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# Disc functionals and Siciak–Zaharyuta extremal functions on singular varieties

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*Dedicated to Józef Siciak on the occasion of his 80th birthday*

**Abstract.** We establish plurisubharmonicity of envelopes of certain classical disc functionals on locally irreducible complex spaces, thereby generalizing the corresponding results for complex manifolds. We also find new formulae expressing the Siciak–Zaharyuta extremal function of an open set in a locally irreducible affine algebraic variety as the envelope of certain disc functionals, similarly to what has been done for open sets in $\mathbb{C}^n$ by Lempert and by Lárusson and Sigurdsson.

**1. Introduction.** Let $\mathbb{D} = \{\zeta \in \mathbb{C} \colon |\zeta| < 1\}$ be the open unit disc in the complex plane $\mathbb{C}$, and let $\mathbb{T} = b\mathbb{D} = \{\zeta \in \mathbb{C} \colon |\zeta| = 1\}$ be its boundary circle.

Let $X$ be a (reduced, paracompact) complex space. Denote by $\mathrm{Psh}(X)$ the set of all plurisubharmonic functions on $X$. (For convenience we agree that the function which is identically equal to $-\infty$ is also plurisubharmonic.) Let $\mathcal{O}(\overline{\mathbb{D}}, X)$ be the set of all maps $f \colon \overline{\mathbb{D}} \to X$ that are holomorphic in an open neighborhood $U_f \subset \mathbb{C}$ of the closed disc $\overline{\mathbb{D}}$ in $\mathbb{C}$. Given a point $x \in X$, set $\mathcal{O}(\overline{\mathbb{D}}, X, x) = \{f \in \mathcal{O}(\overline{\mathbb{D}}, X) \colon f(0) = x\}$; these are holomorphic discs in $X$ centered at $x$. A *disc functional* on $X$ is a function

$$H_X \colon \mathcal{O}(\overline{\mathbb{D}}, X) \to \overline{\mathbb{R}} = [-\infty, +\infty].$$

The *envelope* of $H_X$ is the function $EH_X \colon X \to \overline{\mathbb{R}}$ defined by

$$EH_X(x) = \inf\{H_X(f) \colon f \in \mathcal{O}(\overline{\mathbb{D}}, X, x),\ f(\mathbb{D}) \not\subset X_{\mathrm{sing}}\}, \qquad x \in X. \tag{1.1}$$

Occasionally we consider disc functionals as functions on the larger class $\mathcal{A}_X = \mathcal{A}(\mathbb{D}, X)$ of discs $\overline{\mathbb{D}} \to X$ that are holomorphic in $\mathbb{D}$ and continu-

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ous on $\overline{\mathbb{D}}$. For all functionals treated in this paper, their envelope over $\mathcal{A}_X$ coincides with the envelope (1.1) over the subclass $\mathcal{O}(\overline{\mathbb{D}}, X)$ of $\mathcal{A}_X$.

The theory of disc functionals, initiated by Poletsky in the late 1980s [Po1], offers a different approach to certain extremal functions of pluripotential theory. (For the latter subject see Klimek [Kli].) In several natural examples, the envelope of a disc functional is a plurisubharmonic function. Furthermore, extremal plurisubharmonic functions are usually defined as suprema of classes of plurisubharmonic functions with certain properties, and many of them are envelopes of appropriate disc functionals. As was pointed out by Poletsky in [Po3], one may view this subject as an extension of Kiselman's minimum principle [Kis].

In this paper we extend results on plurisubharmonicity of certain classical disc functionals, obtained by various authors in the manifold case (i.e., when the underlying space $X$ is nonsingular), to complex spaces with singularities.

One of the most important disc functionals is the *Poisson functional*, which associates to an upper semicontinuous function $u$ on $X$ and an analytic disc $f \in \mathcal{A}_X$ the average of the function $u \circ f$ over the circle $\mathbb{T} = b\mathbb{D}$. A fundamental result of Poletsky is that the envelope of the Poisson functional on domains in $\mathbb{C}^n$ is always plurisubharmonic. In §2 we give another proof of our result from [DF] on plurisubharmonicity of Poisson functionals on any locally irreducible complex space, reducing it to Rosay's theorem in the manifold case by using Hironaka desingularization. The same proof also applies to Riesz and Lelong functionals (see §2 for the definitions); so we obtain the following result:

THEOREM 1.1. *Let $X$ be an irreducible and locally irreducible complex space, and let $H_X : \mathcal{O}(\overline{\mathbb{D}}, X) \to \overline{\mathbb{R}}$ be one of the following disc functionals:*

(i) *$P_u$, the Poisson functional corresponding to an upper semicontinuous function $u$ on $X$* (see (2.1));
(ii) *$R_u$, the Riesz functional corresponding to a plurisubharmonic function $u$ on $X$* (see (2.2))*;*
(iii) *$L_\alpha$, the Lelong functional, or $\widetilde{L}_\alpha$, the reduced Lelong functional, associated to a nonnegative function $\alpha$ on $X$* (see (2.4))*.*

*Then the envelope $EH_X$ in* (1.1) *is a plurisubharmonic function on $X$.*

The assumption of local irreducibility cannot be omitted. In §2 we give an example of an irreducible complex curve with a single double point such that the envelopes of the above functionals, corresponding to appropriately chosen functions, are not plurisubharmonic at that point.

