Then $t^*\Omega = \Omega_t$, $n^*\Omega = \Omega_n$.

The bundle V is the Stiefel manifold of ordered pairs of orthogonal unit vectors through 0 in $\mathbb{R}^{2n} \cong \mathbb{C}^n$. Let $f : V \to G$ be the projection.

From the Gysin sequence, we see that the kernel of $f^* : H^{2n-2}(G) \to H^{2n-2}(V)$ is generated by Ω. To find the kernel of f^*, we determine the morphism $f_* : H_{2n-2}(V) \to H_{2n-2}(G)$. A generating $(2n-2)$-cycle of V is $\mathbb{S}^2 \times e$ where $\mathbb{S}^2 \cong \mathbb{C}P^{n-1}$ and e is a point. Let z be any point of S^2. Then from (†), we have

$$hf(z, e) = (z, -z).$$

Therefore, $f_*(S^2 \times e) = S_1 - S_2$. Thus, the kernel of f^* is $\mathbb{Z}$-generated by $S_1^* + S_2^*$.

With a convenient orientation for the fiber of the bundle V, we get $\Omega = S_1^* + S_2^*$. For a suitable orientation of S, we get $\Omega_t.S = \chi_S =$ Euler characteristic of S. We have

$$\Omega_t = t^*(S_1^* + S_2^*) = t^*S_1^* + t^*S_2^*,$$
$$\Omega_n = n^*(S_1^* + S_2^*) = t^*\alpha^*(S_1^* + S_2^*) = t^*(S_1^* - S_2^*) = t^*S_1^* - t^*S_2^*.$$

Since $\Omega_n = 0$, we get

$$(t^*S_1^*).S = (t^*S_2^*).S = \tfrac{1}{2}\chi_S.$$

Local intersection numbers of H and $t(S)$ when all complex points are flat and special. If H is a complex linear $(n-1)$-subspace of G, then it is homologous to one of the S_j, $j = 1, 2$, say S_2 when G has its structure of complex quadric. The intersection number of H and S_1 is 1 and the intersection number of H and S_2 is 0. So, the intersection number of H and $u_1S_1 + u_2S_2$ is u_1.

In a neighborhood of a complex point 0, the manifold S is defined by equation (1) with $w = z_n$ and

$$Q(z) = \sum_{j=1}^{n-1} \mu_j(z_j\overline{z}_j + \lambda_j \operatorname{Re} z_j^2), \qquad \mu_j > 0,\ \lambda_j \geq 0. \tag{1'}$$

Let $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \ldots, n$, with real x_l. Let e_l be the unit vector of the x_l-axis, $l = 1, \ldots, 2n$.

For simplicity assume $n = 3$: $Q(z) = \mu_1(z_1\overline{z}_1 + \lambda_1 \operatorname{Re} z_1^2) + \mu_2(z_2\overline{z}_2 + \lambda_2 \operatorname{Re} z_2^2)$, with $\mu_1 = \mu_2 = 1$. Then, up to higher order terms, S is defined by

$$z_1 = x_1 + ix_2, \qquad z_2 = x_3 + ix_4,$$
$$z_3 = (1+\lambda_1)x_1^2 + (1-\lambda_1)x_2^2 + (1+\lambda_2)x_3^2 + (1-\lambda_2)x_4^2.$$

In a neighborhood of 0, the tangent space to S is defined by the four linearly independent vectors

$$\nu_1 = e_1 + 2(1+\lambda_1)x_1e_5, \qquad \nu_2 = e_2 + 2(1-\lambda_1)x_2e_5,$$
$$\nu_3 = e_3 + 2(1+\lambda_2)x_3e_5, \qquad \nu_4 = e_4 + 2(1-\lambda_2)x_4e_5.$$

Thus, if 0 is special elliptic or special k-hyperbolic with k even, the tangent plane at 0 has the same orientation; if 0 is special elliptic or special k-hyperbolic with k odd, the tangent space has opposite orientation.

PROPOSITION 2.9 (known for $n = 2$ [Bis65], here for $n \geq 3$). *Let S be a smooth, oriented, compact, 2-codimensional, real submanifold of $\mathbb{C}^n$ all of whose complex points are flat and special elliptic or special 1-hyperbolic. Then, on S, $\sharp$(special elliptic points) $-$ $\sharp$(special 1-hyperbolic points) $= \chi(S)$. If S is a sphere, this number is 2.*

Proof. Let $p \in S$ be a complex point and π be the tangent hyperplane to S at p. Assume that

($**$) *the orientation of S induces, on π, the orientation given by its complex structure.*

Then $\pi \in H$.

If p is elliptic, the intersection number of H and $t(S)$ is 1; if p is 1-hyperbolic, the intersection number of H and $t(S)$ is -1 at p.

By the argument preceding ($1'$), the sum of the intersection numbers of H and $t(S)$ at complex points p satisfying ($**$) is u_1. Reversing the condition ($**$), and using Lemma 2.8, we get the proposition. ∎

3. Particular cases: horned sphere, elementary models and their gluings

3.1. We recall the following Harvey–Lawson theorem with a real parameter, to be used later.

Let $E \cong \mathbb{R} \times \mathbb{C}^n$, and $k : \mathbb{R} \times \mathbb{C}^n \to \mathbb{R}$ be the projection. Let $N \subset E$ be a compact (oriented) CR subvariety of $\mathbb{C}^{n+1}$ of real dimension $2n-2$ and CR dimension $n-2$ ($n \geq 3$), of class C^∞, with negligible singularities (i.e. there exists a closed subset $\tau \subset N$ of $(2n-2)$-dimensional Hausdorff measure 0 such that $N \setminus \tau$ is a CR submanifold). Let τ' be the set of all points $z \in N$ such that either $z \in \tau$ or $z \in N \setminus \tau$ and N is not transversal to the complex hyperplane $k^{-1}(k(z))$ at z. Assume that N, as a current of integration, is d-closed and satisfies:

(H) there exists a closed subset $L \subset \mathbb{R}_{x_1}$ with $H^1(L) = 0$ such that for every $x \in k(N) \setminus L$, the fiber $k^{-1}(x) \cap N$ is connected and does not intersect τ'.

THEOREM 3.1 ([DTZ10]; see also [DTZ05]). *Let N satisfy* (H) *with L chosen accordingly. Then there exists, in $E' = E \setminus k^{-1}(L)$, a unique C^∞ Levi-flat $(2n-1)$-subvariety M with negligible singularities in $E' \setminus N$, foliated by complex $(n-1)$-subvarieties, with the properties that M simply* (*or trivially*) *extends to E' as a $(2n-1)$-current* (*still denoted M*) *such that $dM = N$*

in E'. The leaves are the sections by the hyperplanes $E_{x_1^0}$, $x_1^0 \in k(N)\setminus L$, and are the solutions of the "Harvey–Lawson problem" of finding a holomorphic subvariety in $E_{x_1^0} \cong \mathbb{C}^n$ with prescribed boundary $N \cap E_{x_1^0}$.

