Boundaries of Levi-flat hypersurfaces: special hyperbolic points

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Abstract. Let \( S \subset \mathbb{C}^n \), \( n \geq 3 \), be a compact connected 2-codimensional submanifold having the following property: there exists a Levi-flat hypersurface whose boundary is \( S \), possibly as a current. Our goal is to get examples of such \( S \) containing at least one special 1-hyperbolic point: a sphere with two horns, elementary models and their gluings. Some particular cases of \( S \) being a graph are also described.

1. Introduction. Let \( S \subset \mathbb{C}^n \) be a compact connected 2-codimensional submanifold having the following property: there exists a Levi-flat hypersurface \( M \subset \mathbb{C}^n \setminus S \) such that \( dM = S \) (i.e. whose boundary is \( S \), possibly as a current). The case \( n = 2 \) has been intensively studied since the beginning of the eighties, in particular by Bedford, Gaveau, Klingenberg, Shcherbina, Chirka, Tomassini, Słodkowski, Gromov, Eliashberg; it requires global conditions: \( S \) has to be contained in the boundary of a strictly pseudoconvex domain.

We consider the case \( n \geq 3 \); results on this case have been obtained since 2005 by Dolbeault, Tomassini and Zaitsev; local necessary conditions recalled in Section 2 have to be satisfied by \( S \), singular CR points on \( S \) are supposed to be elliptic and the solution \( M \) is obtained in the sense of currents \([DTZ05, DTZ10]\). More recently a regular solution \( M \) has been obtained when \( S \) satisfies a supplementary global condition as in the case \( n = 2 \) \([DTZ11]\), with singular CR points on \( S \) still supposed to be elliptic.

The problem we are interested in is to get examples of such \( S \) containing at least one special 1-hyperbolic point (Section \([2.4]\)). CR orbits near a special 1-hyperbolic point are large and, assuming they are compact, a careful examination has to be done (Sections \([2.6, 2.7]\)). As a topological preliminary, we need a generalization of a theorem of Bishop on the difference of the

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numbers of special elliptic and 1-hyperbolic points (Section 2.8); this result is a particular case of a theorem of Hon-Fei Lai [Lai72].

The first example considered is the sphere with two horns which has one special 1-hyperbolic point and three special elliptic points (Section 3.4). Then we consider elementary models and their gluings to obtain more complicated examples (Section 3.5). The results have been announced in [Dol08], and in a more precise way in [Dol11]; the first aim of this paper is to give complete proofs. Finally, we recall in detail and extend the results of [DTZ11] on regularity of the solution when $S$ is a graph satisfying a supplementary global condition, as in the case $n = 2$, to the case of existence of special 1-hyperbolic points, and to gluing of elementary smooth models (Section 4).

2. Preliminaries: local and global properties of the boundary

2.1. Definitions. A smooth, connected, CR submanifold $M \subset \mathbb{C}^n$ is called minimal at a point $p$ if there does not exist a submanifold $N$ of $M$ of lower dimension through $p$ such that $HN = HM|_N$, where $HN$ is the complex tangent bundle to $N$. By a theorem of Sussmann, all possible submanifolds $N$ such that $HN = HM|_N$ contain, as germs at $p$, one of the minimal possible dimension, defining a so called CR orbit of $p$ in $M$ whose germ at $p$ is uniquely determined.

A smooth compact connected oriented submanifold $S \subset \mathbb{C}^n$ of dimension $2n - 2$ is said to be a locally flat boundary at a point $p$ if it locally bounds a Levi-flat hypersurface near $p$. Assume that $S$ is CR in a small enough neighborhood $U$ of $p \in S$. If all CR orbits of $S$ are 1-codimensional (which will appear as a necessary condition for our problem), the following two conditions are equivalent [DTZ05]:

(i) $S$ is a locally flat boundary on $U$;
(ii) $S$ is nowhere minimal on $U$.

2.2. Complex points of $S$ (i.e. singular CR points on $S$) [DTZ05].

At such a point $p \in S$, $T_pS$ is a complex hyperplane in $T_p\mathbb{C}^n$. In suitable local holomorphic coordinates $(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C}$ vanishing at $p$, with $w = z_n$ and $z = (z_1, \ldots, z_{n-1})$, $S$ is locally given by the equation

$$w = \varphi(z) = Q(z) + O(|z|^3),$$

$$Q(z) = \sum_{1 \leq i, j \leq n-1} (a_{ij}z_i\overline{z}_j + b_{ij}z_i\overline{z}_j + c_{ij}z_i\overline{z}_j).$$

$S$ is called flat at a complex point $p \in S$ if $\sum b_{ij}z_i\overline{z}_j \in \lambda\mathbb{R}$, $\lambda \in \mathbb{C}$. We also say that $p$ is flat.

Let $S \subset \mathbb{C}^n$ be a locally flat boundary with a complex point $p$. Then $p$ is flat.
By making the change of coordinates \((z, w) \mapsto (z, \lambda^{-1} w)\), we get \(\sum b_{ij} z_i \bar{z}_j \in \mathbb{R}\) for all \(z\). By a change of coordinates \((z, w) \mapsto (z, w + \sum a'_{ij} z_i \bar{z}_j)\) we can choose the holomorphic term in (1) to be the conjugate of the antiholomorphic one and so make the whole form \(Q\) real-valued.

We say that \(S\) is in a flat normal form at \(p\) if the coordinates \((z, w)\) as in (1) are chosen such that \(Q(z) \in \mathbb{R}\) for all \(z \in \mathbb{C}^{n-1}\).

Properties of \(Q\). Assume that \(S\) is in a flat normal form; then the quadratic form \(Q\) is real-valued. If \(Q\) is positive definite or negative definite, the point \(p \in S\) is said to be elliptic; if \(p \in S\) is not elliptic, and if \(Q\) is nondegenerate, \(p\) is said to be hyperbolic. From Section 2.4 on, we will only consider particular cases of the quadratic form \(Q\).

2.3. Elliptic points

**Proposition 2.1** ([DTZ05], [DTZ10]). Assume that \(S \subset \mathbb{C}^n (n \geq 3)\) is nowhere minimal at all its CR points and has an elliptic flat complex point \(p\). Then there exists a neighborhood \(V\) of \(p\) such that \(V \setminus \{p\}\) is foliated by compact real \((2n - 3)\)-dimensional CR orbits diffeomorphic to the sphere \(S^{2n-3}\) and there exists a smooth function \(\nu\) having the CR orbits as level surfaces.

**Sketch of proof** (see [DTZ10]). In the case of a quadric \(S_0\) \((w = Q(z))\), the CR orbits are defined by \(w_0 = Q(z)\), where \(w_0\) is constant. Using (1), we approximate the tangent space to \(S\) by the tangent space to \(S_0\) at a point with the same coordinate \(z\); the same is done for the tangent spaces to the CR orbits on \(S\) and \(S_0\); then we construct the global CR orbit on \(S\).

2.4. Special flat complex points. From [Bis65], for \(n = 2\), in suitable local holomorphic coordinates centered at 0, we have \(Q(z) = z \bar{z} + \lambda \text{Re} z^2\), \(\lambda \geq 0\), under the notation of [BK91]; for \(0 \leq \lambda < 1\), \(p\) is said to be elliptic, and for \(\lambda > 1\), it is said to be hyperbolic. The parabolic case \(\lambda = 1\), not generic, will be omitted [BK91]. When \(n \geq 3\), Bishop’s reduction cannot be generalized.

We say that the flat complex point \(p \in S\) is special if in certain holomorphic coordinates centered at 0,

\[
Q(z) = \sum_{j=1}^{n-1} (z_j \bar{z}_j + \lambda_j \text{Re} z_j^2), \quad \lambda_j \geq 0.
\]

Let \(z_j = x_j + iy_j, \ x_j, y_j \text{ real}, \ j = 1, \ldots, n - 1\). Then

\[
Q(z) = \sum_{l=1}^{n-1} ((1 + \lambda_l)x_l^2 + (1 - \lambda_l)y_l^2).
\]

A flat point \(p \in S\) is said to be special elliptic if \(0 \leq \lambda_j < 1\) for any \(j\).
A flat point \( p \in S \) is said to be special \( k \)-hyperbolic if \( \lambda_j > 1 \) for \( j \in J \subset \{1, \ldots, n-1\} \), and \( 0 \leq \lambda_j < 1 \) for \( j \in \{1, \ldots, n-1\} \setminus J \neq \emptyset \), where \( k \) denotes the number of elements of \( J \).

Special elliptic (resp. special \( k \)-hyperbolic) points are elliptic (resp. hyperbolic).

### 2.5. Special hyperbolic points.

For \( S \) given by (1), let \( S_0 \) be the quadric of equation \( w = Q(z) \).

**Lemma 2.2.** Suppose that \( S_0 \) is flat at 0 and that 0 is a special \( k \)-hyperbolic point. Then, in a neighborhood of 0, and with the above local coordinates, \( S_0 \) is CR and nowhere minimal outside 0, and the CR orbits of \( S_0 \) are \((2n-3)\)-dimensional submanifolds given by \( w = \text{const} \neq 0 \).

**Proof.** The submanifolds \( w = \text{const} \neq 0 \) have the same complex tangent space as \( S_0 \) and are of minimal dimension among submanifolds having this property, so they are CR orbits of codimension 1, and from the end of Section 2.1, \( S_0 \) is nowhere minimal outside 0.

The section \( w = 0 \) of \( S_0 \) is a real quadratic cone \( \Sigma_0' \) in \( \mathbb{R}^{2n} \) whose vertex is 0 and, outside 0, it is a CR orbit \( \Sigma_0 \) in a neighborhood of 0. We will call \( \Sigma_0' \) a singular CR orbit.

### 2.6. Foliation by CR orbits in a neighborhood of a special 1-hyperbolic point.

We first imitate and transpose the beginning of the proof of Proposition 2.1, i.e. of 2.4.2 in [DTZ05, DTZ10].

**2.6.1. Local 2-codimensional submanifolds.** In order to use simple notation we will assume \( n = 3 \).

In \( \mathbb{C}^3 \), consider the 4-dimensional submanifold \( S \) locally defined by the equation

\[
(1) \quad w = \varphi(z) = Q(z) + O(|z|^3),
\]

and the 4-dimensional submanifold \( S_0 \) of equation

\[
(4) \quad w = Q(z)
\]

with

\[
Q(z) = (\lambda_1 + 1)x_1^2 - (\lambda_1 - 1)y_1^2 + (1 + \lambda_2)x_2^2 + (1 - \lambda_2)y_2^2
\]

having a special 1-hyperbolic point at 0 \((\lambda_1 > 1, 0 \leq \lambda_2 < 1)\), and the cone \( \Sigma_0' \) whose equation is \( Q = 0 \). On \( S_0 \), a CR orbit is a 3-dimensional submanifold \( \mathcal{K}_{w_0} \) whose equation is \( w_0 = Q(z) \). If \( w_0 > 0 \), then \( \mathcal{K}_{w_0} \) does not meet the line \( L = \{x_1 = x_2 = y_2 = 0\} \); if \( w_0 < 0 \), then \( \mathcal{K}_{w_0} \) cuts \( L \) at two points.