On a complex manifold $X$, the envelopes of the disc functionals mentioned above are the following extremal plurisubharmonic functions:

- The envelope $EP_u$ of the Poisson functional is the largest plurisubharmonic minorant of the upper semicontinuous function $u$.
- The envelope $ER_u$ of the Riesz functional is
$$ER_u = \sup\{v \in \mathrm{Psh}(X)\colon v \le 0,\, dd^c v \ge dd^c u\},$$
the largest nonpositive plurisubharmonic function on $X$ whose Levi form is bounded below by the Levi form of $u$ [LS1].
- The envelope $EL_\alpha$ of the Lelong functional is the largest nonpositive plurisubharmonic function whose Lelong number at each point $x \in X$ is $\ge \alpha(x)$ (see §3 below).

In §3 we give a new treatment of the Lelong functional, simplifying the proof of plurisubharmonicity of its envelope that was given by Lárusson and Sigurdsson in [LS1, LS2]. The key point is obtained by the method of gluing holomorphic sprays of discs, similarly to what was done in [DF] for the Poisson functional. Our proof also applies to locally irreducible complex spaces without having to use the desingularization theorem.

In §4 we find a formula expressing the Siciak–Zaharyuta extremal function $V_{\Omega,X}$ of a nonempty open set $\Omega$ in a locally irreducible affine algebraic variety $X \subset \mathbb{C}^n$ as the envelope of appropriate Poisson functionals, obtained from Green functions on complex curves in $X$ with boundaries in $\Omega$. For open sets in $X = \mathbb{C}^n$ such formulas have been obtained by Lempert (in the case when $\Omega$ is convex) and by Lárusson and Sigurdsson.

**2. Plurisubharmonicity of envelopes of disc functionals.** Let $X$ be a complex space. Given an upper semicontinuous function $u\colon X \to \mathbb{R} \cup \{-\infty\}$, the associated *Poisson functional* is defined by

$$P_u(f) = \frac{1}{2\pi}\int_0^{2\pi} u(f(e^{it}))\,dt, \qquad f \in \mathcal{O}(\overline{\mathbb{D}}, X). \tag{2.1}$$

Let $u\colon X \to \mathbb{R} \cup \{-\infty\}$ be a plurisubharmonic function. The associated *Riesz functional* is given by

$$R_u(f) = \frac{1}{2\pi}\int_{\mathbb{D}} \log|\cdot|\,\Delta(u \circ f), \qquad f \in \mathcal{O}(\overline{\mathbb{D}}, X). \tag{2.2}$$

The Laplacian $\Delta g$ of the subharmonic function $g = u \circ f$ is a positive Borel measure on $\overline{\mathbb{D}}$. There is a close connection with the Poisson functional which derives from the following *Riesz representation formula* on the disc:

$$g(0) = \frac{1}{2\pi}\int_0^{2\pi} g(e^{it})\,dt + \frac{1}{2\pi}\int_{\mathbb{D}} \log|\cdot|\,\Delta g.$$

Applying this to the function $g = u \circ f$ on $\overline{\mathbb{D}}$, where $u$ and $f$ are as in (2.2), we obtain

$$u(f(0)) = P_u(f) + R_u(f).$$

Setting $x = f(0) \in X$, this can be rewritten as $R_u(f) = u(x) + P_{-u}(f)$. Taking the infimum over all $f \in \mathcal{O}(\overline{\mathbb{D}}, X, x)$ yields the following relation between the Riesz and the Poisson envelopes:

$$ER_u = u + EP_{-u}. \tag{2.3}$$

Therefore, to prove that $ER_u$ is plurisubharmonic, we need to show that $EP_{-u} < \infty$ and that $EP_{-u}$ is plurisubharmonic on $X$.

If $v \le 0$ is a plurisubharmonic function on $X$ such that $dd^c v \ge dd^c u$, then for every disc $f \in \mathcal{O}(\overline{\mathbb{D}}, X)$ we have $\Delta(v \circ f) \ge \Delta(u \circ f)$, and hence $R_v(f) \le R_u(f)$. Since $P_v(f) \le 0$, the Riesz formula gives

$$v(f(0)) = P_v(f) + R_v(f) \le R_v(f) \le R_u(f).$$

By taking the infimum over all discs with a given center we get $v \le ER_u$. Once we know that $ER_u$ is plurisubharmonic, it follows that it is the biggest plurisubharmonic function $v \le 0$ satisfying $dd^c v \ge dd^c u$.