REMARK 3.2. Theorem 3.1 is valid in the space $E \cap \{\alpha_1 < x_1 < \alpha_2\}$, with the corresponding condition (H). Moreover, since N is compact, for a suitable parameter x_1, we can assume $x_1 \in [0,1]$.

To solve the boundary problem by Levi-flat hypersurfaces, S has to satisfy necessary and sufficient local conditions. A way to prove that these conditions can occur is to construct an example for which the solution is obvious.

3.2. Sphere with one special 1-hyperbolic point (sphere with two horns): Example. In $\mathbb{C}^3$, let z_j, $j = 1,2,3$, be the complex coordinates and $z_j = x_j + iy_j$. In $\mathbb{R}^6 \cong \mathbb{C}^3$, consider the 4-dimensional subvariety (with negligible singularities) S defined by

$$\begin{aligned}
&y_3 = 0,\\
&0 \le x_3 \le 1,\\
&x_3(x_1^2 + y_1^2 + x_2^2 + y_2^2 + x_3^2 - 1)\\
&\qquad + (1 - x_3)(x_1^4 + y_1^4 + x_2^4 + y_2^4 + 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2) = 0,\\
&-1 \le x_3 \le 0,\\
&x_3 = x_1^4 + y_1^4 + x_2^4 + y_2^4 + 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2.
\end{aligned}$$

The singular set of S is the 3-dimensional section $x_3 = 0$ along which the tangent space is not everywhere (uniquely) defined. S being in the real hyperplane $\{y_3 = 0\}$, the complex tangent spaces to S are $\{x_3 = x^0\}$ for suitable x^0.

Since the tangent space to the hypersurface $f(x_1, y_1, x_2, y_2, x_3) = 0$ in $\mathbb{R}^5$ is

$$X_1 f'_{x_1} + Y_1 f'_{y_1} + X_2 f'_{x_2} + Y_2 f'_{y_2} + X_3 f'_{x_3} = 0,$$

the tangent space to S in the hyperplane $\{y_3 = 0\}$ is, for $x_3 \ge 0$,

$$\begin{aligned}
&2x_1[x_3 + 2(1 - x_3)(x_1^2 + 2)]X_1 + 2y_1[x_3 + 2(1 - x_3)(y_1^2 - 1)]Y_1\\
&\qquad + 2x_2[x_3 + (1 - x_3)(2x_2^2 + 1)]X_2 + 2y_2[x_3 + (1 - x_3)(2y_2^2 + 1)]Y_2\\
&\qquad\qquad + [(x_1^2 + y_1^2 + x_2^2 + y_2^2 + 3x_3^2 - 1)\\
&\qquad\qquad\quad - (x_1^4 + y_1^4 + x_2^4 + y_2^4 + 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2)]X_3 = 0;
\end{aligned}$$

and for $x_3 \le 0$,

$$4(x_1^2 + 2)x_1 X_1 + 4(y_1^2 - 1)y_1 Y_1 + 2(2x_2^2 + 1)x_2 X_2 + 2(2y_2^2 + 1)y_2 Y_2 - X_3 = 0.$$

The complex points of S are defined by the vanishing of the coefficients of X_j, $j = 1,2,3,4$, in the equations of the tangent spaces. For $0 \le x_3 \le 1$, this yields

$$\begin{aligned} x_1[x_3 + 2(1 - x_3)(x_1^2 + 2)] = 0, \\ y_1[x_3 + 2(1 - x_3)(y_1^2 - 1)] = 0, \\ x_2[x_3 + (1 - x_3)(2x_2^2 + 1)] = 0, \\ y_2[x_3 + (1 - x_3)(2y_2^2 + 1)] = 0. \end{aligned}$$

We have the solutions

$$\begin{aligned} h : x_j = 0,\ y_j = 0\ (j = 1, 2),\ x_3 = 0; \\ e_3 : x_j = 0,\ y_j = 0\ (j = 1, 2),\ x_3 = 1. \end{aligned}$$

For $x_3 \leq 0$, the vanishing of the coefficients yields

$$\begin{aligned} (x_1^2 + 2)x_1 = 0, \\ (y_1^2 - 1)y_1 = 0, \\ (2x_2^2 + 1)x_2 = 0, \\ (2y_2^2 + 1)y_2 = 0. \end{aligned}$$

We have the solutions

$$\begin{aligned} &h :\ x_j = 0,\ y_j = 0\ (j = 1, 2),\ x_3 = 0; \\ &e_1, e_2 :\ x_1 = 0,\ y_1 = \pm 1,\ x_2 = 0,\ y_2 = 0,\ x_3 = -1. \end{aligned}$$

Note that the tangent space to S at h is well defined. Moreover, the set S will be smoothed along its section by the hyperplane $\{x_3 = 0\}$ by a small deformation leaving h unchanged. *In the following, S will denote this smooth submanifold.*

Lemma 3.3. *The points e_1, e_2, e_3 are special elliptic; the point h is special 1-hyperbolic.*

Proof. Point e_3: Let $x_3' = 1 - x_3$, then the equation of S in a neighborhood of e_3 is

$$\begin{aligned} &(1 - x_3')(x_1^2 + y_1^2 + x_2^2 + y_2^2 + x_3'^2 - 2x_3') \\ &\qquad - x_3'(x_1^4 + y_1^4 + x_2^4 + y_2^4 + 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2) = 0, \text{ i.e.} \\ &2x_3' = x_1^2 + y_1^2 + x_2^2 + y_2^2 + O(|z|^3), \quad \text{or} \quad w = z\overline{z} + O(|z|^3), \end{aligned}$$

so e_3 is special elliptic.

Points e_1, e_2: Let $y_1' = y_1 \pm 1$, $x_3' = x_3 + 1$. Then the equation of S in a neighborhood of e_1, e_2 is

$$\begin{aligned} x_3' - 1 &= x_1^4 + (y_1' \mp 1)^4 + x_2^4 + y_2^4 + 4x_1^2 - 2(y_1' \mp 1)^2 + x_2^2 + y_2^2 \\ &= x_1^4 + y_1'^4 \mp 4y_1'^3 + 6y_1'^2 \mp 4y_1' + 1 + x_2^4 + y_2^4 \\ &\quad + 4x_1^2 - 2(y_1' \mp 1)^2 + x_2^2 + y_2^2, \end{aligned}$$

so

$$x_3' = x_1^4 + y_1'^4 \mp 4y_1'^3 + 4y_1'^2 + x_2^4 + y_2^4 + 4x_1^2 + x_2^2 + y_2^2, \quad \text{i.e.}$$
$$x_3' = 4x_1^2 + 4y_1'^2 + x_2^2 + y_2^2 + O(|z|^3), \quad \text{or} \quad w = 4z_1\overline{z}_1 + z_2\overline{z}_2.$$

Hence e_1, e_2 are special elliptic.