**Lemma 2.3.** \( \Sigma_0 = \Sigma_0' \setminus 0 \) has two connected components in a neighborhood of 0.
2.6.2. CR orbits. By differentiating (1), we get for the tangent spaces the asymptotics

\[ T_{(z,\varphi(z))}S = T_{(z,Q(z))}S_0 + O(|z|^2), \quad z \in \mathbb{C}^2. \]

Here both \( T_{(z,\varphi(z))}S \) and \( T_{(z,Q(z))}S_0 \) depend continuously on \( z \) near the origin. Consider

(i) the hyperboloid \( H_- = \{ Q = -1 \} \), (then \( Q(z/(-Q(z))^{1/2}) = -1 \)), and the projection
\[
\pi_- : \mathbb{C}^3 \setminus \{ z = 0 \} \rightarrow H_- , \quad (z, w) \mapsto z/(-Q(z))^{1/2} ,
\]

(ii) for every \( z \in H_- \), a real orthonormal basis \( e_1(z), \ldots, e_6(z) \) of \( \mathbb{C}^3 \cong \mathbb{R}^6 \) such that
\[
e_1(z), e_2(z) \in H_zH_- , \quad e_3(z) \in T_zH_- ,
\]

where \( HH_- \) is the complex tangent bundle to \( H_- \).

Locally such a basis can be chosen continuously depending on \( z \). For every \( (z, w) \in \mathbb{C}^3 \setminus \{ z = 0 \} \), consider the basis \( e_1(\pi_-(z, w)), \ldots, e_6(\pi_-(z, w)) \). The unit vectors \( e_1(\pi_-(z, w)), e_2(\pi_-(z, w)), e_3(\pi_-(z, w)) \) are tangent to the CR orbit \( \mathcal{K}_{w_0} \) at \( (z, w) \) for \( w_0 < 0 \). Then, from (5), we have

\[ H_{(z,\varphi(z))}S = H_{(z,Q(z))}S_0 + O(|z|^2), \quad z \neq 0, z \rightarrow 0. \]

As in \([DTZ10]\), in a neighborhood of 0, denote by \( E(q), q \in S \setminus \{ 0 \}, w < 0, \) the tangent space to the local CR orbit \( \mathcal{K} \) on \( S \) through \( q \), and by \( E_0(q_0), q_0 \in S_0 \setminus \{ 0 \}, w < 0, \) the analogous object for \( S_0 \). We have

\[ E(z, \varphi(z)) = E_0(z, Q(z)) + O(|z|^2), \quad z \neq 0, z \rightarrow 0. \]

Given \( q \in S \) of integration of \( E(q), q \in S \), we get, locally, the CR orbit (leaf) on \( \tilde{S} \) through \( q \); given \( q_0 \in S_0 \), by integration of \( E_0(q_0), q_0 \in S_0 \), we get, locally, the CR orbit (leaf) on \( S_0 \) through \( q_0 \) (theorem of Sussmann). On \( S_0 \), a leaf is the 3-dimensional submanifold \( \mathcal{K}_{q_0} = \mathcal{K}_{w_0} = \mathcal{K} \) whose equation is \( w_0 = Q(z) \), with \( q = (z_0, w_0 = Q(z_0)) \). Moreover, \( \pi_- \) projects each \( E_0(q), q \in S_0, w < 0, \) bijectively onto \( T_{\pi(q)}H_- \), so \( \pi_-|_{\mathcal{K}_0} \) is a diffeomorphism onto \( H_- \); this implies, from (7), that, in a suitable neighborhood of the origin, the restriction of \( \pi_- \) to each local CR orbit of \( S \) is a local diffeomorphism.

We have \( \varphi(z) = Q(z) + \Phi(z) \) with \( \Phi(z) = O(|z|^3) \).

2.6.3. Behavior of local CR orbits. We follow the construction of \( E(z, \varphi(z)) \); compare with \( E_0(z, Q(z)) \). We know the integral manifold, the
orbit of $E_0(z, Q(z))$; and we deduce an evaluation of the integral manifold $\mathcal{K}$ of $E(z, \varphi(z))$.

**Lemma 2.4.** Under the above hypotheses, the local orbit $\Sigma$ corresponding to $\Sigma_0$ has two connected components in a neighborhood of 0.

**Proof.** Using the real coordinates, as for Lemma 2.3, consider $\Sigma' \cap \{y_1 = 0\}$. Locally, the connected components are obtained for $y_1 > 0$ and $y_1 < 0$ respectively, from formula (1). ■

We will call $\Sigma' = \Sigma$ a singular CR orbit and a singular leaf of the foliation.

We intend to prove that:

1) $\mathcal{K}$ does not cross the singular leaf through 0;
2) the only separatrix is the singular leaf through 0.

From the orbit $\mathcal{K}_0$, we will construct the differential equation defining it, and using (7), we will construct the differential equation defining $\mathcal{K}$.

In $\mathbb{C}^3$, we use the notation $x = x_1$, $y = y_1$, $u = x_2$, $v = y_2$; it suffices to consider the particular case $Q = 3x^2 - y^2 + u^2 + v^2$. On $S_0$, the orbit $\mathcal{K}_0$ issuing from the point $(c, 0, 0, 0)$ is defined by $3x^2 - y^2 + u^2 + v^2 = 3c^2$, i.e., for $x \geq 0, \quad x = \frac{1}{\sqrt{3}}(y^2 - u^2 - v^2 + 3c^2)^{1/2} = A(y, u, v)$; the local coordinates on the orbit are $(y, u, v)$. The orbit $\mathcal{K}_0$ satisfies the differential equation $dx = dA$. From (7), the orbit $\mathcal{K}$ issuing from $(c, 0, 0, 0)$ satisfies $dx = dA + \Psi$ with $\Psi(y, u, v; c) = O(|z|^2)$; hence $\Psi = d\Phi$, so $x = A + \Phi$ with $\Phi = O(|z|^3)$. More explicitly, $\mathcal{K}$ is defined by

$$x = x_{\mathcal{K},c} = \frac{1}{\sqrt{3}}(y^2 - u^2 - v^2 + 3c^2)^{1/2} + \Phi(y, u, v; c), \quad \Phi(y, u, v; c) = O(|z|^3).$$

The cone $\Sigma'_0$ whose equation is $Q = 0$ is a separatrix for the orbits $\mathcal{K}_0$. The corresponding object $\Sigma' = \{\varphi(z) = 0\}$ for $S$ has the singular point 0 and for $x > 0, y > 0, u > 0, v > 0$ it is defined by the differential equation $dx = d(A + \Phi)$ with $c = 0$, i.e. the local equation of $\Sigma'$ is

$$x = x_{\mathcal{K},0} = \frac{1}{\sqrt{3}}(y^2 - u^2 - v^2)^{1/2} + \Phi(y, u, v; 0), \quad \Phi(y, u, v; 0) = O(|z|^3).$$

For given $(y, u, v)$, $x_{\mathcal{K},c} - x_{\mathcal{K},0} = x_{\mathcal{K},c} - x_{\mathcal{K},0} + \Phi(y, u, v; c) - \Phi(y, u, v; 0)$. But $x_{\mathcal{K},c} - x_{\mathcal{K},0} = O(1)$ and $\Phi(y, u, v; c) - \Phi(y, u, v; 0) = O(|z|^3)$.

As a consequence, for $x > 0, y > 0, u > 0, v > 0$, locally, $\Sigma'$ is the unique separatrix for the orbits $\mathcal{K}$. The same holds for $x < 0$.

What has been done for the hyperboloid $H_- = \{Q = -1\}$ can be repeated for the hyperboloid $H_+ = \{Q = 1\}$. As at the beginning of Section 2.6.2, we consider

(i) the hyperboloid $H_+ = \{Q = 1\}$ and the projection

$$\pi_+: \mathbb{C}^3 \setminus \{z = 0\} \to H_+, \quad (z, w) \mapsto z/(Q(z))^{1/2},$$
Then there exists a neighborhood $p$ such that
\[ e_1(z), e_2(z) \in H_2H_+, \quad e_3(z) \in T_2H_+, \]
where $HH_+$ is the complex tangent bundle to $H_+$. 

**Lemma 2.5.** Given $\varphi$, there exists $R > 0$ such that, in $B(0,R) \cap \{x > 0, y > 0, u > 0, v > 0\} \subset \mathbb{C}^2$, the CR orbits $K$ have $\Sigma'$ as a unique separatrix.

**Proof.** When $c$ tends to zero, $x_{K,c} - x_{K,0} = x_{K,0,c} - x_{K,0} + \Phi(y,u,v;c) - \Phi(y,u,v;0) = O(|z|)$ and $\Phi(y,u,v;c) - \Phi(y,u,v;0) = O(|z|^3)$. For $\varphi(z) = Q(z) + \Phi(z)$ with $\Phi(z) = O(|z|^3)$ given, in (7), $E(z, \varphi(z)) - E_0(z, Q(z)) = O(|z|^2)$ and $\Phi(y,u,v;c) - \Phi(y,u,v;0)$ are also given. Then there exists $R$ such that, for $|z| < R$, $x_{K,c} - x_{K,0} > 0$. ■

### 2.7. CR orbits near a subvariety containing a special 1-hyperbolic point.

In this section we will impose conditions on $S$ and give a local property in a neighborhood of a compact $(2n - 3)$-subvariety of $S$.

Assume that $S \subset \mathbb{C}^n$ $(n \geq 3)$ is a locally closed $(2n - 2)$-submanifold, nowhere minimal at all its CR points, which has a unique 1-hyperbolic flat complex point $p$, and such that:

(i) if $\Sigma$ is the orbit whose closure $\Sigma'$ contains $p$, then $\Sigma'$ is compact.