The *Lelong functional* associated to a nonnegative real function $\alpha$ on $X$ is defined by

$$L_\alpha(f) = \sum_{z \in \mathbb{D}} \alpha(f(z)) m_f(z) \log|z|, \qquad f \in \mathcal{O}(\overline{\mathbb{D}}, X), \tag{2.4}$$

where $m_f(z)$ denotes the multiplicity of $f$ at $z$. We get the *reduced Lelong functional*, $\widetilde{L}_\alpha$, by removing the multiplicities $m_f(z)$ from the above formula.

Plurisubharmonicity of envelopes of these functionals on domains in $\mathbb{C}^n$ was established by Poletsky [Po1, Po2]; similar results for the Poisson functional were found by Bu and Schachermayer [BS]. Poletsky's theorem was extended to all complex manifolds for the Poisson functional by Rosay [Ro1, Ro2] (see also Edigarian [Ed3]), and then for the other functionals mentioned above by Lárusson and Sigurdsson [LS1, LS2] and Edigarian [Ed2]. The envelope of the Lelong functional coincides with the envelope of the corresponding reduced Lelong functional [LS2]; see Theorem 3.1 below.

In [DF] we proved that the envelope of the Poisson functional is plurisubharmonic if $X$ is a locally irreducible complex space.

THEOREM 2.1 ([DF, Theorem 1.1]). *Let $X$ be a locally irreducible complex space and $u\colon X \to \mathbb{R} \cup \{-\infty\}$ an upper semicontinuous function. Then the envelope*

$$\widehat{u}(x) = \inf\left\{ \int_0^{2\pi} u(f(e^{it}))\,\frac{dt}{2\pi} : f \in \mathcal{O}(\overline{\mathbb{D}}, X, x)\right\}, \qquad x \in X, \tag{2.5}$$

*of the Poisson functional $P_u$ is the largest plurisubharmonic minorant of $u$.*

The proof given in [DF] is no more difficult than Rosay's proofs in [Ro1, Ro2] for the case when $X$ is a complex manifold; it combines Poletsky's proof on $X = \mathbb{C}^n$ with the method of gluing holomorphic sprays of discs. (For an exposition of the latter method we refer to [For, §5.8–§5.9].) We use this opportunity to give another proof of Theorem 2.1, reducing it to the case when $X$ is a complex manifold by applying the Hironaka desingularization theorem [Hir, AHV, BM]. The latter states that for every paracompact reduced complex space, $X$, there is a proper holomorphic surjection $\pi : M \to X$ with the following properties:

- $M$ is a complex manifold,
- $\pi : M \setminus \pi^{-1}(X_{\mathrm{sing}}) \to X \setminus X_{\mathrm{sing}}$ is a biholomorphism, and
- $\pi^{-1}(X_{\mathrm{sing}})$ is a complex hypersurface in $M$.

We will prove the following more general result on envelopes of disc functionals, showing that the only problem with plurisubharmonicity is at points where the complex space is locally reducible.

THEOREM 2.2. *Let $X$ be a complex space and let $\pi : M \to X$ be a desingularization of $X$. Given a disc functional $H_X : \mathcal{O}(\overline{\mathbb{D}}, X) \to \overline{\mathbb{R}}$, we define a disc functional $H_M = \pi^* H_X : \mathcal{O}(\overline{\mathbb{D}}, M) \to \overline{\mathbb{R}}$ by*

$$H_M(g) = H_X(\pi \circ g) \quad \text{for each } g \in \mathcal{O}(\overline{\mathbb{D}}, M). \tag{2.6}$$

*If the envelope $EH_M$ is plurisubharmonic on $M$, then the envelope $EH_X$ given by* (1.1) *is plurisubharmonic on the regular part $X_{\mathrm{reg}}$ of $X$, and*

$$EH_X(x) = \inf\{EH_M(p) : p \in \pi^{-1}(x)\} \quad \text{for each } x \in X_{\mathrm{sing}}. \tag{2.7}$$

*If $X$ is locally irreducible at a point $x \in X_{\mathrm{sing}}$, then $EH_X$ is plurisubharmonic in a neighborhood of $x$ in $X$.*

We shall need the following lemma on lifting holomorphic discs to a desingularization. For the sake of completeness we include the proof.