Point h: The equation of S in a neighborhood of h is for $x_3 \geq 0$,

$$x_3(x_1^2 + y_1^2 + x_2^2 + y_2^2 + x_3^2 - 1) + (1 - x_3)(x_1^4 + y_1^4 + x_2^4 + y_2^4 + 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2) = 0,$$

and for $x_3 \leq 0$,

$$x_3 = x_1^4 + y_1^4 + x_2^4 + y_2^4 + 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2,$$

i.e. $x_3 = 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2 + O(|z|^3)$ in both cases, up to third order terms. Hence $w = z_1\overline{z}_1 + z_2\overline{z}_2 + 3\operatorname{Re} z_1^2$, so h is special 1-hyperbolic. ∎

The section $\Sigma' = S \cap \{x_3 = 0\}$. Up to a small smooth deformation, its equation is

$$x_1^4 + y_1^4 + x_2^4 + y_2^4 + 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2 = 0 \quad \text{in } \{x_3 = 0\}.$$

The tangent cone to Σ' at 0 is $4x_1^2 - 2y_1^2 + x_2^2 + y_2^2 = 0$. Locally, the section of S by the coordinate 3-space x_1, y_1, x_3 is

$$x_3 = 4x_1^2 - 2y_1^2 + O(|z|^3),$$

and the section by the x_2, y_2, x_3-space is $x_3 = x_2^2 + y_2^2 + O(|z|^3)$.

Lemma 3.4. *Under the above hypotheses and notation:*

(i) *$\Sigma = \Sigma' \setminus 0$ has two connected components σ_1, σ_2.*

(ii) *The closures of the three connected components of $S \setminus \Sigma'$ are submanifolds with boundaries and corners.*

Proof. (i) The only singular point of Σ' is 0. We work in the ball $B(0, A)$ of $\mathbb{C}^2_{x_1,y_1,x_2,y_2}$ for small A and in the 3-space $\pi_\lambda = \{y_2 = \lambda x_2\}$, $\lambda \in \mathbb{R}$. For λ fixed, $\pi_\lambda \cong \mathbb{R}^3_{x_1,y_1,x_2}$, and $\Sigma' \cap \pi_\lambda$ is the cone of equation $4x_1^2 - 2y_1^2 + (1 + \lambda^2)x_2^2 + O(|z|^3) = 0$ with vertex 0 and basis the hyperboloid H_λ of equation $4x_1^2 - 2y_1^2 + (1 + \lambda^2)x_2^{02} + O(|z|^3) = 0$ in the plane $x_2 = x_2^0$; the curves H_λ have no common point outside 0. So, when λ varies, the surfaces $\Sigma' \cap \pi_\lambda$ are disjoint outside 0. The set Σ' is clearly connected; $\Sigma' \cap \{y_1 = 0\} = \{0\}$, the origin of $\mathbb{C}^3$; by the above, $\sigma_1 = \Sigma \cap \{y_1 > 0\}$, and $\sigma_2 = \Sigma \cap \{y_1 < 0\}$.

(ii) The three connected components of $S \setminus \Sigma'$ contain, respectively, e_1, e_2, e_3 and their boundaries are $\overline{\sigma}_1$, $\overline{\sigma}_2$, $\overline{\sigma}_1 \cup \overline{\sigma}_2$; these boundaries have corners as shown in the first part of the proof. ∎

The connected component of $\mathbb{C}^2 \times \mathbb{R} \setminus S$ containing $(0, 0, 0, 0, 1/2)$ is the Levi-flat solution, the complex leaves being the sections by the hyperplanes $x_3 = x_3^0$, $-1 < x_3^0 < 1$.

The section by the hyperplanes $x_3 = x_3^0$ is diffeomorphic to a 3-sphere for $0 < x_3^0 < 1$ and to the union of two disjoint 3-spheres for $-1 < x_3^0 < 0$, as can be shown intersecting S by lines through the origin in the hyperplane $x_3 = x_3^0$; Σ' is homeomorphic to the union of two 3-spheres with a common point.

3.3. Sphere with one special 1-hyperbolic point (sphere with two horns), general case. The example of Section 3.2 shows that the necessary conditions of Section 2 can be realised. Moreover, from Proposition 2.9, the hypothesis on the number of complex points is meaningful.

PROPOSITION 3.5 (cf. [Dol08, Proposition 2.6.1]). *Let $S \subset \mathbb{C}^n$ be a compact connected real* 2*-codimensional manifold such that the following holds:*

- (i) *S is a topological sphere, nonminimal at every CR point;*
- (ii) *every complex point of S is flat; there exist three special elliptic points e_j, $j = 1, 2, 3$, and one special* 1*-hyperbolic point h;*
- (iii) *S does not contain complex manifolds of dimension $n - 2$;*
- (iv) *the singular CR orbit Σ' through h on S is compact and $\Sigma' \setminus \{h\}$ has two connected components σ_1 and σ_2 whose closures are homeomorphic to spheres of dimension $2n - 3$;*
- (v) *the closures S_1, S_2, S_3 of the three connected components S_1', S_2', S_3' of $S \setminus \Sigma'$ are submanifolds with* (*singular*) *boundary.*

Then each $S_j \setminus (e_j \cup \Sigma')$, $j = 1, 2, 3$, carries a foliation $\mathcal{F}_j$ of class C^∞ with 1*-codimensional CR orbits as compact leaves.*

Proof. From conditions (i) and (ii), S satisfies the hypotheses of Proposition 2.1 near any elliptic flat point e_j, and of Proposition 2.6 near Σ', all CR orbits being diffeomorphic to the sphere $\mathbb{S}^{2n-3}$. Assumption (iii) guarantees that all CR orbits in S must be of real dimension $2n-3$. Hence, by removing small connected open saturated neighborhoods of all special elliptic points, and of Σ', we obtain, from $S \setminus \Sigma'$, three compact manifolds S_j'', $j = 1, 2, 3$, with boundary and with the foliation $\mathcal{F}_j$ of codimension 1 given by its CR orbits, near e_j; the first cohomology group with values in $\mathbb{R}$ of these orbits is 0. It is easy to show that this foliation is transversely oriented. ∎

Recall Thurston's Stability Theorem ([CaC, Theorem 6.2.1].