Let $q \in S$, $q \neq p$; then, in a neighborhood $U$ of $q$ not containing $p$, $S$ is CR, $\text{CR-dim } S = n - 2$, $S$ is nonminimal and $\Sigma$ is 1-codimensional. We show that the CR orbits constitute a foliation on $S$ whose separatrix is $\Sigma'$. This is true in $U$ since $\Sigma \cap U$ is a leaf. Moreover, let $U_0$ be the ball $B(0,R)$ centered at $p = 0$ as in Lemma 2.5. If $U \cap U_0 \neq \emptyset$, the leaves in $U$ glue with the leaves in $U_0$ on $U \cap U_0$. Since $\Sigma'$ is compact, there exist a finite number of points $q_j \in \Sigma'$, $j = 0, 1, \ldots, J$, and open neighborhoods $U_j$, as above, such that $(U_j)_{j=0}^J$ is an open covering of $\Sigma'$. Moreover the leaves in $U_j$ glue respectively with the leaves in $U_k$ if $U_j \cap U_k \neq \emptyset$.

**Proposition 2.6.** Assume that $S \subset \mathbb{C}^n$ $(n \geq 3)$ is a locally closed $(2n - 2)$-submanifold, nowhere minimal at all its CR points, which has a unique special 1-hyperbolic flat complex point $p$, and such that:

(i) if $\Sigma$ is the orbit whose closure $\Sigma'$ contains $p$, then $\Sigma'$ is compact;

(ii) $\Sigma$ has two connected components $\sigma_1$, $\sigma_2$ whose closures are homeomorphic to spheres of dimension $2n - 3$.

Then there exists a neighborhood $V$ of $\Sigma'$ such that $V \setminus \Sigma'$ is foliated by compact real $(2n - 3)$-dimensional CR orbits whose equation in a neighborhood of $p$ is (3), and, the $w(= x_n)$-axis being assumed to be vertical, each orbit is diffeomorphic to either
• the sphere $S^{2n-3}$ above $\Sigma'$, or
• the union of two spheres $S^{2n-3}$ under $\Sigma'$,

and there exists a smooth function $\nu$ having the CR orbits as level surfaces.

Proof. This follows from the above and the following remark:

When $x_n$ tends to 0, the orbits tend to $\Sigma'$, and because of the geometry of the orbits near $p$, they are diffeomorphic to a sphere above $\Sigma'$, and to the union of two spheres under $\Sigma'$. The existence of $\nu$ is proved as in Proposition 2.1, namely, consider a smooth curve $\gamma : [0, \varepsilon) \to S$ such that $\gamma(0) = q$, where $q$ is a point of $\Sigma$ close to $p$, and $\gamma$ is a diffeomorphism onto its image $\Gamma = \gamma([0, \varepsilon))$. Let $\nu = \gamma^{-1}$ on the image of $\gamma$. Then, close enough to $q$, every CR orbit cuts $\Gamma$ at a unique point $q(t)$, $t \in [0, \varepsilon)$. Hence there is a unique extension of $\nu$ from $\gamma([0, \varepsilon))$ to $V \setminus p$ where $V$ is a neighborhood of $\Sigma'$ having CR orbits as its level surfaces. As $\nu$ is smooth away from $p$, it is smooth on the orbit $\Sigma$ and, if we set $\nu(p) = \nu(q) = 0$, $\nu$ is smooth on a neighborhood of $\Sigma \cup \{p\} = \Sigma'$.

2.8. Geometry of the complex points of $S$. The results of Section 2.8 are particular cases of theorems of Lai [Lai72], which I learnt from F. Forstnerič in July 2011.

In [BK91], E. Bedford and W. Klingenberg cite the following theorem of E. Bishop [Bis65, Section 4, p. 15]: On a 2-sphere embedded in $\mathbb{C}^2$, the difference between the numbers of elliptic points and of hyperbolic points is the Euler–Poincaré characteristic, i.e. 2. For the proof, Bishop uses a theorem of [CS51, Section 4].

We extend this result to $n \geq 3$ and give proofs which are essentially similar to the proofs of the general case [Lai72, Lai74] but simpler.

Let $S$ be a smooth compact connected oriented submanifold of dimension $2n-2$. Let $G$ be the manifold of oriented real linear $(2n-2)$-subspaces of $\mathbb{C}^n$. The submanifold $S$ of $\mathbb{C}^n$ has a given orientation which defines an orientation $o(p)$ of the tangent space to $S$ at any point $p \in S$. By mapping each point of $S$ to its oriented tangent space, we get a smooth Gauss map

$$t : S \to G.$$ 

Denote by $-t(p)$ the tangent space to $S$ at $p$ with the opposite orientation $-o(p)$.

Properties of $G$

(a) $\dim G = 2(2n-2)$.

Proof. $G$ is a two-fold covering of the Grassmannian $M_{m,k}$ of linear $k$-subspaces of $\mathbb{R}^m$ [Ste99, Part I, Section 7.9], for $m = 2n$ and $k = 2n-2$; they have the same dimension. We have

$$M_{m,k} \cong O_m/O_k \times O_{m-k}.$$
But $\dim O_k = \frac{1}{2}k(k - 1)$, hence
\[
\dim M_{m,k} = \frac{1}{2}(m(m - 1) - k(k - 1) - (m - k)(m - k - 1)) = k(m - k).
\]

(b) $G$ has the complex structure of a smooth quadric of complex dimension $2n - 2$ in $\mathbb{CP}^{2n-1}$ [Lai74, Pol08].

c) There exists a canonical isomorphism $h : G \to \mathbb{CP}^{n-1} \times \mathbb{CP}^{n-1}$.

d) Homology of $G$: Let $S_1, S_2$ be generators of $H_{2n-2}(G, \mathbb{Z})$; we assume that $S_1$ and $S_2$ are fundamental cycles of complex projective subspaces of complex dimension $n - 1$ of the complex quadric $G$. We also denote $S_1, S_2$ the ordered two factors $\mathbb{CP}^{n-1}$, so that $h : G \to S_1 \times S_2$.

**Proposition 2.7.** For $n \geq 2$, in general, $S$ has isolated complex points.

**Proof.** Let $\pi \in G$ be a complex hyperplane of $\mathbb{C}^n$ whose orientation is induced by its complex structure; the set of such $\pi$ is $H = G_{C|_{n-1,n}}^{\mathbb{C}} = \mathbb{CP}^{n-1*} \subset G$, as a real submanifold. If $p$ is a complex point of $S$, then $t(p) \in H$ or $-t(p) \in H$. The set of complex points of $S$ is the inverse image under $t$ of the intersections $t(S) \cap H$ and $-t(S) \cap H$ in $G$. Since $\dim t(S) = 2n - 2$, $\dim H = 2(n - 1)$, $\dim G = 2(2n - 2)$, it follows that the intersection is 0-dimensional in general. 

Denoting also by $S$ the fundamental cycle of the submanifold $S$ and by $t_*$ the homomorphism defined by $t$, we have
\[
t_*(S) \sim u_1S_1 + u_2S_2
\]
where $\sim$ means “homologous to”.

**Lemma 2.8** (proved for $n = 2$ in [CS51]). With the above notation, we have $u_1 = u_2$ and $u_1 + u_2 = \chi(S)$, the Euler–Poincaré characteristic of $S$.

The proof for $n = 2$ works for any $n \geq 3$, namely:

Let $G'$ be the manifold of oriented real linear 2-subspaces of $\mathbb{C}^n$. Let $\alpha : G \to G'$ map each oriented 2-$(n-1)$-subspace $R$ to its normal 2-subspace $R'$ oriented so that $R, R'$ determine the orientation of $\mathbb{C}^n$. Then $\alpha$ is a canonical isomorphism. Let $n : S \to G'$ be the map defined by taking oriented normal planes. Then $n = \alpha t$ and $t = \alpha^{-1}n$, hence we have the mapping $h\alpha h^{-1} : S_1 \times S_2 \to S_1 \times S_2$. Let $(x, y) \in S_1 \times S_2$. Then
\[
h\alpha h^{-1}(x, y) = (x, -y).
\]

Over $G$, there is a bundle $V$ of spheres with fiber over a real oriented linear $(2n - 2)$-subspace of $\mathbb{C}^n$ through 0 being the unit sphere $S^{2n-3}$ of this subspace. Let $\Omega$ be the characteristic class of $V$, and let $\Omega_t, \Omega_n$ denote the characteristic classes of the tangent and normal bundles of $S$. Then $t^*\Omega = \Omega_t$, $n^*\Omega = \Omega_n$. 
The bundle $V$ is the Stiefel manifold of ordered pairs of orthogonal unit vectors through 0 in $\mathbb{R}^{2n} \cong \mathbb{C}^n$. Let $f : V \to G$ be the projection.

From the Gysin sequence, we see that the kernel of $f^* : H^{2n-2}(G) \to H^{2n-2}(V)$ is generated by $\Omega$. To find the kernel of $f^*$, we determine the morphism $f_* : H_{2n-2}(V) \to H_{2n-2}(G)$. A generating $(2n-2)$-cycle of $V$ is $S^2 \times e$ where $S^2 \cong \mathbb{C}P^{n-1}$ and $e$ is a point. Let $z$ be any point of $S^2$. Then from (†), we have

$$hf(z, e) = (z, -z).$$

Therefore, $f_*(S^2 \times e) = S_1 - S_2$. Thus, the kernel of $f^*$ is $\mathbb{Z}$-generated by $S_1^* + S_2^*$.

With a convenient orientation for the fiber of the bundle $V$, we get $\Omega = S_1^* + S_2^*$. For a suitable orientation of $S$, we get $\Omega_t. S = \chi_S$ = Euler characteristic of $S$. We have

$$\Omega_t = t^*(S_1^* + S_2^*) = t^*S_1^* + t^*S_2^*,$$

$$\Omega_n = n^*(S_1^* + S_2^*) = t^*\alpha^*(S_1^* + S_2^*) = t^*(S_1^* - S_2^*) = t^*S_1^* - t^*S_2^*.$$

Since $\Omega_n = 0$, we get

$$(t^*S_1^*). S = (t^*S_2^*). S = \frac{1}{2}\chi_S.$$

**Local intersection numbers of $H$ and $t(S)$ when all complex points are flat and special.** If $H$ is a complex linear $(n - 1)$-subspace of $G$, then it is homologous to one of the $S_j$, $j = 1, 2$, say $S_2$ when $G$ has its structure of complex quadric. The intersection number of $H$ and $S_1$ is 1 and the intersection number of $H$ and $S_2$ is 0. So, the intersection number of $H$ and $u_1S_1 + u_2S_2$ is $u_1$.

In a neighborhood of a complex point 0, the manifold $S$ is defined by equation (1) with $w = z_n$ and

$$Q(z) = \sum_{j=1}^{n-1} \mu_j(z_jz_j + \lambda_j \Re z_j^2), \quad \mu_j > 0, \lambda_j \geq 0.$$ 

Let $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \ldots, n$, with real $x_l$. Let $e_l$ be the unit vector of the $x_l$-axis, $l = 1, \ldots, 2n$.