LEMMA 2.3. *Let $X$ be a complex space and let $\pi : M \to X$ be a desingularization of $X$. Given a holomorphic disc $f \in \mathcal{O}(\overline{\mathbb{D}}, X)$ such that $f(\mathbb{D}) \not\subset X_{\mathrm{sing}}$ there exists a unique holomorphic disc $g \in \mathcal{O}(\overline{\mathbb{D}}, M)$ such that $\pi \circ g = f$.*

*Proof.* Fix $f \in \mathcal{O}(\overline{\mathbb{D}}, X)$ such that $f(\mathbb{D}) \not\subset X_{\mathrm{sing}}$. Let $U_f$ be an open connected neighborhood of $\overline{\mathbb{D}}$ on which $f$ is holomorphic. Then $S = f^{-1}(X_{\mathrm{sing}})$ is a discrete subset of $U_f$. Since the map $\pi : M \to X$ is biholomorphic on $M \setminus \pi^{-1}(X_{\mathrm{sing}})$, there is a unique holomorphic map $g = \pi^{-1} \circ f : U_f \setminus S \to M$ such that $\pi \circ g = f$ on $U_f \setminus S$. We need to show that $g$ extends holomorphically across $S$. Pick a point $s \in S$ and a connected open neighborhood $U \subset U_f$ of $s$ such that $U \cap S = \{s\}$. By shrinking $U$ around $s$ we may assume that the image $C = f(U) \subset X$ is an irreducible closed complex curve in an open neighborhood $V \subset X$ of the image point $f(s) = x \in X_{\rm sing}$. Its preimage $\pi^{-1}(C)$ is then a closed complex subvariety of the open set $\pi^{-1}(V) \subset M$. Observe that $\pi^{-1}(C) \setminus \pi^{-1}(x) = g(U \setminus \{s\})$. Since the closure of the difference of two subvarieties is again a subvariety (see e.g. [Chi, Corollary, p. 53]), we infer that $\Sigma := \overline{g(U \setminus \{s\})}$ is a pure one-dimensional complex subvariety of $\pi^{-1}(V)$ projecting onto $C$. (This is the proper transform of $C$ in $M$.) Since $g(U\setminus\{s\})$ is connected, the set $\Sigma \cap \pi^{-1}(x)$ consists of precisely one point, say $p$, and setting $g(s) = p$ extends the map $g$ holomorphically to the point $s$. ∎

*Proof of Theorem 2.2.* Assume that the envelope $EH_M$ is plurisubharmonic. Since $\pi$ is biholomorphic over $X_{\rm reg}$, the function $EH_M|\pi^{-1}(X_{\rm reg})$ passes down to a plurisubharmonic function on $X_{\rm reg}$:

$$EH_M = w \circ \pi \tag{2.8}$$

for some function $w : X_{\rm reg} \to \mathbb{R} \cup \{-\infty\}$. To see that $w = EH_X$ on $X_{\rm reg}$, choose a point $x \in X_{\rm reg}$ and let $p \in M$ be the unique point with $\pi(p) = x$. Every analytic disc $f \in \mathcal{O}(\overline{\mathbb{D}}, X)$ with $f(0) = x$ satisfies $f(\mathbb{D}) \not\subset X_{\rm sing}$, and by Lemma 2.3 it lifts to a disc $g \in \mathcal{O}(\overline{\mathbb{D}}, M)$ centered at $g(0) = p$ so that $\pi \circ g = f$. In particular, every disc $g \in \mathcal{O}(\overline{\mathbb{D}}, M)$ with $g(0) = p \in \pi^{-1}(X_{\rm reg})$ is the unique lifting of its projection $f = \pi \circ g$. By taking the infimum over all discs $f \in \mathcal{O}(\overline{\mathbb{D}}, X, x)$, the above implies in view of (2.6) and (2.8) that $w(x) = EH_M(p) = EH_X(x)$. This shows that $w = EH_X$ on $X_{\rm reg}$.

Consider now a point $x \in X_{\rm sing}$. Since any disc $f \in \mathcal{O}(\overline{\mathbb{D}}, X, x)$ such that $f(\mathbb{D}) \not\subset X_{\rm sing}$ lifts to a disc $g \in \mathcal{O}(\overline{\mathbb{D}}, M)$ centered at some point $p \in \pi^{-1}(x)$, we deduce by (2.6) and (2.8) that $EH_M(p) \le H_M(g) = H_X(f)$. Taking the infimum over all $f \in \mathcal{O}(\overline{\mathbb{D}}, X, x)$ with $f(\mathbb{D}) \not\subset X_{\rm sing}$ we infer that

$$\alpha := \inf\{EH_M(p) : p \in \pi^{-1}(x)\} \le EH_X(x).$$

To get the converse inequality, pick $\epsilon > 0$ and choose a point $p \in \pi^{-1}(x) \in M$ such that $EH_M(p) < \alpha + \epsilon$. There is disc $g \in \mathcal{O}(\overline{\mathbb{D}}, M, p)$ satisfying

$$H_M(g) < EH_M(p) + \epsilon < \alpha + 2\epsilon.$$

By moving $g$ slightly, keeping its center fixed, we may assume that $g(\mathbb{D})$ is not contained in the hypersurface $\pi^{-1}(X_{\rm sing})$. Then the image of the disc $f = \pi \circ g \in \mathcal{O}(\overline{\mathbb{D}}, X, x)$ is not contained in $X_{\rm sing}$. The above inequality together with (2.6) and (2.8) implies

$$H_X(f) = H_M(g) < EH_M(p) + \epsilon < \alpha + 2\epsilon,$$

and therefore $EH_X(x) < \alpha + 2\epsilon$. Since $\epsilon > 0$ was arbitrary, we have $EH_X(x) \le \alpha = \inf\{EH_M(p) : p \in \pi^{-1}(x)\}$, which proves (2.7).