PROPOSITION 3.6. *Let $(M, \mathcal{F})$ be a compact, connected, transversely orientable, foliated manifold with boundary or corners, of codimension* 1*, of class C^1. If there is a compact leaf L with $H^1(L, \mathbb{R}) = 0$, then every leaf is homeomorphic to L, and M is homeomorphic to $L \times [0, 1]$, foliated as a product.*

From the above theorem, S_j'' is homeomorphic to $\mathbb{S}^{2n-3} \times [0, 1]$ with CR orbits being of the form $\mathbb{S}^{2n-3} \times \{x\}$ for $x \in [0, 1]$. Then the full manifold

S_j is homeomorphic to a half-sphere supported by $\mathbb{S}^{2n-2}$ and $\mathcal{F}_j$ extends to S_j, with S_3 having its boundary pinched at the point h.

THEOREM 3.7. *Let $S \subset \mathbb{C}^n$, $n \geq 3$, be a compact connected smooth real* 2-*codimensional submanifold satisfying conditions* (i) *to* (v) *of Proposition* 3.5. *Then there exists a Levi-flat $(2n-1)$-subvariety $\tilde{M} \subset \mathbb{C} \times \mathbb{C}^n$ with boundary $\tilde{S}$ (in the sense of currents) such that the natural projection $\pi : \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$ restricts to a bijection which is a CR diffeomorphism between $\tilde{S}$ and S outside the complex points of S.*

Proof. By Proposition 2.1, for every e_j, a continuous function ν_j', C^∞ outside e_j, can be constructed in a neighborhood U_j of e_j, $j = 1, 2, 3$, and by Proposition 2.6, we have an analogous result in a neighborhood of Σ'. Furthermore, from Proposition 3.6, a smooth function ν_j'' whose level sets are leaves of $\mathcal{F}_j$ can be obtained globally on $S_j' \setminus (e_j \cup \Sigma')$. With the functions ν_j' and ν_j'', and analogous functions near Σ', using a partition of unity, we obtain a global smooth function $\nu_j \colon S_j \to \mathbb{R}$ without critical points away from the complex points e_j and from Σ'.

Let σ_1, resp. σ_2 be the two connected, relatively compact components of $\Sigma \setminus \{h\}$, according to condition (iv); $\overline{\sigma}_1$, resp. $\overline{\sigma}_2$ is the boundary of S_1, resp. S_2, and $\overline{\sigma}_1 \cup \overline{\sigma}_2$ is the boundary of S_3. We can assume that the three functions ν_j are finite-valued and get the same values on $\overline{\sigma}_1$ and $\overline{\sigma}_2$. Then the functions ν_j are induced by a unique function $\nu : S \to \mathbb{R}$.

The submanifold S, being locally the boundary of a Levi-flat hypersurface, is orientable. We now set $\tilde{S} = N = \mathrm{graph}(\nu) = \{(\nu(z), z) : z \in S\}$. Let $S_s = \{e_1, e_2, e_3, \overline{\sigma_1 \cup \sigma_2}\}$.

The map $\lambda : S \to \tilde{S}$ $(z \mapsto (\nu(z), z))$ is bicontinuous; $\lambda|_{S \setminus S_s}$ is a diffeomorphism; moreover λ is a CR map. Choose an orientation on S. Then N is an (oriented) CR subvariety with the negligible set of singularities $\tau = \lambda(S_s)$.

At every point of $S \setminus S_s$, $d_{x_1}\nu \neq 0$, so condition (H) (Section 3.1) is satisfied at every point of $N \setminus \tau$.

All the assumptions of Theorem 3.1 being satisfied by $N = \tilde{S}$, in a particular case, we conclude that N is the boundary of a Levi-flat $(2n-2)$-variety (with negligible singularities) $\tilde{M}$ in $\mathbb{R} \times \mathbb{C}^n$.

Taking $\pi : \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$ to be the standard projection, we obtain the conclusion. ∎

3.4. Generalizations: elementary models and their gluings. The examples and the proofs of the theorems when S is homeomorphic to a sphere (Section 3.3) suggest the following definitions.

Let T' be a smooth, locally closed (i.e. closed in an open set), connected submanifold of $\mathbb{C}^n$, $n \geq 3$. We assume that T' has the following properties:

(i) T' is relatively compact, not necessarily compact, and of codimension 2.
(ii) T' is nonminimal at every CR point.
(iii) T' does not contain complex manifolds of dimension $n-2$.
(iv) T' has exactly two complex points which are flat and either special elliptic or special 1-hyperbolic.
(v) If $p \in T'$ is special 1-hyperbolic, then the singular orbit Σ' through p is compact, and $\Sigma' \backslash p$ has two connected components σ_1, σ_2 whose closures are homeomorphic to spheres of dimension $2n-3$.
(vi) If $p \in T'$ is special 1-hyperbolic, then in a neighborhood of p with convenient coordinates, the equation of T' up to third order terms is

$$z_n = \sum_{j=1}^{n-1} (z_j \overline{z}_j + \lambda_j \operatorname{Re} z_j^2), \qquad \lambda_1 > 1,\ 0 \leq \lambda_j < 1 \text{ for } j \neq 1,$$

or in real coordinates x_j, y_j with $z_j = x_j + iy_j$,

$$x_n = \big((\lambda_1{+}1)x_1^2 - (\lambda_1{-}1)y_1^2\big) + \sum_{j=2}^{n-1} \big((1{+}\lambda_j)x_j^2 + (1{-}\lambda_j)y_j^2\big) + O(|z|^3).$$

(vii) The closures (in T') T_1, T_2, T_3 of the three connected components T_1', T_2', T_3' of $T' \setminus \Sigma'$ are submanifolds with (singular) boundary. Let T_j'', $j = 1, 2, 3$, be a neighborhood of T_j' in T'.

Up- and down-1-hyperbolic points. Let τ be the $(2n-2)$-submanifold with (singular) boundary contained into T' such that either $\overline{\sigma}_1$ (resp. $\overline{\sigma}_2$) or Σ' is the boundary of τ near p. In the first case, we say that p is 1-*up* (resp. 2-*up*), in the second it is *down*. If T' is contained in a small enough neighborhood of Σ' in $\mathbb{C}^n$, such a T' will be called a *local elementary model*, more precisely it defines a *germ of elementary model around Σ*.

The union T of T_1, T_2, T_3 and of the germ of elementary model around the singular orbit at every special 1-hyperbolic point is called an *elementary model*. It behaves as a locally closed submanifold still denoted T.

Examples of elementary models. We will say that T is an *elementary model of type*:

(a) if it has two elliptic points;
(b) if it has one special elliptic point and one down-1-hyperbolic point;
(c_1) if it has one special elliptic point and one 1-up-1-hyperbolic point;
(c_2) if it has one special elliptic point and one 2-up-1-hyperbolic point;
(d_1) if it has two special 1-up-1-hyperbolic points;
(d_2) if it has two special 2-up-1-hyperbolic points;
(e) if it has two special down-1-hyperbolic points;

Other configurations can be easily imagined.