For simplicity assume $n = 3$: $Q(z) = \mu_1(z_1z_1 + \lambda_1 \Re z_1^2) + \mu_2(z_2z_2 + \lambda_2 \Re z_2^2)$, with $\mu_1 = \mu_2 = 1$. Then, up to higher order terms, $S$ is defined by

$$z_1 = x_1 + ix_2, \quad z_2 = x_3 + ix_4,$$

$$z_3 = (1 + \lambda_1)x_1^2 + (1 - \lambda_1)x_2^2 + (1 + \lambda_2)x_3^2 + (1 - \lambda_2)x_4^2.$$ 

In a neighborhood of 0, the tangent space to $S$ is defined by the four linearly independent vectors

$$v_1 = e_1 + 2(1 + \lambda_1)x_1e_5, \quad v_2 = e_2 + 2(1 - \lambda_1)x_2e_5,$$

$$v_3 = e_3 + 2(1 + \lambda_2)x_3e_5, \quad v_4 = e_4 + 2(1 - \lambda_2)x_4e_5.$$
Thus, if 0 is special elliptic or special $k$-hyperbolic with $k$ even, the tangent plane at 0 has the same orientation; if 0 is special elliptic or special $k$-hyperbolic with $k$ odd, the tangent space has opposite orientation.

**Proposition 2.9** (known for $n = 2$ [Bis65], here for $n \geq 3$). Let $S$ be a smooth, oriented, compact, 2-codimensional, real submanifold of $\mathbb{C}^n$ all of whose complex points are flat and special elliptic or special $1$-hyperbolic. Then, on $S$, $\sharp$ (special elliptic points) − $\sharp$ (special 1-hyperbolic points) = $\chi(S)$. If $S$ is a sphere, this number is 2.

**Proof.** Let $p \in S$ be a complex point and $\pi$ be the tangent hyperplane to $S$ at $p$. Assume that $\pi$ induces, on $\pi$, the orientation given by its complex structure. Then $\pi \in H$.

If $p$ is elliptic, the intersection number of $H$ and $t(S)$ is 1; if $p$ is 1-hyperbolic, the intersection number of $H$ and $t(S)$ is $-1$ at $p$.

By the argument preceding (1'), the sum of the intersection numbers of $H$ and $t(S)$ at complex points $p$ satisfying (**) is $u_1$. Reversing the condition (**), and using Lemma 2.8 we get the proposition.

3. Particular cases: horned sphere, elementary models and their gluings

3.1. We recall the following Harvey–Lawson theorem with a real parameter, to be used later.

Let $E \cong \mathbb{R} \times \mathbb{C}^n$, and $k : \mathbb{R} \times \mathbb{C}^n \rightarrow \mathbb{R}$ be the projection. Let $N \subset E$ be a compact (oriented) CR subvariety of $\mathbb{C}^{n+1}$ of real dimension $2n - 2$ and CR dimension $n - 2$ ($n \geq 3$), of class $C^\infty$, with negligible singularities (i.e. there exists a closed subset $\tau \subset N$ of $(2n - 2)$-dimensional Hausdorff measure 0 such that $N \setminus \tau$ is a CR submanifold). Let $\tau'$ be the set of all points $z \in N$ such that either $z \in \tau$ or $z \in N \setminus \tau$ and $N$ is not transversal to the complex hyperplane $k^{-1}(k(z))$ at $z$. Assume that $N$, as a current of integration, is $d$-closed and satisfies:

(\text{H}) there exists a closed subset $L \subset \mathbb{R}_{x_1}$ with $H^1(L) = 0$ such that for every $x \in k(N) \setminus L$, the fiber $k^{-1}(x) \cap N$ is connected and does not intersect $\tau'$.

**Theorem 3.1** ([DTZ10]; see also [DTZ05]). Let $N$ satisfy (H) with $L$ chosen accordingly. Then there exists, in $E' = E \setminus k^{-1}(L)$, a unique $C^\infty$ Levi-flat $(2n - 1)$-subvariety $M$ with negligible singularities in $E' \setminus N$, foliated by complex $(n - 1)$-subvarieties, with the properties that $M$ simply (or trivially) extends to $E'$ as a $(2n - 1)$-current (still denoted $M$) such that $dM = N$.
in $E'$. The leaves are the sections by the hyperplanes $E_{x_1^0}$, $x_1^0 \in k(N) \setminus L$, and are the solutions of the “Harvey–Lawson problem” of finding a holomorphic subvariety in $E_{x_1^0} \cong \mathbb{C}^n$ with prescribed boundary $N \cap E_{x_1^0}$.

**Remark 3.2.** Theorem 3.1 is valid in the space $E \cap \{ \alpha_1 < x_1 < \alpha_2 \}$, with the corresponding condition (H). Moreover, since $N$ is compact, for a suitable parameter $x_1$, we can assume $x_1 \in [0, 1]$.

To solve the boundary problem by Levi-flat hypersurfaces, $S$ has to satisfy necessary and sufficient local conditions. A way to prove that these conditions can occur is to construct an example for which the solution is obvious.

**3.2. Sphere with one special 1-hyperbolic point (sphere with two horns): Example.** In $\mathbb{C}^3$, let $z_j$, $j = 1, 2, 3$, be the complex coordinates and $z_j = x_j + iy_j$. In $\mathbb{R}^6 \cong \mathbb{C}^3$, consider the 4-dimensional subvariety (with negligible singularities) $S$ defined by

$$y_3 = 0,$$

$$0 \leq x_3 \leq 1,$$

$$x_3(x_1^2 + y_1^2 + x_2^2 + y_2^2 + x_3^2 - 1)$$

$$+ (1 - x_3)(x_1^4 + y_1^4 + x_2^4 + y_2^4 + 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2) = 0,$$

$$-1 \leq x_3 \leq 0,$$

$$x_3 = x_1^4 + y_1^4 + x_2^4 + y_2^4 + 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2.$$

The singular set of $S$ is the 3-dimensional section $x_3 = 0$ along which the tangent space is not everywhere (uniquely) defined. $S$ being in the real hyperplane $\{ y_3 = 0 \}$, the complex tangent spaces to $S$ are $\{ x_3 = x_0 \}$ for suitable $x_0$.

Since the tangent space to the hypersurface $f(x_1, y_1, x_2, y_2, x_3) = 0$ in $\mathbb{R}^5$ is

$$X_1 f'_{x_1} + Y_1 f'_{y_1} + X_2 f'_{x_2} + Y_2 f'_{y_2} + X_3 f'_{x_3} = 0,$$

the tangent space to $S$ in the hyperplane $\{ y_3 = 0 \}$ is, for $x_3 \geq 0$,

$$2x_1 [x_3 + 2(1 - x_3)(x_1^2 + 2)]X_1 + 2y_1 [x_3 + 2(1 - x_3)(y_1^2 - 1)]Y_1$$

$$+ 2x_2 [x_3 + (1 - x_3)(2x_2^2 + 1)]X_2 + 2y_2 [x_3 + (1 - x_3)(2y_2^2 + 1)]Y_2$$

$$+ [(x_1^2 + y_1^2 + x_2^2 + y_2^2 + 3x_3^2 - 1)$$

$$- (x_1^4 + y_1^4 + x_2^4 + y_2^4 + 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2)]X_3 = 0;$$

and for $x_3 \leq 0$,

$$4(x_1^2 + 2)x_1 X_1 + 4(y_1^2 - 1)y_1 Y_1 + 2(2x_2^2 + 1)x_2 X_2 + 2(2y_2^2 + 1)y_2 Y_2 - X_3 = 0.$$

The complex points of $S$ are defined by the vanishing of the coefficients of $X_j$, $j = 1, 2, 3, 4$, in the equations of the tangent spaces. For $0 \leq x_3 \leq 1,$
this yields
\[ x_1[x_3 + 2(1 - x_3)(x_1^2 + 2)] = 0, \]
\[ y_1[x_3 + 2(1 - x_3)(y_1^2 - 1)] = 0, \]
\[ x_2[x_3 + (1 - x_3)(2x_2^2 + 1)] = 0, \]
\[ y_2[x_3 + (1 - x_3)(2y_2^2 + 1)] = 0. \]
We have the solutions
\[ h : x_j = 0, y_j = 0 \ (j = 1, 2), \ x_3 = 0; \]
\[ e_3 : x_j = 0, y_j = 0 \ (j = 1, 2), \ x_3 = 1. \]
For \( x_3 \leq 0 \), the vanishing of the coefficients yields
\[ (x_1^2 + 2)x_1 = 0, \]
\[ (y_1^2 - 1)y_1 = 0, \]
\[ (2x_2^2 + 1)x_2 = 0, \]
\[ (2y_2^2 + 1)y_2 = 0. \]
We have the solutions
\[ h : x_j = 0, y_j = 0 \ (j = 1, 2), \ x_3 = 0; \]
\[ e_1, e_2 : x_1 = 0, y_1 = \pm 1, \ x_2 = 0, \ y_2 = 0, \ x_3 = -1. \]
Note that the tangent space to \( S \) at \( h \) is well defined. Moreover, the set \( S \) will be smoothed along its section by the hyperplane \( \{x_3 = 0\} \) by a small deformation leaving \( h \) unchanged. In the following, \( S \) will denote this smooth submanifold.

**Lemma 3.3.** The points \( e_1, e_2, e_3 \) are special elliptic; the point \( h \) is special 1-hyperbolic.

**Proof.** Point \( e_3 \): Let \( x_3' = 1 - x_3 \), then the equation of \( S \) in a neighborhood of \( e_3 \) is
\[
(1 - x_3')(x_1^2 + y_1^2 + x_2^2 + y_2^2 + x_3'^2 - 2x_3')
- x_3'(x_1^4 + y_1^4 + x_2^4 + y_2^4 + 4x_1^2 + 2y_1^2 + x_2^2 + y_2^2) = 0, \]
i.e.
\[
2x_3' = x_1^2 + y_1^2 + x_2^2 + y_2^2 + O(|z|^3), \quad \text{or} \quad w = z\overline{z} + O(|z|^3),
\]
so \( e_3 \) is special elliptic.