Assume now that $X$ is locally irreducible at a point $x \in X_{\rm sing}$. Pick a point $p \in M$ with $\pi(p) = x$. Since $EH_M$ is upper semicontinuous and

$\pi^{-1}(X_{\rm sing})$ is a hypersurface in $M$, we have

$$EH_M(p) = \limsup_{q\in\pi^{-1}(X_{\rm reg}),\, q\to p} EH_M(q).$$

Local irreducibility of $X$ at the point $x$ implies that the fiber $\pi^{-1}(x)$ is a connected compact analytic set, and therefore any plurisubharmonic function on $M$ is constant on $\pi^{-1}(x)$. This implies in view of (2.7) that

$$EH_X(x) = \limsup_{x'\in X_{\rm reg},\, x'\to x} EH_X(x').$$

A theorem of Demailly [Dem, Théorème 1.7] now shows that the function $EH_X$, being plurisubharmonic on $X_{\rm reg}$, is also plurisubharmonic in a neighborhood of $x$ in $X$. ∎

Theorem 1.1 now follows from Theorem 2.2 and the following lemma.

LEMMA 2.4. *Let $M$ and $X$ be complex spaces and $\pi : M \to X$ a holomorphic map. Let $H_X : \mathcal{O}(\overline{\mathbb{D}}, X) \to \overline{\mathbb{R}}$ be one of the following disc functionals:*

- (i) *$P_u$, the Poisson functional corresponding to an upper semicontinuous function $u$ on $X$,*
- (ii) *$P_v$, the Poisson functional corresponding to a plurisuperharmonic function $v$ on $X$,*
- (iii) *$\widetilde{L}_\alpha$, the reduced Lelong functional associated to a nonnegative function $\alpha$ on $X$.*

*The disc functional $H_M : \mathcal{O}(\overline{\mathbb{D}}, M) \to \overline{\mathbb{R}}$ defined by*

$$H_M(g) = H_X(\pi\circ g), \qquad g \in \mathcal{O}(\overline{\mathbb{D}}, M), \tag{2.9}$$

*is then of the same kind* (i)–(iii), *and, if $M$ is nonsingular, then $EH_M$ is plurisubharmonic on $M$.*

*Proof.* If $H_X = P_u$ is the Poisson functional corresponding to an upper semicontinuous function $u$ on $X$, then $H_M$ is the Poisson functional corresponding to the upper semicontinuous function $u\circ\pi$ on $M$, since

$$P_{u\circ\pi}(g) = \frac{1}{2\pi}\int_0^{2\pi} u(\pi(g(e^{it})))\,dt = P_u(\pi\circ g)$$

for each $g \in \mathcal{O}(\overline{\mathbb{D}}, M)$. A similar argument applies in case (ii). If $H_X = \widetilde{L}_\alpha$ is the reduced Lelong functional associated to a nonnegative function $\alpha$ on $X$, then $H_M$ is the reduced Lelong functional corresponding to the nonnegative function $\alpha\circ\pi$ on $M$:

$$\widetilde{L}_{\alpha\circ\pi}(g) = \sum_{\zeta\in\mathbb{D}} \alpha(\pi(g(\zeta)))\log|\zeta| = \widetilde{L}_\alpha(\pi\circ g), \qquad g\in\mathcal{O}(\overline{\mathbb{D}}, M).$$

By the results cited above, the envelopes of all these disc functionals are plurisubharmonic on $M$ if $M$ is smooth. ∎

Remark 2.5. In this paper we defined the envelope of a disc functional at a point $x \in X$ as the infimum over all discs centered at $x$ and not entirely lying in the singular locus $X_{\mathrm{sing}}$, whereas in [DF] we considered the infimum with respect to *all discs* centered at $x$. The two envelopes clearly coincide on $X_{\mathrm{reg}}$, and if they are plurisubharmonic then they coincide on all of $X$.