The prescribed boundary of a Levi-flat hypersurface of $\mathbb{C}^n$ in [DTZ05] and [DTZ10], whose complex points are flat and elliptic, is an elementary model of type (a).

Properties of elementary models. For instance, if T is 1-up and has one special elliptic point, we solve the boundary problem as in S_1 in the proof of Theorem 3.7.

PROPOSITION 3.8. *Let T be a local elementary model. Then T carries a foliation $\mathcal{F}$ of class C^∞ with 1-codimensional CR orbits as compact leaves.*

Proof. From the definition and Proposition 2.6. ∎

THEOREM 3.9. *Let T be an elementary model. There exists an open neighborhood T'' in T' carrying a smooth function $\nu : T'' \to \mathbb{R}$ whose level sets are leaves of a smooth foliation.*

Proof. By removing small connected open saturated neighborhoods of every special elliptic point, and of Σ', the singular orbit through every special 1-hyperbolic point p, we obtain, from $T \setminus \Sigma'$, three manifolds T_j'', $j = 1, 2, 3$, with boundary:

- T_1 and T_2 containing one special elliptic point e or one special 1-hyperbolic point with the foliations $\mathcal{F}_1$, $\mathcal{F}_2$, from Propositions 2.1 and 3.8,
- T_3'' with the foliation $\mathcal{F}_3$ of codimension 1 given by its CR orbits whose first cohomology group with values in $\mathbb{R}$ is 0, near e, or p. It is easy to show that this later foliation is transversely oriented.

From Thurston's Stability Theorem (Proposition 3.6), T_3'' is homeomorphic to $\mathbb{S}^{2n-3} \times [0, 1]$, foliated as a product, with CR orbits being of the form $\mathbb{S}^{2n-3} \times \{x\}$ for $x \in [0, 1]$; hence we obtain smooth functions ν_1, ν_2, ν_3 whose level sets are leaves of the foliations $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ respectively, and using a partition of unity we get the desired function ν on T. ∎

THEOREM 3.10. *Let T be an elementary model. Then there exists a Levi-flat $(2n-1)$-subvariety $\tilde{M} \subset \mathbb{C} \times \mathbb{C}^n$ with boundary $\tilde{T}$ (in the sense of currents) such that the natural projection $\pi : \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$ restricts to a bijection which is a CR diffeomorphism between $\tilde{T}$ and T outside the complex points of T.*

Proof. The submanifold T, being locally the boundary of a Levi-flat hypersurface, is orientable. We now set $\tilde{T} = N = \mathrm{graph}(\nu) = \{(\nu(z), z) : z \in S\} \subset E \cong \mathbb{R} \times \mathbb{C}^{n-1}$. Let T_s be the set of all flat complex points of T.

The map $\lambda : T \to \tilde{T}$ $(z \mapsto (\nu(z), z))$ is bicontinuous; $\lambda|_{T \setminus T_s}$ is a diffeomorphism; moreover λ is a CR map. Choose an orientation on T. Then N is an (oriented) CR subvariety with the negligible set of singularities $\tau = \lambda(T_s)$.

Using Remark 3.2, at every point of $T \setminus T_s$, $d_{x_1}\nu \neq 0$, so condition (H) (Section 3.1) is satisfied at every point of $N \setminus \tau$.

All the assumptions of Theorem 3.1 being satisfied by $N = \tilde{T}$, in a particular case, we conclude that N is the boundary of a Levi-flat $(2n-2)$-variety (with negligible singularities) $\tilde{M}$ in $\mathbb{R} \times \mathbb{C}^n$.

Taking $\pi : \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$ to be the standard projection, we obtain the conclusion. ■

3.5. Gluing of elementary models. The gluing happens between two compatible elementary models along boundaries, for instance down and 1-up. Note that the gluing can only be made at special 1-hyperbolic points. More precisely, it can be defined as follows.

The properties of the submanifold S of $\mathbb{C}^n$ assumed in Section 2 have a meaning in any complex analytic manifold X of complex dimension $n \geq 3$, and are kept under any holomorphic isomorphism.

We will define a submanifold S' of X obtained by gluing of elementary models by induction on the number m of models. An elementary model T in X is the image of an elementary model T_0 in $\mathbb{C}^n$ under an analytic isomorphism of a neighborhood of T_0 in $\mathbb{C}^n$ into X.

Let S' be a closed smooth real submanifold of X of dimension $2n-2$ which is nonminimal at every CR point. Assume that S' is obtained by gluing m elementary models. Then S' has the following properties:

- S' has a finite number of flat complex points, some special elliptic and the others special 1-hyperbolic;
- for every special 1-hyperbolic p', there exists a CR-isomorphism h induced by a holomorphic isomorphism of the ambient space $\mathbb{C}^n$ from a neighborhood of p in T' onto a neighborhood of p' in S';
- for every CR orbit $\Sigma_{p'}$ whose closure contains a special 1-hyperbolic point p', there exists a CR-isomorphism h induced by a holomorphic isomorphism of the ambient space $\mathbb{C}^n$ from a neighborhood of $\Sigma_p = \Sigma'_p \setminus p$ in T' onto a neighborhood V of $\Sigma_{p'}$ in S'.

Every special 1-hyperbolic point of S' which belongs to only one elementary model in S' will be called *free.*

We will define the gluing of one more elementary model to S'.

Gluing an elementary model T of type (d_1) to a free down-1-hyperbolic point of S'. Let h_1 be a CR-isomorphism from a neighborhood V_1 of $\overline{\sigma}'_1$ induced by a holomorphic isomorphism of the ambient space $\mathbb{C}^n$ onto a neighborhood of σ_1 in S'. Let k_1 be a CR-isomorphism from a neighborhood T''_1 of T'_1 into X such that $k_1|_{V_1} = h_1$.

Theorem 3.11. *The compact manifold or the manifold with singular boundary S', obtained by the gluing of a finite number of elementary models, is the boundary of a Levi-flat hypersurface of X in the sense of currents.*

Proof. From Theorem 3.10 and the definition of gluing. ■

3.6. Examples of gluing. Denoting the gluing of two models of type (d_1) and (d_2) to a free down-1-hyperbolic point of S' by: $\to (d_1) - (d_2)$, and the converse by: $(d_1) - (d_2) \to$, and, also, analogous configurations in the same way, we get:

- torus: $(b) \to (d_1) - (d_2) \to (b)$; the Euler–Poincaré characteristic of a torus is $\chi(\mathbf{T}^k) = 0$; two special elliptic and two special 1-hyperbolic points;
- bitorus: $(b) \to (d_1) - (d_2) \to (e) \to (d_1) - (d_2) \to (b)$.

4. Case of graphs (see [DTZ11] for the case of elliptic points only, and dropping the property of the function solution to be Lipschitz).