Points \( e_1, e_2 \): Let \( y_1' = y_1 \pm 1 \), \( x_3' = x_3 + 1 \). Then the equation of \( S \) in a neighborhood of \( e_1, e_2 \) is
\[
x_3' - 1 = x_1^4 + (y_1' \mp 1)^4 + x_2^4 + y_2^4 + 4x_1^2 - 2(y_1' \mp 1)^2 + x_2^2 + y_2^2
= x_1^4 + y_1'^4 + 4y_1'^3 + 6y_1'^2 + 4y_1'^2 + 4y_1'^2 + 1 + x_2^4 + y_2^4
+ 4x_1^2 - 2(y_1' \mp 1)^2 + x_2^2 + y_2^2.
The tangent cone to the equation of $S$ at the origin of $\Sigma$ has two connected components $\sigma_1, \sigma_2$. Therefore
$$x_3(x_1^2 + y_1^2 + x_2^2 + y_2^2 + x_3^2 - 1) + (1 - x_3)(x_1^4 + y_1^4 + x_2^4 + y_2^4 + 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2) = 0,$$
and for $x_3 \leq 0$,
$$x_3 = x_1^4 + y_1^4 + x_2^4 + y_2^4 + 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2,$$
i.e. $x_3 = 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2 + O(|z|^3)$ in both cases, up to third order terms.
Hence $w = z_1\bar{z}_1 + z_2\bar{z}_2 + 3\Re z_1^2$, so $h$ is special 1-hyperbolic.

The section $\Sigma' = S \cap \{x_3 = 0\}$. Up to a small smooth deformation, its equation is
$$x_1^4 + y_1^4 + x_2^4 + y_2^4 + 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2 = 0 \quad \text{in } \{x_3 = 0\}.$$
The tangent cone to $\Sigma'$ at 0 is $4x_1^2 - 2y_1^2 + x_2^2 + y_2^2 = 0$. Locally, the section of $S$ by the coordinate 3-space $x_1, y_1, x_3$ is
$$x_3 = 4x_1^2 - 2y_1^2 + O(|z|^3),$$
and the section by the $x_2, y_2, x_3$-space is $x_3 = x_2^2 + y_2^2 + O(|z|^3)$.

**Lemma 3.4.** Under the above hypotheses and notation:

(i) $\Sigma = \Sigma' \setminus \{0\}$ has two connected components $\sigma_1, \sigma_2$.
(ii) The closures of the three connected components of $S \setminus \Sigma'$ are submanifolds with boundaries and corners.

**Proof.** (i) The only singular point of $\Sigma'$ is 0. We work in the ball $B(0, A)$ of $\mathbb{C}^2_{x_1, y_1, x_2, y_2}$ for small $A$ and in the 3-space $\pi_\lambda = \{y_2 = \lambda x_2\}, \lambda \in \mathbb{R}$. For $\lambda$ fixed, $\pi_\lambda \cong \mathbb{R}^3_{x_1, y_1, x_2}$, and $\Sigma' \cap \pi_\lambda$ is the cone of equation $4x_1^2 - 2y_1^2 + (1 + \lambda^2)x_2^2 + O(|z|^3) = 0$ with vertex 0 and basis the hyperboloid $H_\lambda$ of equation $4x_1^2 - 2y_1^2 + (1 + \lambda^2)x_2^2 + O(|z|^3) = 0$ in the plane $x_2 = x_2^0$; the curves $H_\lambda$ have no common point outside 0. So, when $\lambda$ varies, the surfaces $\Sigma' \cap \pi_\lambda$ are disjoint outside 0. The set $\Sigma'$ is clearly connected; $\Sigma' \cap \{y_1 = 0\} = \{0\}$, the origin of $\mathbb{C}^3$; by the above, $\sigma_1 = \Sigma \cap \{y_1 > 0\}$, and $\sigma_2 = \Sigma \cap \{y_1 < 0\}$.

(ii) The three connected components of $S \setminus \Sigma'$ contain, respectively, $e_1, e_2, e_3$ and their boundaries are $\overline{\sigma}_1, \overline{\sigma}_2, \overline{\sigma}_1 \cup \overline{\sigma}_2$; these boundaries have corners as shown in the first part of the proof.

The connected component of $\mathbb{C}^2 \times \mathbb{R} \setminus S$ containing $(0, 0, 0, 0, 1/2)$ is the Levi-flat solution, the complex leaves being the sections by the hyperplanes $x_3 = x_3^0, -1 < x_3^0 < 1$. 
The section by the hyperplanes \( x_3 = x_0^0 \) is diffeomorphic to a 3-sphere for \( 0 < x_0^0 < 1 \) and to the union of two disjoint 3-spheres for \( -1 < x_0^0 < 0 \), as can be shown intersecting \( S \) by lines through the origin in the hyperplane \( x_3 = x_0^0 \); \( \Sigma' \) is homeomorphic to the union of two 3-spheres with a common point.

### 3.3. Sphere with one special 1-hyperbolic point (sphere with two horns), general case.

The example of Section 3.2 shows that the necessary conditions of Section 2 can be realised. Moreover, from Proposition 2.9, the hypothesis on the number of complex points is meaningful.

**Proposition 3.5 (cf. \([Dol08, \text{Proposition 2.6.1}]\)).** Let \( S \subset \mathbb{C}^n \) be a compact connected real 2-codimensional manifold such that the following holds:

(i) \( S \) is a topological sphere, nonminimal at every CR point;

(ii) every complex point of \( S \) is flat; there exist three special elliptic points \( e_j, j = 1, 2, 3 \), and one special 1-hyperbolic point \( h \);

(iii) \( S \) does not contain complex manifolds of dimension \( n - 2 \);

(iv) the singular CR orbit \( \Sigma' \) through \( h \) on \( S \) is compact and \( \Sigma' \setminus \{h\} \) has two connected components \( \sigma_1 \) and \( \sigma_2 \) whose closures are homeomorphic to spheres of dimension \( 2n - 3 \);

(v) the closures \( S_1, S_2, S_3 \) of the three connected components \( S'_1, S'_2, S'_3 \) of \( S \setminus \Sigma' \) are submanifolds with (singular) boundary.

Then each \( S_j \setminus (e_j \cup \Sigma') \), \( j = 1, 2, 3 \), carries a foliation \( F_j \) of class \( C^\infty \) with 1-codimensional CR orbits as compact leaves.

**Proof.** From conditions (i) and (ii), \( S \) satisfies the hypotheses of Proposition 2.1 near any elliptic flat point \( e_j \), and of Proposition 2.6 near \( \Sigma' \), all CR orbits being diffeomorphic to the sphere \( S^{2n-3} \). Assumption (iii) guarantees that all CR orbits in \( S \) must be of real dimension \( 2n - 3 \). Hence, by removing small connected open saturated neighborhoods of all special elliptic points, and of \( \Sigma' \), we obtain, from \( S \setminus \Sigma' \), three compact manifolds \( S''_j \), \( j = 1, 2, 3 \), with boundary and with the foliation \( F_j \) of codimension 1 given by its CR orbits, near \( e_j \); the first cohomology group with values in \( \mathbb{R} \) of these orbits is 0. It is easy to show that this foliation is transversely oriented. ■

Recall Thurston’s Stability Theorem (\([CaC, \text{Theorem 6.2.1}]\)).

**Proposition 3.6.** Let \((M, \mathcal{F})\) be a compact, connected, transversely orientable, foliated manifold with boundary or corners, of codimension 1, of class \( C^1 \). If there is a compact leaf \( L \) with \( H^1(L, \mathbb{R}) = 0 \), then every leaf is homeomorphic to \( L \), and \( M \) is homeomorphic to \( L \times [0, 1] \), foliated as a product.

From the above theorem, \( S''_j \) is homeomorphic to \( S^{2n-3} \times [0, 1] \) with CR orbits being of the form \( S^{2n-3} \times \{x\} \) for \( x \in [0, 1] \). Then the full manifold
$S_j$ is homeomorphic to a half-sphere supported by $S^{2n-2}$ and $F_j$ extends to $S_j$, with $S_3$ having its boundary pinched at the point $h$.

**Theorem 3.7.** Let $S \subset \mathbb{C}^n$, $n \geq 3$, be a compact connected smooth real 2-codimensional submanifold satisfying conditions (i) to (v) of Proposition 3.5. Then there exists a Levi-flat $(2n-1)$-subvariety $\tilde{M} \subset \mathbb{C} \times \mathbb{C}^n$ with boundary $\tilde{S}$ (in the sense of currents) such that the natural projection $\pi : \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$ restricts to a bijection which is a CR diffeomorphism between $\tilde{S}$ and $S$ outside the complex points of $S$.

**Proof.** By Proposition 2.1, for every $e_j$, a continuous function $\nu'_j$, $C^\infty$ outside $e_j$, can be constructed in a neighborhood $U_j$ of $e_j$, $j = 1, 2, 3$, and by Proposition 2.6 we have an analogous result in a neighborhood of $\Sigma'$. Furthermore, from Proposition 3.6, a smooth function $\nu''_j$ whose level sets are leaves of $F_j$ can be obtained globally on $S'_j \setminus (e_j \cup \Sigma')$. With the functions $\nu'_j$ and $\nu''_j$, and analogous functions near $\Sigma'$, using a partition of unity, we obtain a global smooth function $\nu_j : S_j \to \mathbb{R}$ without critical points away from the complex points $e_j$ and from $\Sigma'$.

Let $\sigma_1$, resp. $\sigma_2$ be the two connected, relatively compact components of $\Sigma \setminus \{h\}$, according to condition (iv); $\bar{\sigma}_1$, resp. $\bar{\sigma}_2$ is the boundary of $S_1$, resp. $S_2$, and $\bar{\sigma}_1 \cup \bar{\sigma}_2$ is the boundary of $S_3$. We can assume that the three functions $\nu_j$ are finite-valued and get the same values on $\bar{\sigma}_1$ and $\bar{\sigma}_2$. Then the functions $\nu_j$ are induced by a unique function $\nu : S \to \mathbb{R}$.

The submanifold $S$, being locally the boundary of a Levi-flat hypersurface, is orientable. We now set $\tilde{S} = N = \text{graph}(\nu) = \{(\nu(z), z) : z \in S\}$. Let $\tilde{S}_s = \{e_1, e_2, e_3, \sigma_1 \cup \sigma_2\}$.

The map $\lambda : S \to \tilde{S}$ ($z \mapsto (\nu(z), z)$) is bicontinuous; $\lambda|_{\tilde{S}_s}$ is a diffeomorphism; moreover $\lambda$ is a CR map. Choose an orientation on $\tilde{S}$. Then $N$ is an (oriented) CR subvariety with the negligible set of singularities $\tau = \lambda(S_s)$.

At every point of $S \setminus S_s$, $dx_r\nu \neq 0$, so condition (H) (Section 3.1) is satisfied at every point of $N \setminus \tau$.

All the assumptions of Theorem 3.1 being satisfied by $N = \tilde{S}$, in a particular case, we conclude that $N$ is the boundary of a Levi-flat $(2n-2)$-variety (with negligible singularities) $\tilde{M}$ in $\mathbb{R} \times \mathbb{C}^n$.