Example 2.6. We show that the conclusion of Theorem 1.1 fails in general at a point where $X$ is not locally irreducible. Such examples can already be found in the simplest case when $X$ is a complex curve with a single double point. To be explicit, consider the map $f : \mathbb{C} \to \mathbb{C}^2$ given by $f(z) = (z^3, z^2 + z)$. It is easily seen that $f$ is a proper holomorphic immersion whose only double point is $p = f(\omega_1) = f(\omega_2)$, where $\omega_1 \neq \omega_2$ are the two nonreal solutions of the equation $z^3 = 1$. The immersed complex curve $X = f(3\mathbb{D}) \subset \mathbb{C}^2$ has exactly one self-intersection (at the point $p$), and $f : 3\mathbb{D} \to X$ is a desingularization of $X$.

Choose a smooth convex function $v \leq 0$ on the disc $3\overline{\mathbb{D}}$ such that $v = 0$ on $b(3\mathbb{D})$ and $v(\omega_1) \neq v(\omega_2)$, and a linear function $A : \mathbb{C} \to \mathbb{R}$ such that $v + A \leq 0$ on $3\mathbb{D}$ and $(v + A)(\omega_1) = (v + A)(\omega_2)$. It is easily seen that $EP_{-(v+A)} = -A$ on $3\mathbb{D}$. The subharmonic function $v + A$ descends to a smooth subharmonic function $v_X \leq 0$ on $X$, and we deduce by (2.7) that

$$EP_{-v_X}(p) = \min\{EP_{-(v+A)}(\omega_1), EP_{-(v+A)}(\omega_2)\} = \min\{-A(\omega_1), -A(\omega_2)\}.$$

Since these two values are different, we see that the Poisson envelope $EP_{-v_X}$ is not upper semicontinuous at the point $p$. As $ER_{v_X} = v_X + EP_{-v_X}$ and $v_X$ is continuous on $X$, we also see that the envelope of the Riesz functional, $ER_{v_X}$, is not upper semicontinuous at $p$.

For the Lelong functional, choose a function $\alpha \geq 0$ on $3\mathbb{D}$ with $\alpha(\omega_1) = \alpha(\omega_2) = 0$ such that $EL_\alpha(\omega_1) \neq EL_\alpha(\omega_2)$. Then $\alpha$ descends to a function $\alpha_X \geq 0$ on $X$ and $EL_{\alpha_X}(p) = \min\{EL_\alpha(\omega_1), EL_\alpha(\omega_2)\}$, so $EL_{\alpha_X}$ is not upper semicontinuous at $p$.

**3. Envelopes of Lelong functionals.** In this section we give a new and simpler treatment of Lelong functionals (2.4) that have been considered earlier by Poletsky [Po2, Po3] any by Lárusson and Sigurdsson [LS1, LS2]. For simplicity we consider the case when $X$ is a complex manifold, although the methods and results also apply on the regular locus of any complex space. (In the latter case, with $X$ locally irreducible at each point, the proof of plurisubharmonicity of Lelong envelopes can be concluded as in Theorem 2.2.)

Given a plurisubharmonic function $u \in \mathrm{Psh}(X)$, we denote by $\nu_u(x) \in [0, +\infty]$ its Lelong number at a point $x \in X$. Recall that in any local coordinate system $z$ on $X$, with $z(x) = 0$, we have

$$\nu_u(x) = \lim_{r\to 0} \frac{\sup_{|z|\le r} u(z)}{\log r}.$$

(We consider the function $u = -\infty$ as plurisubharmonic and set $\nu_{-\infty} = +\infty$. Lelong numbers can also be defined for functions on complex spaces.)

Given a nonnegative function $\alpha : X \to \mathbb{R}_+$, let

$$\mathcal{F}_\alpha = \{u \in \mathrm{Psh}(X) : u \le 0,\ \nu_u \ge \alpha\}.$$

The goal is to identify the corresponding extremal function

$$v_\alpha = \sup\{u : u \in \mathcal{F}_\alpha\} \tag{3.1}$$

as the envelope of certain disc functionals which arise naturally from the following considerations. (Compare with [LS1, §5].)

If $u \in \mathcal{F}_\alpha$ and $f \in \mathcal{A}_X$, then clearly $u \circ f \le 0$ is a subharmonic function on the disc $\mathbb{D}$ whose Lelong number at any point $z \in \mathbb{D}$ satisfies

$$\nu_{u\circ f}(z) \ge \alpha(f(z))m_f(z),$$

where $m_f(z)$ denotes the multiplicity of $f$ at the point $z$. Hence $u \circ f$ is bounded above by the largest subharmonic function $v = v_{\alpha,f} \le 0$ on $\mathbb{D}$ satisfying $\nu_v \ge (\alpha \circ f)m_f$. This extremal function $v$ is the weighted sum of Green functions with coefficients $(\alpha \circ f)m_f$:

$$v(\zeta) = \sum_{z\in\mathbb{D}} \alpha(f(z))m_f(z) \log\left|\frac{\zeta - z}{1 - \bar{z}\zeta}\right|, \qquad \zeta \in \mathbb{D}.$$

(If the sum is divergent then $v \equiv -\infty$.) Indeed, the difference between $v$ and the right hand side above is subharmonic on $\overline{\mathbb{D}}$, except perhaps at the points $z$ where $\alpha(f(z)) > 0$; near these points it is bounded above, so it extends to a subharmonic function on $\mathbb{D}$. Since it is clearly $\le 0$ on $b\mathbb{D}$, the maximum principle implies that it is $\le 0$ on all of $\mathbb{D}$, which proves the claim.