4.1. We want to add the following hypothesis: S is embedded into the boundary of a strictly pseudoconvex domain of $\mathbb{C}^n$, $n \geq 3$, and more precisely, let (z, w) be the coordinates in $\mathbb{C}^{n-1} \times \mathbb{C}$, with $z = (z_1, \ldots, z_{n-1}), w = u + iv = z_n$, let Ω be a strictly pseudoconvex domain in $\mathbb{C}^{n-1} \times \mathbb{R}_u$ (i.e. the second fundamental form of the boundary $b\Omega$ of Ω is everywhere positive definite); let S be the graph graph(g) of a smooth function $g : b\Omega \to \mathbb{R}_v$. Notice that $b\Omega \times \mathbb{R}_v$ contains S and is strictly pseudoconvex.

Assume that S is a *horned sphere* (Section 3.3), *satisfying the hypotheses of Theorem* 3.7. Denote by p_j, $j = 1, \ldots, 4$, the complex points of S.

4.2. Our aim is to prove

Theorem 4.1. *Let S be the graph of a smooth function $g : b\Omega \to \mathbb{R}_v$. Let $Q = (q_1, \ldots, q_4) \in b\Omega$ be the projections of the complex points $P = (p_1, \ldots, p_4)$ of S, respectively. Then there exists a continuous function $f : \overline{\Omega} \to \mathbb{R}_v$ which is smooth on $\overline{\Omega} \setminus Q$ and such that $f_{|b\Omega} = g$, and $M_0 =$ graph$(f)\setminus S$ is a smooth Levi-flat hypersurface of $\mathbb{C}^n$. Moreover, each complex leaf of M_0 is the graph of a holomorphic function $\phi : \Omega' \to \mathbb{C}$ where $\Omega' \subset \mathbb{C}^{n-1}$ is a domain with smooth boundary* (*that depends on the leaf*) *and ϕ is smooth on $\overline{\Omega'}$.*

The natural candidate to be the graph M of f is $\pi(\tilde{M})$ where $\tilde{M}$ and π are as in Theorem 3.7. We prove that this is the case, proceeding in several steps.

4.3. Behavior near S. Assume that D is a strictly pseudoconvex domain such that $S \subset bD$.

Recall ([HL75, Theorem 10.4]): *Let D be a strictly pseudoconvex domain in $\mathbb{C}^n$, $n \geq 3$, with boundary bD, and let $\Sigma \subset bD$ be a compact connected maximally complex smooth $(2d-1)$-submanifold with $d \geq 2$. Then Σ is the boundary of a uniquely determined relatively compact subset $V \subset \overline{D}$ such that $\overline{V} \setminus \Sigma$ is a complex analytic subset of D with finitely many singularities of pure dimension $\leq d-1$, and near Σ, $\overline{V}$ is a d-dimensional complex manifold with boundary.*

V is said to be *the solution of the boundary problem for Σ.*

LEMMA 4.2 ([DTZ11]). *Let Σ_1, Σ_2 be compact connected maximally complex $(2d-1)$-submanifolds of bD. Let V_1, V_2 be the corresponding solutions of the boundary problem. If $d \geq 2$, $2d \geq n+1$ and $\Sigma_1 \cap \Sigma_2 = \emptyset$, then $V_1 \cap V_2 = \emptyset$.*

Let Σ be a CR orbit of the foliation of $S \setminus P$. Then Σ is a compact maximally complex $(2n-3)$-dimensional real submanifold of $\mathbb{C}^n$ contained in bD. Let $V = V_\Sigma$ be the solution of the boundary problem corresponding to Σ. From Theorem 3.7, $V = \pi(\tilde{V})$, where $\tilde{V} = (\tilde{M} \setminus \tilde{S}) \cap (\mathbb{C}^n \times \{x\})$ for suitable $x \in (0,1)$ (the projection on the x-axis being finite, we can always assume that x lies in $(0,1)$). Moreover $\pi_{|\tilde{V}}$ is a biholomorphism $\tilde{V} \cong V$ and $M \setminus S \subset D$.

Let Σ_1, Σ_2 be two distinct orbits of the foliation of $S \setminus P$, and $\overline{V}_1, \overline{V}_2$ the corresponding leaves. Then, from Lemma 4.2, $\overline{V}_1 \cap \overline{V}_2 = \emptyset$.

Assume that S satisfies the full hypotheses of Theorem 4.1. Set $m_1 = \min_S g$, $m_2 = \max_S g$ and pick $r \gg 0$ such that

$$D = \Omega \times [m_1, m_2] \subset\subset \mathbf{B}(r) \cap (\Omega \times i\mathbb{R}_v)$$

where $\mathbf{B}(r)$ is the ball $\{|(z,w)| < r\}$.

LEMMA 4.3. *Let $p \in S$ be a CR point. Then, near p, M is the graph of a function ϕ on a domain $U \subset \mathbb{C}_z^{n-1} \times \mathbb{R}_u$ which is smooth up to the boundary of U.*

Proof. Near p, each CR orbit Σ is smooth and can be represented as the graph of a CR function over a strictly pseudoconvex hypersurface and V_Σ as the graph of the local holomorphic extension of this function. From the Hopf lemma, V is transversal to the strictly pseudoconvex hypersurface $d\Omega \times i\mathbb{R}_v$ near p. Hence the family of the V_Σ, near p, forms a smooth real hypersurface with boundary on S that is the graph of a smooth function ϕ from a relatively open neighborhood U of p on $\overline{\Omega}$ into $\mathbb{R}_v$. Finally, Lemma 4.2 guarantees that this family does not intersect any other leaf V from M. ∎

COROLLARY 4.4. *If $p \in S$ is a CR point, then each complex leaf V of M, near p, is the graph of a holomorphic function on a domain $\Omega_V \subset \mathbb{C}_z^{n-1}$, which is smooth up to the boundary of Ω_V.*

4.4. Solution as the graph of a continuous function. We recall some results of Shcherbina [Sh93].

His Main Theorem is the following:

Let G be a bounded strictly convex domain in $\mathbb{C}_z \times \mathbb{R}_u$ ($z \in \mathbb{C}$) and $\varphi : bG \to \mathbb{R}_v$ be a continuous function. Then the following properties hold, where $\Gamma =$ graph, and $\hat{\Gamma}(\varphi)$ means the polynomial hull of $\Gamma(\varphi)$:

- (a_i) *$\hat{\Gamma}(\varphi) \backslash \Gamma(\varphi)$ is the union of a disjoint family $\{D_\alpha\}$ of complex discs;*
- (a_{ii}) *for each α, there is a simply connected domain $\Omega_\alpha \subset \mathbb{C}_z$ and a holomorphic function $w = f_\alpha$, defined on Ω_α, such that D_α is the graph of f_α;*
- (a_{iii}) *for each f_α, there exists an extension $f_\alpha^* \in C(\overline{\Omega}_\alpha)$ and $bD_\alpha = \{(z, w) \in b\Omega_\alpha \times \mathbb{C}_w : w = f_\alpha^*(z)\}$.*