Taking $\pi : \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$ to be the standard projection, we obtain the conclusion.

**3.4. Generalizations: elementary models and their gluings.** The examples and the proofs of the theorems when $S$ is homeomorphic to a sphere (Section 3.3) suggest the following definitions.

Let $T'$ be a smooth, locally closed (i.e. closed in an open set), connected submanifold of $\mathbb{C}^n$, $n \geq 3$. We assume that $T'$ has the following properties:
(i) \( T' \) is relatively compact, not necessarily compact, and of codimension 2.
(ii) \( T' \) is nonminimal at every CR point.
(iii) \( T' \) does not contain complex manifolds of dimension \( n - 2 \).
(iv) \( T' \) has exactly two complex points which are flat and either special elliptic or special 1-hyperbolic.
(v) If \( p \in T' \) is special 1-hyperbolic, then the singular orbit \( \Sigma' \) through \( p \) is compact, and \( \Sigma' \setminus p \) has two connected components \( \sigma_1, \sigma_2 \) whose closures are homeomorphic to spheres of dimension \( 2n - 3 \).
(vi) If \( p \in T' \) is special 1-hyperbolic, then in a neighborhood of \( p \) with convenient coordinates, the equation of \( T' \) up to third order terms is

\[
z_n = \sum_{j=1}^{n-1} (z_j \bar{z}_j + \lambda_j \text{Re} z_j^2), \quad \lambda_1 > 1, \quad 0 \leq \lambda_j < 1 \text{ for } j \neq 1,
\]

or in real coordinates \( x_j, y_j \) with \( z_j = x_j + iy_j \),

\[
x_n = ((\lambda_1 + 1)x_1^2 - (\lambda_1 - 1)y_1^2) + \sum_{j=2}^{n-1} ((1 + \lambda_j)x_j^2 + (1 - \lambda_j)y_j^2) + O(|z|^3).
\]

(vii) The closures (in \( T' \)) \( T_1, T_2, T_3 \) of the three connected components \( T'_1, T'_2, T'_3 \) of \( T' \setminus \Sigma' \) are submanifolds with (singular) boundary. Let \( T''_j, j = 1, 2, 3 \), be a neighborhood of \( T'_j \) in \( T' \).

Up- and down-1-hyperbolic points. Let \( \tau \) be the \((2n - 2)\)-submanifold with (singular) boundary contained into \( T' \) such that either \( \bar{\sigma}_1 \) (resp. \( \bar{\sigma}_2 \)) or \( \Sigma' \) is the boundary of \( \tau \) near \( p \). In the first case, we say that \( p \) is 1-up (resp. 2-up), in the second it is down. If \( T' \) is contained in a small enough neighborhood of \( \Sigma' \) in \( \mathbb{C}^n \), such a \( T' \) will be called a local elementary model, more precisely it defines a germ of elementary model around \( \Sigma \).

The union \( T \) of \( T_1, T_2, T_3 \) and of the germ of elementary model around the singular orbit at every special 1-hyperbolic point is called an elementary model. It behaves as a locally closed submanifold still denoted \( T \).

Examples of elementary models. We will say that \( T \) is an elementary model of type:

(a) if it has two elliptic points;
(b) if it has one special elliptic point and one down-1-hyperbolic point;
(c_1) if it has one special elliptic point and one 1-up-1-hyperbolic point;
(c_2) if it has one special elliptic point and one 2-up-1-hyperbolic point;
(d_1) if it has two special 1-up-1-hyperbolic points;
(d_2) if it has two special 2-up-1-hyperbolic points;
(e) if it has two special down-1-hyperbolic points;

Other configurations can be easily imagined.
The prescribed boundary of a Levi-flat hypersurface of $\mathbb{C}^n$ in [DTZ05] and [DTZ10], whose complex points are flat and elliptic, is an elementary model of type (a).

**Properties of elementary models.** For instance, if $T$ is 1-up and has one special elliptic point, we solve the boundary problem as in $S_1$ in the proof of Theorem 3.7.

**Proposition 3.8.** Let $T$ be a local elementary model. Then $T$ carries a foliation $F$ of class $C^\infty$ with 1-codimensional CR orbits as compact leaves.

**Proof.** From the definition and Proposition 2.6.

**Theorem 3.9.** Let $T$ be an elementary model. There exists an open neighborhood $T''$ in $T'$ carrying a smooth function $\nu : T'' \to \mathbb{R}$ whose level sets are leaves of a smooth foliation.

**Proof.** By removing small connected open saturated neighborhoods of every special elliptic point, and of $\Sigma'$, the singular orbit through every special 1-hyperbolic point $p$, we obtain, from $T \setminus \Sigma'$, three manifolds $T''_j$, $j = 1, 2, 3$, with boundary:

- $T_1$ and $T_2$ containing one special elliptic point $e$ or one special 1-hyperbolic point with the foliations $F_1, F_2$, from Propositions 2.1 and 3.8,
- $T''_3$ with the foliation $F_3$ of codimension 1 given by its CR orbits whose first cohomology group with values in $\mathbb{R}$ is 0, near $e$, or $p$. It is easy to show that this later foliation is transversely oriented.

From Thurston’s Stability Theorem (Proposition 3.6), $T''_3$ is homeomorphic to $S^{2n-3} \times [0, 1]$, foliated as a product, with CR orbits being of the form $S^{2n-3} \times \{x\}$ for $x \in [0, 1]$; hence we obtain smooth functions $\nu_1, \nu_2, \nu_3$ whose level sets are leaves of the foliations $F_1, F_2, F_3$ respectively, and using a partition of unity we get the desired function $\nu$ on $T$.

**Theorem 3.10.** Let $T$ be an elementary model. Then there exists a Levi-flat $(2n - 1)$-subvariety $\tilde{M} \subset \mathbb{C} \times \mathbb{C}^n$ with boundary $\tilde{T}$ (in the sense of currents) such that the natural projection $\pi : \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$ restricts to a bijection which is a CR diffeomorphism between $\tilde{T}$ and $T$ outside the complex points of $T$.

**Proof.** The submanifold $T$, being locally the boundary of a Levi-flat hypersurface, is orientable. We now set $\tilde{T} = N = \text{graph}(\nu) = \{(\nu(z), z) : z \in S\} \subset E \cong \mathbb{R} \times \mathbb{C}^{n-1}$. Let $T_s$ be the set of all flat complex points of $T$.

The map $\lambda : T \to \tilde{T}$ ($z \mapsto (\nu(z), z)$) is bicontinuous; $\lambda|_{T \setminus T_s}$ is a diffeomorphism; moreover $\lambda$ is a CR map. Choose an orientation on $T$. Then $N$ is an (oriented) CR subvariety with the negligible set of singularities $\tau = \lambda(T_s)$. 
Using Remark 3.2 at every point of $T \setminus T_{s}$, $d_{x_{1}}\nu \neq 0$, so condition (H) (Section 3.1) is satisfied at every point of $N \setminus \tau$.

All the assumptions of Theorem 3.1 being satisfied by $N = \tilde{T}$, in a particular case, we conclude that $N$ is the boundary of a Levi-flat $(2n - 2)$-variety (with negligible singularities) $\tilde{M}$ in $\mathbb{R} \times \mathbb{C}^{n}$.

Taking $\pi : \mathbb{C} \times \mathbb{C}^{n} \to \mathbb{C}^{n}$ to be the standard projection, we obtain the conclusion.

3.5. Gluing of elementary models. The gluing happens between two compatible elementary models along boundaries, for instance down and 1-up. Note that the gluing can only be made at special 1-hyperbolic points. More precisely, it can be defined as follows.

The properties of the submanifold $S$ of $\mathbb{C}^{n}$ assumed in Section 2 have a meaning in any complex analytic manifold $X$ of complex dimension $n \geq 3$, and are kept under any holomorphic isomorphism.

We will define a submanifold $S'$ of $X$ obtained by gluing of elementary models by induction on the number $m$ of models. An elementary model $T$ in $X$ is the image of an elementary model $T_{0}$ in $\mathbb{C}^{n}$ under an analytic isomorphism of a neighborhood of $T_{0}$ in $\mathbb{C}^{n}$ into $X$.

Let $S'$ be a closed smooth real submanifold of $X$ of dimension $2n - 2$ which is nonminimal at every CR point. Assume that $S'$ is obtained by gluing $m$ elementary models. Then $S'$ has the following properties:

- $S'$ has a finite number of flat complex points, some special elliptic and the others special 1-hyperbolic;
- for every special 1-hyperbolic $p'$, there exists a CR-isomorphism $h$ induced by a holomorphic isomorphism of the ambient space $\mathbb{C}^{n}$ from a neighborhood of $p$ in $T'$ onto a neighborhood of $p'$ in $S'$;
- for every CR orbit $\Sigma_{p'}$ whose closure contains a special 1-hyperbolic point $p'$, there exists a CR-isomorphism $h$ induced by a holomorphic isomorphism of the ambient space $\mathbb{C}^{n}$ from a neighborhood of $\Sigma_{p} = \Sigma_{p'} \setminus p$ in $T'$ onto a neighborhood $V$ of $\Sigma_{p'}$ in $S'$.

Every special 1-hyperbolic point of $S'$ which belongs to only one elementary model in $S'$ will be called free.

We will define the gluing of one more elementary model to $S'$.

Gluing an elementary model $T$ of type $(d_{1})$ to a free down-1-hyperbolic point of $S'$. Let $h_{1}$ be a CR-isomorphism from a neighborhood $V_{1}$ of $\tilde{\sigma}_{1}$ induced by a holomorphic isomorphism of the ambient space $\mathbb{C}^{n}$ onto a neighborhood of $\sigma_{1}$ in $S'$. Let $k_{1}$ be a CR-isomorphism from a neighborhood $T_{1}''$ of $T_{1}'$ into $X$ such that $k_{1}|_{V_{1}} = h_{1}$. 

**Theorem 3.11.** The compact manifold or the manifold with singular boundary $S'$, obtained by the gluing of a finite number of elementary models, is the boundary of a Levi-flat hypersurface of $X$ in the sense of currents.

**Proof.** From Theorem 3.10 and the definition of gluing. ■

**3.6. Examples of gluing.** Denoting the gluing of two models of type $(d_1)$ and $(d_2)$ to a free down-1-hyperbolic point of $S'$ by: $→ (d_1) - (d_2)$, and the converse by: $(d_1) - (d_2) →$, and, also, analogous configurations in the same way, we get:

- torus: $(b) → (d_1) - (d_2) → (b)$; the Euler–Poincaré characteristic of a torus is $\chi(T^k) = 0$; two special elliptic and two special 1-hyperbolic points;
- bitorus: $(b) → (d_1) - (d_2) → (e) → (d_1) - (d_2) → (b)$.