Setting $\zeta = 0$, we see that for every $u \in \mathcal{F}_\alpha$ and $f \in \mathcal{A}_X$ we have

$$u(f(0)) \le \sum_{z\in\mathbb{D}} \alpha(f(z))m_f(z)\log|z| \le \sum_{z\in\mathbb{D}} \alpha(f(z))\log|z| \le \inf_{z\in\mathbb{D}} \alpha(f(z))\log|z|.$$

(The second and the third inequalities are trivial.) These expressions determine the following disc functionals on $X$ with values in $[-\infty, 0]$:

$$L_\alpha(f) = \sum_{z\in\mathbb{D}} \alpha(f(z))m_f(z)\log|z|,$$
$$\widetilde{L}_\alpha(f) = \sum_{z\in\mathbb{D}} \alpha(f(z))\log|z|,$$
$$K_\alpha(f) = \inf_{z\in\mathbb{D}} \alpha(f(z))\log|z|.$$

The first two are the *Lelong functional*, $L_\alpha$, and the *reduced Lelong functional*, $\widetilde{L}_\alpha$, which have already been mentioned in §2. By taking infima over all analytic discs $f$ in $X$ with a fixed center $f(0) = x \in X$ we obtain the corresponding inequalities for their envelopes:

$$u \le EL_\alpha \le E\widetilde{L}_\alpha \le EK_\alpha =: k_\alpha, \quad \forall u \in \mathcal{F}_\alpha. \tag{3.2}$$

The function $k_\alpha = EK_\alpha : X \to [-\infty, 0]$, which is denoted $k_X^\alpha$ in [LS1, p. 21], is related to a certain function studied by Edigarian [Ed1]. It is easily seen that $k_\alpha$ is upper semicontinuous (it suffices to move the center $f(0)$ of the test disc, while at the same time fixing the value $f(z) \in X$ at a point $z \in \mathbb{D}$ where $\alpha(f(z)) \log|z|$ is close to optimal), and that its Lelong numbers are bounded below by $\alpha$ [LS1, Proposition 5.2]. Hence the Poisson envelope $EP_{k_\alpha}$ (which is the largest plurisubharmonic minorant of $k_\alpha$ according to Theorem 2.1) also has Lelong numbers bounded below by $\alpha$, and so it belongs to the class $\mathcal{F}_\alpha$. Since $u \le k_\alpha$ for every $u \in \mathcal{F}_\alpha$ by (3.2), it follows that $EP_{k_\alpha} = v_\alpha$ is the extremal function (3.1). Furthermore, and this is the only nontrivial thing that remains to be seen, the envelopes $EL_\alpha$ and $E\widetilde{L}_\alpha$ also equal the extremal function $v_\alpha$.

Summarizing the above discussion, we have the following result.

Theorem 3.1. *For every function $\alpha \ge 0$ on a complex manifold $X$ the extremal function $v_\alpha$ in* (3.1) *is plurisubharmonic and equals the envelope of both the Lelong and the generalized Lelong functionals:*

$$v_\alpha = EL_\alpha = E\widetilde{L}_\alpha = EP_{k_\alpha}. \tag{3.3}$$

*If $X$ is a complex space then these equalities hold on the regular locus $X_{\rm reg}$; if $X$ is locally irreducible at every point, then they hold on all of $X$.*

On manifolds this was proved by Lárusson and Sigurdsson, first for domains in Stein manifolds [LS1], and then, following the work of Rosay [Ro1, Ro2], on all complex manifolds [LS2]. We find their proof rather difficult even for domains in Stein manifolds. Here we give a direct proof of the equalities (3.3) on the regular locus of any complex space. On locally irreducible complex spaces the result then follows by the arguments in §2.

We shall need the following elementary lemma.