LEMMA 4.5 ([Sh93]). *Let $\{G_n\}_{n=0}^\infty$ be a sequence of bounded strictly convex domains $G_n \subset \mathbb{C}_z \times \mathbb{R}_u$ such that $G_n \to G_0$. Let $\{\varphi_n\}_{n=0}^\infty$ be a sequence of continuous functions $\varphi_n : bG_n \to \mathbb{R}_v$ such that $\Gamma(\varphi_n) \to \Gamma(\varphi_0)$ in the Hausdorff metric. Then, if Φ_n is the continuous function $\overline{G}_n \to \mathbb{R}_v$ such that $\hat{\Gamma}(\varphi_n) = \Gamma(\Phi_n)$, we have $\Gamma(\Phi_n) \to \Gamma(\Phi_0)$ in the Hausdorff metric.*

LEMMA 4.6 ([Sh93]). *Let $\mathcal{U}$ be a smooth connected surface which is properly embedded into some convex domain $G \subset \mathbb{C}_z \times \mathbb{R}_u$. Suppose that near each of its points, u can be defined locally by the equation $u = u(z)$. Then the surface $\mathcal{U}$ can be represented globally as the graph of some function $u = U(z)$, defined on some domain $\Omega \subset \mathbb{C}_z$.*

PROPOSITION 4.7. *M is the graph of a continuous function $f : \overline{\Omega} \to \mathbb{R}_v$.*

Proof. We will intersect the graph S with a convenient affine subspace of real dimension 4 to go back to the situation studied by Shcherbina.

Fix $a \in \mathbb{C}_z^{n-1} \setminus 0$ and, for a given point $(\zeta, \xi) \in \Omega$ with $\zeta \in \mathbb{C}_z^{n-1}$ and $\xi \in \mathbb{R}_u$, let $H_{(\zeta,\xi)} \subset \mathbb{C}_z^{n-1} \times \{\xi\}$ be the complex line through (ζ, ξ) in the direction $(a, 0)$. Set

$$L_{(\zeta,\xi)} = H_{(\zeta,\xi)} + \mathbb{R}_u(0,1), \qquad \Omega_{(\zeta,\xi)} = L_{(\zeta,\xi)} \cap \Omega,$$
$$S_{(\zeta,\xi)} = (H_{(\zeta,\xi)} + \mathbb{C}_w(0,1)) \cap S.$$

Then $S_{(\zeta,\xi)}$ is contained in the strictly convex cylinder

$$(H_{(\zeta,\xi)} + \mathbb{C}_w(0,1)) \cap (b\Omega \times i\mathbb{R}_v)$$

and is the graph of $g_{|b\Omega_{(\zeta,\xi)}}$.

From (a_{ii}), the polynomial hull of $S_{(\zeta,\xi)}$ is a continuous graph over $\overline{\Omega}_{(\zeta,\xi)}$. Consider $M = \pi(\tilde{M})$ and set

$$M_{\zeta,\xi)} = (H_{(\zeta,\xi)} + \mathbb{C}_w(0,1)) \cap M.$$

It follows that $M_{\zeta,\xi)}$ is contained in the polynomial hull $\hat{S}_{(\zeta,\xi)}$. From (a_{iii}), $\hat{S}_{(\zeta,\xi)}$ is a graph over $\overline{\Omega}_{(\zeta,\xi)}$ foliated by analytic discs, so $M_{\zeta,\xi)}$ is a graph over a subset U of $\overline{\Omega}_{(\zeta,\xi)}$.

Every analytic disc Δ of $\hat{S}_{(\zeta,\xi)}$ has its boundary on $S_{(\zeta,\xi)}$. Since all the complex points of S are isolated, $b\Delta$ contains a CR point p of S; from Lemma 4.3, near p, $M_{\zeta,\xi)}$ is a graph over $\overline{\Omega}_{(\zeta,\xi)}$. Near p, Δ is contained in $M_{\zeta,\xi)}$, hence in a closed complex analytic leaf V_Σ of M; so $\Delta \subset V_\Sigma \subset M$; but $\Delta \subset H_{(\zeta,\xi)} + \mathbb{C}_w(0,1)$, so $\Delta \subset M_{\zeta,\xi)}$. Consequently, $M_{\zeta,\xi)} = \hat{S}_{(\zeta,\xi)}$ near p.

It follows that M is the graph of a function $f : \overline{\Omega} \to \mathbb{R}_v$.

One proves, using Lemma 4.5, that f is continuous on Ω, whence on $\overline{\Omega} \setminus Q$, by Lemma 4.3. Then continuity at every q_j is proved using the Kontinuitätssatz on the domain of holomorphy $\Omega \times i\mathbb{R}_v$. ∎

4.5. Regularity. The property that $M \setminus P = (p_1, \ldots, p_4)$ *is a smooth manifold with boundary* results from:

LEMMA 4.8. *Let U be a domain in $\mathbb{C}_z^{n-i} \times \mathbb{R}_u$, $n \geq 2$, and $f : U \to \mathbb{R}_v$ a continuous function. Let $A \subset \operatorname{graph}(f)$ be a germ of complex analytic set of codimension 1. Then A is a germ of complex manifold which is a graph over $\mathbb{C}_z^{n-i}$.*

Proof. Assume that A is a germ at 0. Let $h \in \mathcal{O}_{n+1}$, $h \not\equiv 0$, be such that $A = \{h = 0\}$. For $\varepsilon \ll 1$, let $\mathbf{D}_\varepsilon$ be the disc $\{z = 0\} \cap \{|w| < \varepsilon\}$. Then $A \cap \mathbf{D}_\varepsilon = \{0\}$, i.e. A is w-regular.

Let $\pi : \mathbb{C}^n_{z,w} \to \mathbb{C}^{n-1}_z$ be the projection. The local structure theorem for analytic sets gives:

- *for some neighborhood U of 0 in $\mathbb{C}^{n-1}_z$, there exists an analytic hypersurface $\Delta \subset U$ such that $A_\Delta = A \cap ((U \setminus \Delta) \times \mathbf{D}_\varepsilon)$ is a manifold;*
- $\pi : A_\Delta \to U \setminus \Delta$ *is a d-sheeted covering* ($d \in \mathbb{N}$).

It is easy to show that the covering $\pi : A_\Delta \to U \setminus \Delta$ is trivial.