**4. Case of graphs** (see [DTZ11] for the case of elliptic points only, and dropping the property of the function solution to be Lipschitz).

**4.1.** We want to add the following hypothesis: $S$ is embedded into the boundary of a strictly pseudoconvex domain of $\mathbb{C}^n$, $n \geq 3$, and more precisely, let $(z, w)$ be the coordinates in $\mathbb{C}^{n-1} \times \mathbb{C}$, with $z = (z_1, \ldots, z_{n-1}), w = u + iv = z_n$, let $\Omega$ be a strictly pseudoconvex domain in $\mathbb{C}^{n-1} \times \mathbb{R} u$ (i.e. the second fundamental form of the boundary $b\Omega$ of $\Omega$ is everywhere positive definite); let $S$ be the graph graph$(g)$ of a smooth function $g : b\Omega \rightarrow \mathbb{R} v$.

Notice that $b\Omega \times \mathbb{R} v$ contains $S$ and is strictly pseudoconvex.

Assume that $S$ is a horned sphere (Section 3.3), satisfying the hypotheses of Theorem 3.7. Denote by $p_j, j = 1, \ldots, 4$, the complex points of $S$.

**4.2.** Our aim is to prove

**Theorem 4.1.** Let $S$ be the graph of a smooth function $g : b\Omega \rightarrow \mathbb{R} v$. Let $Q = (q_1, \ldots, q_4) \in b\Omega$ be the projections of the complex points $P = (p_1, \ldots, p_4)$ of $S$, respectively. Then there exists a continuous function $f : \overline{\Omega} \rightarrow \mathbb{R} v$ which is smooth on $\overline{\Omega} \setminus Q$ and such that $f_{|b\Omega} = g$, and $M_0 = \text{graph}(f) \setminus S$ is a smooth Levi-flat hypersurface of $\mathbb{C}^n$. Moreover, each complex leaf of $M_0$ is the graph of a holomorphic function $\phi : \Omega' \rightarrow \mathbb{C}$ where $\Omega' \subset \mathbb{C}^{n-1}$ is a domain with smooth boundary (that depends on the leaf) and $\phi$ is smooth on $\overline{\Omega'}$.

The natural candidate to be the graph $M$ of $f$ is $\pi(\tilde{M})$ where $\tilde{M}$ and $\pi$ are as in Theorem 3.7. We prove that this is the case, proceeding in several steps.

**4.3. Behavior near $S$.** Assume that $D$ is a strictly pseudoconvex domain such that $S \subset bD$.
Recall ([HL75, Theorem 10.4]): Let $D$ be a strictly pseudoconvex domain in $\mathbb{C}^n$, $n \geq 3$, with boundary $bD$, and let $\Sigma \subset bD$ be a compact connected maximally complex smooth $(2d-1)$-submanifold with $d \geq 2$. Then $\Sigma$ is the boundary of a uniquely determined relatively compact subset $V \subset \overline{D}$ such that $\overline{V} \setminus \Sigma$ is a complex analytic subset of $D$ with finitely many singularities of pure dimension $\leq d-1$, and near $\Sigma$, $\overline{V}$ is a $d$-dimensional complex manifold with boundary.

$V$ is said to be the solution of the boundary problem for $\Sigma$.

**Lemma 4.2** ([DTZ11]). Let $\Sigma_1, \Sigma_2$ be compact connected maximally complex $(2d-1)$-submanifolds of $bD$. Let $V_1, V_2$ be the corresponding solutions of the boundary problem. If $d \geq 2$, $2d \geq n+1$ and $\Sigma_1 \cap \Sigma_2 = \emptyset$, then $V_1 \cap V_2 = \emptyset$.

Let $\Sigma$ be a CR orbit of the foliation of $S \setminus P$. Then $\Sigma$ is a compact maximally complex $(2n-3)$-dimensional real submanifold of $\mathbb{C}^n$ contained in $bD$. Let $V = V_\Sigma$ be the solution of the boundary problem corresponding to $\Sigma$. From Theorem 3.7, $V = \pi(\tilde{V})$, where $\tilde{V} = (M \setminus \tilde{S}) \cap (\mathbb{C}^n \times \{x\})$ for suitable $x \in (0, 1)$ (the projection on the $x$-axis being finite, we can always assume that $x$ lies in $(0, 1)$). Moreover $\pi|_{\tilde{V}}$ is a biholomorphism $\tilde{V} \cong V$ and $M \setminus S \subset D$.

Let $\Sigma_1, \Sigma_2$ be two distinct orbits of the foliation of $S \setminus P$, and $\overline{V}_1, \overline{V}_2$ the corresponding leaves. Then, from Lemma 4.2 $\overline{V}_1 \cap \overline{V}_2 = \emptyset$.

Assume that $S$ satisfies the full hypotheses of Theorem 4.1. Set $m_1 = \min_S g$, $m_2 = \max_S g$ and pick $r \gg 0$ such that

$$D = \Omega \times [m_1, m_2] \subset \subset B(r) \cap (\Omega \times i\mathbb{R}_v)$$

where $B(r)$ is the ball $\{|(z,w)| < r\}$.

**Lemma 4.3.** Let $p \in S$ be a CR point. Then, near $p$, $M$ is the graph of a function $\phi$ on a domain $U \subset \mathbb{C}_z^{n-1} \times \mathbb{R}_w$ which is smooth up to the boundary of $U$.

**Proof.** Near $p$, each CR orbit $\Sigma$ is smooth and can be represented as the graph of a CR function over a strictly pseudoconvex hypersurface and $V_\Sigma$ as the graph of the local holomorphic extension of this function. From the Hopf lemma, $V$ is transversal to the strictly pseudoconvex hypersurface $d\Omega \times i\mathbb{R}_v$ near $p$. Hence the family of the $V_\Sigma$, near $p$, forms a smooth real hypersurface with boundary on $S$ that is the graph of a smooth function $\phi$ from a relatively open neighborhood $U$ of $p$ on $\overline{\Omega}$ into $\mathbb{R}_v$. Finally, Lemma 4.2 guarantees that this family does not intersect any other leaf $V$ from $M$.

**Corollary 4.4.** If $p \in S$ is a CR point, then each complex leaf $V$ of $M$, near $p$, is the graph of a holomorphic function on a domain $\Omega_V \subset \mathbb{C}_z^{n-1}$, which is smooth up to the boundary of $\Omega_V$. 

4.4. Solution as the graph of a continuous function. We recall some results of Shcherbina [Sh93].

His Main Theorem is the following:

Let $G$ be a bounded strictly convex domain in $\mathbb{C}_z \times \mathbb{R}_u$ ($z \in \mathbb{C}$) and $\varphi : bG \to \mathbb{R}_v$ be a continuous function. Then the following properties hold, where $\Gamma = \text{graph}$, and $\hat{\Gamma}(\varphi)$ means the polynomial hull of $\Gamma(\varphi)$:

(a) $\hat{\Gamma}(\varphi) \setminus \Gamma(\varphi)$ is the union of a disjoint family $\{D_\alpha\}$ of complex discs;

(b) for each $\alpha$, there is a simply connected domain $\Omega_\alpha \subset \mathbb{C}_z$ and a holomorphic function $w = f_\alpha$, defined on $\Omega_\alpha$, such that $D_\alpha$ is the graph of $f_\alpha$;

(c) for each $f_\alpha$, there exists an extension $f_\alpha^* \in C(\overline{\Omega}_\alpha)$ and $bD_\alpha = \{(z,w) \in b\Omega_\alpha \times \mathbb{C}_w : w = f_\alpha^*(z)\}$.

Lemma 4.5 ([Sh93]). Let $\{G_n\}_{n=0}^\infty$ be a sequence of bounded strictly convex domains $G_n \subset \mathbb{C}_z \times \mathbb{R}_u$ such that $G_n \to G_0$. Let $\{\varphi_n\}_{n=0}^\infty$ be a sequence of continuous functions $\varphi_n : bG_n \to \mathbb{R}_v$ such that $\Gamma(\varphi_n) \to \Gamma(\varphi_0)$ in the Hausdorff metric. Then, if $\Phi_n$ is the continuous function $\overline{G}_n \to \mathbb{R}_v$ such that $\hat{\Gamma}(\varphi_n) = \Gamma(\Phi_n)$, we have $\Gamma(\Phi_n) \to \Gamma(\Phi_0)$ in the Hausdorff metric.

Lemma 4.6 ([Sh93]). Let $\mathcal{U}$ be a smooth connected surface which is properly embedded into some convex domain $G \subset \mathbb{C}_z \times \mathbb{R}_u$. Suppose that near each of its points, $u$ can be defined locally by the equation $u = u(z)$. Then the surface $\mathcal{U}$ can be represented globally as the graph of some function $u = U(z)$, defined on some domain $\Omega \subset \mathbb{C}_z$.

Proposition 4.7. $M$ is the graph of a continuous function $f : \overline{\Omega} \to \mathbb{R}_v$.

Proof. We will intersect the graph $S$ with a convenient affine subspace of real dimension 4 to go back to the situation studied by Shcherbina.

Fix $a \in \mathbb{C}_z^{-1} \setminus 0$ and, for a given point $(\zeta, \xi) \in \Omega$ with $\zeta \in \mathbb{C}_z^{-1}$ and $\xi \in \mathbb{R}_u$, let $H(\zeta, \xi) \subset \mathbb{C}_z^{n-1} \times \{\xi\}$ be the complex line through $(\zeta, \xi)$ in the direction $(a,0)$. Set

$$L(\zeta, \xi) = H(\zeta, \xi) + \mathbb{R}_u(0,1), \quad \Omega(\zeta, \xi) = L(\zeta, \xi) \cap \Omega,$$

$$S(\zeta, \xi) = (H(\zeta, \xi) + \mathbb{C}_w(0,1)) \cap S.$$

Then $S(\zeta, \xi)$ is contained in the strictly convex cylinder

$$H(\zeta, \xi) + \mathbb{C}_w(0,1) \cap (b\Omega \times i\mathbb{R}_v)$$

and is the graph of $g_{|b\Omega(\zeta, \xi)}$.

From (a), the polynomial hull of $S(\zeta, \xi)$ is a continuous graph over $\overline{H}(\zeta, \xi)$. Consider $M = \pi(\hat{M})$ and set

$$M(\zeta, \xi) = (H(\zeta, \xi) + \mathbb{C}_w(0,1)) \cap M.$$
It follows that $M_{\zeta,\xi}$ is contained in the polynomial hull $\hat{S}_{(\zeta,\xi)}$. From (a_{iii}), $\hat{S}_{(\zeta,\xi)}$ is a graph over $\overline{\Omega}_{(\zeta,\xi)}$ foliated by analytic discs, so $M_{\zeta,\xi}$ is a graph over a subset $U$ of $\overline{\Omega}_{(\zeta,\xi)}$.