Then we may define holomorphic functions $\tau_1, \ldots, \tau_d : U \setminus \Delta \to \mathbb{C}$ such that A_Δ is the union of the graphs of the τ_j. By the Riemann extension theorem, the functions τ_j extend as holomorphic functions $\tau_j \in \mathcal{O}(U)$. Suppose that $\tau_j \neq \tau_k$ for $j \neq k$. Then for some disc $\mathbf{D} \subset U$ centered at 0, we have $\tau_j|_\mathbf{D} \neq \tau_k|_\mathbf{D}$, so $(\tau_j - \tau_k)|_\mathbf{D}$ vanishes only at 0. But, from the hypothesis, on restriction to $\mathbf{D}$, $\{\operatorname{Re}(\tau_j - \tau_k) = 0\} \subset \{\tau_j - \tau_k = 0\}|_\mathbf{D} = \{0\}$, impossible. ∎

4.6. Proof of Theorem 4.1. Consider the foliation of $S \setminus P$ given by the level sets of the smooth function $\nu : S \to [0,1]$ (Sections 2.3 and 2.7) and set $L_t = \{\nu = t\}$ for $t \in (0.1)$. Let $V_t \subset \overline{\Omega} \times i\mathbb{R}_v \subset \mathbb{C}^n$ be the complex leaf of M bounded by L_t.

By Proposition 4.7, M is the graph of a continuous function over Ω, and, by Lemma 4.8, each leaf V_t is a complex smooth hypersurface and $\pi|_{V_t}$ is a submersion.

Since Ω is strictly convex, as in the situation studied by Shcherbina (see Lemma 4.6), $\pi_{|V_t}$ is 1-1, so, by Corollary 4.4, π sends V_t onto a domain $\Omega_t \subset \mathbb{C}_z^{n-1}$ with smooth boundary. Let

$$\pi_u : (\mathbb{C}_z^{n-1} \times \mathbb{R}_u) \times i\mathbb{R}_v \to \mathbb{R}_u, \qquad \pi_v : (\mathbb{C}_z^{n-1} \times \mathbb{R}_u) \times i\mathbb{R}_v \to \mathbb{R}_v.$$

Then $\pi_{u|L_t} = a_t.\pi_{|L_t}$ and $\pi_{v|L_t} = b_t.\pi_{|L_t}$ where a_t, b_t are smooth functions on $b\Omega_t$. Moreover $b\Omega_t$, a_t, b_t depend smoothly on t.

If $(z_t, w_t) \in M$, then w_t varies on V_t, so w_t is a holomorphic extension of $a_t + ib_t$ to Ω_t. In particular u_t and v_t are smooth in (z,t), from the Bochner–Martinelli formula. The function $\partial u_t/\partial t$ is harmonic on Ω_t for each t and has a smooth extension on $b\Omega_t$.

From Lemma 4.3 and Corollary 4.4, $\partial u_t/\partial t$ does not vanish on $b\Omega_t$. Since the CR orbits L_t are connected from Proposition 3.5, $b\Omega_t$ is also connected, hence $\partial u_t/\partial t$ has constant sign on $b\Omega_t$, so, by the maximum principle, also on Ω_t and, in particular, it does not vanish. This implies that $M \setminus S$ is the graph of a smooth function over Ω which smoothly extends to $\overline{\Omega} \setminus Q$.

By Proposition 4.7, M is the graph of a continuous function over $\overline{\Omega}$.

4.7. Elementary smooth models. *An elementary smooth model in* $\mathbb{C}^n$ is an elementary model in the sense of Section 3.4 and satisfying the further condition which makes sense by Theorem 4.1:

(G) Let (z, w) be the coordinates in $\mathbb{C}^{n-1} \times \mathbb{C}$, with $z = (z_1, \dots, z_{n-1})$, $w = u + iv = z_n$, and let Ω be a strictly pseudoconvex domain in $\mathbb{C}^{n-1} \times \mathbb{R}_u$; assume that T' is the graph of a smooth function $g : b\Omega \to \mathbb{R}_v$.

THEOREM 4.9. *Let T be an elementary smooth model. Then there exists a continuous function $f : \overline{\Omega} \to \mathbb{R}_v$ which is smooth on $\overline{\Omega} \setminus Q$ and such that $f_{|b\Omega} = g$, and $M_0 = \mathrm{graph}(f) \setminus S$ is a smooth Levi-flat hypersurface of $\mathbb{C}^n$; in particular, S is the boundary of the hypersurface $M = \mathrm{graph}(f)$.*

Proof. Similar to the proof of Theorem 4.1. ∎

Gluing of elementary smooth models. In an open set of $\mathbb{C}^n$, a coordinate system (z, w) of $\mathbb{C}_z^{n-1} \times \mathbb{R}_u$ defines an $(n-1, 1)$-*frame*.

To define the gluing of elementary models (Section 3.5) we considered a CR-isomorphism from an open set of $\mathbb{C}^n$ induced by a holomorphic isomorphism of the ambient space $\mathbb{C}^n$ onto an open set in $\mathbb{C}^n$. To define the gluing of elementary smooth models, we have to consider a holomorphic isomorphism of the ambient space $\mathbb{C}^n$ onto an open set in $\mathbb{C}^n$ sending an $(n-1, 1)$-frame of $\mathbb{C}_z^{n-1} \times \mathbb{R}_u$ onto an $(n-1, 1)$-frame of $\mathbb{C}_{z'}^{n-1} \times \mathbb{R}_{u'}$.

As in Section 3.5, we will define a submanifold S' of X obtained by gluing elementary smooth models by induction on the number m of models. An elementary smooth model T in X is the image of an elementary smooth model T_0 of $\mathbb{C}^n$ under an analytic isomorphism of a neighborhood of T_0 in $\mathbb{C}^n$ into X.

Gluing an elementary smooth model T *of type* (d_1) *to a free down-1-hyperbolic point of* S'. Every elementary smooth model is contained in a cylinder $b\Omega \times \mathbb{R}_v$ determined by Ω and an $(n-1, 1)$-frame. Two sets Ω are *compatible* if either they coincide or one is part of the other.

The announced gluing is defined in the following way: there exists a CR-isomorphism h_1 from a neighborhood V_1 of $\overline{\sigma}'_1$ induced by a holomorphic isomorphism of the ambient space $\mathbb{C}^n$ onto a neighborhood of σ_1 in S'. Let k_1 be a CR-isomorphism from a neighborhood T''_1 of T'_1 into X such that $k_1|_{V_1} = h_1$, and there exists a common $(n-1, 1)$-frame on which the corresponding sets Ω are compatible. Such a situation is possible as the example of the horned (almost everywhere) smooth sphere shows (Theorem 4.1).

Note that the gluing implies that the submanifold S' is C^0 and smooth except at the complex points.

Other gluings are obtained in a similar way. Hence:

THEOREM 4.10. *The manifold* S' *obtained by gluing elementary smooth models is of class* C^0, *and smooth except at the complex points.*

COROLLARY 4.11. *The manifold* S' *is the boundary of a manifold* M *of class* C^0 *whose interior is a Levi-flat smooth hypersurface.*

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