Every analytic disc $\Delta$ of $\hat{S}_{(\zeta,\xi)}$ has its boundary on $S_{(\zeta,\xi)}$. Since all the complex points of $S$ are isolated, $b\Delta$ contains a CR point $p$ of $S$; from Lemma 4.3 near $p$, $M_{\zeta,\xi}$ is a graph over $\overline{\Omega}_{(\zeta,\xi)}$. Near $p$, $\Delta$ is contained in $M_{\zeta,\xi}$, hence in a closed complex analytic leaf $V_{\Sigma}$ of $M$; so $\Delta \subset V_{\Sigma} \subset M$; but $\Delta \subset H_{(\zeta,\xi)} + \mathbb{C}w(0,1)$, so $\Delta \subset M_{\zeta,\xi}$. Consequently, $M_{\zeta,\xi} = \hat{S}_{(\zeta,\xi)}$ near $p$.

It follows that $M$ is the graph of a function $f : \overline{\Omega} \rightarrow \mathbb{R}_v$.

One proves, using Lemma 4.5, that $f$ is continuous on $\Omega$, whence on $\overline{\Omega} \setminus Q$, by Lemma 4.3 Then continuity at every $q_j$ is proved using the Kontinuitätssatz on the domain of holomorphy $\Omega \times i\mathbb{R}_v$.

4.5. Regularity. The property that $M \setminus P = (p_1, \ldots, p_4)$ is a smooth manifold with boundary results from:

**Lemma 4.8.** Let $U$ be a domain in $\mathbb{C}^{n-i}_z \times \mathbb{R}_w$, $n \geq 2$, and $f : U \rightarrow \mathbb{R}_v$ a continuous function. Let $A \subset \text{graph}(f)$ be a germ of complex analytic set of codimension 1. Then $A$ is a germ of complex manifold which is a graph over $\mathbb{C}^{n-i}_z$.

**Proof.** Assume that $A$ is a germ at 0. Let $h \in \mathcal{O}_{n+1}$, $h \neq 0$, be such that $A = \{h = 0\}$. For $\varepsilon \ll 1$, let $D_\varepsilon$ be the disc $\{z = 0\} \cap \{|w| < \varepsilon\}$. Then $A \cap D_\varepsilon = \{0\}$, i.e. $A$ is $w$-regular.

Let $\pi : \mathbb{C}^n_{z,w} \rightarrow \mathbb{C}^{n-1}_z$ be the projection. The local structure theorem for analytic sets gives:

- for some neighborhood $U$ of 0 in $\mathbb{C}^{n-1}_z$, there exists an analytic hypersurface $\Delta \subset U$ such that $A_{\Delta} = A \cap ((U \setminus \Delta) \times D_\varepsilon)$ is a manifold;
- $\pi : A_{\Delta} \rightarrow U \setminus \Delta$ is a $d$-sheeted covering ($d \in \mathbb{N}$).

It is easy to show that the covering $\pi : A_{\Delta} \rightarrow U \setminus \Delta$ is trivial.

Then we may define holomorphic functions $\tau_1, \ldots, \tau_d : U \setminus \Delta \rightarrow \mathbb{C}$ such that $A_{\Delta}$ is the union of the graphs of the $\tau_j$. By the Riemann extension theorem, the functions $\tau_j$ extend as holomorphic functions $\tau_j \in \mathcal{O}(U)$. Suppose that $\tau_j \neq \tau_k$ for $j \neq k$. Then for some disc $D \subset U$ centered at 0, we have $\tau_j|_D \neq \tau_k|_D$, so $(\tau_j - \tau_k)|_D$ vanishes only at 0. But, from the hypothesis, on restriction to $D$, $\{\text{Re}(\tau_j - \tau_k) = 0\} \subset \{\tau_j - \tau_k = 0\}|_D = \{0\}$, impossible.

4.6. Proof of Theorem 4.1. Consider the foliation of $S \setminus P$ given by the level sets of the smooth function $\nu : S \rightarrow [0,1]$ (Sections 2.3 and 2.7) and set $L_t = \{\nu = t\}$ for $t \in (0,1)$. Let $V_t \subset \overline{\Omega} \times i\mathbb{R}_v \subset \mathbb{C}^n$ be the complex leaf of $M$ bounded by $L_t$. 
By Proposition 4.7, $M$ is the graph of a continuous function over $\Omega$, and, by Lemma 4.8, each leaf $V_t$ is a complex smooth hypersurface and $\pi|_{V_t}$ is a submersion.

Since $\Omega$ is strictly convex, as in the situation studied by Shcherbina (see Lemma 4.6), $\pi|_{V_t}$ is 1-1, so, by Corollary 4.4, $\pi$ sends $V_t$ onto a domain $\Omega_t \subset \mathbb{C}^{n-1}_{z}$ with smooth boundary. Let

$$\pi_u : (\mathbb{C}^{n-1}_z \times \mathbb{R}_u) \times i\mathbb{R}_v \to \mathbb{R}_u, \quad \pi_v : (\mathbb{C}^{n-1}_z \times \mathbb{R}_u) \times i\mathbb{R}_v \to \mathbb{R}_v.$$  

Then $\pi_u|_{L_t} = a_t, \pi|_{L_t}$ and $\pi_v|_{L_t} = b_t, \pi|_{L_t}$ where $a_t, b_t$ are smooth functions on $b\Omega_t$. Moreover $b\Omega_t, a_t, b_t$ depend smoothly on $t$.

If $(z_t, w_t) \in M$, then $w_t$ varies on $V_t$, so $w_t$ is a holomorphic extension of $a_t + ib_t$ to $\Omega_t$. In particular $u_t$ and $v_t$ are smooth in $(z, t)$, from the Bochner–Martinelli formula. The function $\partial u_t/\partial t$ is harmonic on $\Omega_t$ for each $t$ and has a smooth extension on $b\Omega_t$.

From Lemma 4.3 and Corollary 4.4, $\partial u_t/\partial t$ does not vanish on $b\Omega_t$. Since the CR orbits $L_t$ are connected from Proposition 3.5, $b\Omega_t$ is also connected, hence $\partial u_t/\partial t$ has constant sign on $b\Omega_t$, so, by the maximum principle, also on $\Omega_t$ and, in particular, it does not vanish. This implies that $M \setminus S$ is the graph of a smooth function over $\Omega$ which smoothly extends to $\Omega \setminus Q$.

By Proposition 4.7, $M$ is the graph of a continuous function over $\Omega$.

4.7. Elementary smooth models. An elementary smooth model in $\mathbb{C}^n$ is an elementary model in the sense of Section 3.4 and satisfying the further condition which makes sense by Theorem 4.1:

(G) Let $(z, w)$ be the coordinates in $\mathbb{C}^{n-1}_n \times \mathbb{C}$, with $z = (z_1, \ldots, z_{n-1})$, $w = u + iv = z_n$, and let $\Omega$ be a strictly pseudoconvex domain in $\mathbb{C}^{n-1}_n \times \mathbb{R}_u$; assume that $T'$ is the graph of a smooth function $g : b\Omega \to \mathbb{R}_v$.

**Theorem 4.9.** Let $T$ be an elementary smooth model. Then there exists a continuous function $f : \overline{\Omega} \to \mathbb{R}_v$ which is smooth on $\overline{\Omega} \setminus Q$ and such that $f|_{b\Omega} = g$, and $M_0 = \text{graph}(f) \setminus S$ is a smooth Levi-flat hypersurface of $\mathbb{C}^n$; in particular, $S$ is the boundary of the hypersurface $M = \text{graph}(f)$.

**Proof.** Similar to the proof of Theorem 4.1.

Gluing of elementary smooth models. In an open set of $\mathbb{C}^n$, a coordinate system $(z, w)$ of $\mathbb{C}^{n-1}_n \times \mathbb{R}_u$ defines an $(n-1, 1)$-frame.

To define the gluing of elementary models (Section 3.5) we considered a CR-isomorphism from an open set of $\mathbb{C}^n$ induced by a holomorphic isomorphism of the ambient space $\mathbb{C}^n$ onto an open set in $\mathbb{C}^n$. To define the gluing of elementary smooth models, we have to consider a holomorphic isomorphism of the ambient space $\mathbb{C}^n$ onto an open set in $\mathbb{C}^n$ sending an $(n-1, 1)$-frame of $\mathbb{C}^{n-1}_n \times \mathbb{R}_u$ onto an $(n-1, 1)$-frame of $\mathbb{C}^{n-1}_n \times \mathbb{R}_{w'}$. 

As in Section 3.5, we will define a submanifold $S'$ of $X$ obtained by gluing elementary smooth models by induction on the number $m$ of models. An elementary smooth model $T$ in $X$ is the image of an elementary smooth model $T_0$ of $\mathbb{C}^n$ under an analytic isomorphism of a neighborhood of $T_0$ in $\mathbb{C}^n$ into $X$.

**Gluing an elementary smooth model $T$ of type $(d_1)$ to a free down-1-hyperbolic point of $S'$.** Every elementary smooth model is contained in a cylinder $b\Omega \times \mathbb{R}_v$ determined by $\Omega$ and an $(n-1,1)$-frame. Two sets $\Omega$ are compatible if either they coincide or one is part of the other.

The announced gluing is defined in the following way: there exists a CR-isomorphism $h_1$ from a neighborhood $V_1$ of $\sigma_1'$ induced by a holomorphic isomorphism of the ambient space $\mathbb{C}^n$ onto a neighborhood of $\sigma_1$ in $S'$. Let $k_1$ be a CR-isomorphism from a neighborhood $T_1''$ of $T_1'$ into $X$ such that $k_1|_{V_1} = h_1$, and there exists a common $(n-1,1)$-frame on which the corresponding sets $\Omega$ are compatible. Such a situation is possible as the example of the horned (almost everywhere) smooth sphere shows (Theorem 4.1).

Note that the gluing implies that the submanifold $S'$ is $C^0$ and smooth except at the complex points.

Other gluings are obtained in a similar way. Hence:

**Theorem 4.10.** The manifold $S'$ obtained by gluing elementary smooth models is of class $C^0$, and smooth except at the complex points.

**Corollary 4.11.** The manifold $S'$ is the boundary of a manifold $M$ of class $C^0$ whose interior is a Levi-flat smooth hypersurface.

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