Lemma 3.2. *Let $J$ be a union of finitely many closed arcs in the circle $\mathbb{T} = b\mathbb{D}$, let $U \subset \mathbb{C}$ be an open set containing $J$, and let $\zeta : U \cap \overline{\mathbb{D}} \to \mathbb{C}$ be a continuous function satisfying $0 < |\zeta(z)| < 1$ for $z \in U \cap \overline{\mathbb{D}}$. Given $\epsilon > 0$, the following inequality holds for all sufficiently large integers $k \in \mathbb{N}$:*

$$\sum_{z \in U\cap\mathbb{D},\, z^k = \zeta(z)} \log|z| < \int_{e^{it}\in J} \log|\zeta(e^{it})| \frac{dt}{2\pi} + \epsilon. \tag{3.4}$$

*Proof.* This is obvious when $\zeta(z) = a$ is a constant function. In that case the equation $z^k = a$ has $k$ solutions

$$z_j(a) = \sqrt[k]{|a|} e^{i(\theta_0 + j/2k\pi)}, \quad j = 1, \dots, k,$$

where $a = |a|e^{ik\theta_0}$. Since the points $z_1(a), \ldots, z_k(a)$ are equidistributed along the circle $|z| = \sqrt[k]{|a|}$, at least $k|J|$ of them belong to $U \cap \mathbb{D}$ if $k$ is large enough. (Here $|J|$ denotes the normalized arc length of $J$.) This gives

$$\sum_{z_j(a) \in U \cap \mathbb{D}} \log|z_j(a)| \le k|J| \log \sqrt[k]{|a|} = \log|a| \cdot |J| = \int_{e^{it} \in J} \log|a| \,\frac{dt}{2\pi}.$$

In the general case we break $J$ into pairwise disjoint closed segments $J_1, \ldots, J_m$ (separated by short gaps) such that, for some choice of points $e^{it_j} \in J_j$ and open pairwise disjoint sets $U_j$ with $J_j \subset U_j \Subset U$, we have

$$\sum_{j=1}^{m} \log|\zeta(e^{it_j})| \cdot |J_j| < \int_{e^{it} \in J} \log|\zeta(e^{it})| \,\frac{dt}{2\pi} + \frac{\epsilon}{2} \tag{3.5}$$

and

$$|\zeta(z) - \zeta(e^{it_j})| < \frac{\epsilon}{2} |\zeta(e^{it_j})|, \qquad z \in U_j \cap \overline{\mathbb{D}}. \tag{3.6}$$

It suffices to show that for every $j = 1, \ldots, m$ we have

$$\sum_{z \in U_j,\ z^k = \zeta(z)} \log|z| < \log|\zeta(e^{it_j})| \cdot |J_j| + \frac{\epsilon}{2}|J_j|. \tag{3.7}$$

Indeed, by summing the inequalities (3.7) over all $j = 1, \ldots, m$ and using (3.5) we obtain the estimate (3.4).

We now prove (3.7). Fix $j \in \{1, \ldots, m\}$ and write $a_j = \zeta(e^{it_j})$, so $0 < |a_j| < 1$. Let $\Delta_j \subset \mathbb{C}$ be the open disc of radius $r_j = (\epsilon/2)|a_j|$ centered at $a_j$. By choosing $\epsilon > 0$ small enough we may assume that $\overline{\Delta}_j \subset \mathbb{D}^* = \mathbb{D} \setminus \{0\}$ for each $j$. Since the map $\mathbb{D}^* \to \mathbb{D}^*$, $z \mapsto z^k$, is a $k$-fold covering, the preimage of $\overline{\Delta}_j$ is a disjoint union of $k$ simply connected closed domains (discs) $\overline{\Delta}_{j,l} \subset \mathbb{D}^*$, $l = 1, \ldots, k$. As $k \to +\infty$, the discs $\Delta_{j,l}$ converge to the circle $\mathbb{T}$ and are equidistributed around $\mathbb{T}$. For $k$ large enough at least the proportional number $k|J_j|$ of the discs $\overline{\Delta}_{j,l}$ are contained in $U_j \cap \mathbb{D}$. Let $\Delta_{j,l}$ be such a disc. As the point $z$ traces the boundary $b\Delta_{j,l}$ in the positive direction, the image point $z^k$ traces $b\Delta_j$ once in the positive direction. From the estimate (3.6) we infer that the function $z \mapsto z^k - \zeta(z)$ has winding number 1 around $b\Delta_{j,l}$, and hence the equation $z^k = \zeta(z)$ has a solution $z = z_{j,l}$ in $\Delta_{j,l}$. (If the function $z \mapsto \zeta(z)$ is holomorphic, as will be the case in our application, then there is precisely one solution in $\Delta_{j,l}$ by Rouché's theorem.) Clearly this solution satisfies

$$\log|z_{j,l}| = \frac{1}{k} \log|\zeta(z_{j,l})| < \frac{1}{k} \log\left(|a_j|\left(1 + \frac{\epsilon}{2}\right)\right) < \frac{1}{k}\left(\log|a_j| + \frac{\epsilon}{2}\right).$$

Since there are at least $k|J_j|$ solutions $z_{j,l}$, the sum of their logarithms is bounded above by $|J_j|(\log|a_j| + \epsilon/2)$, which gives (3.7). ∎

*Proof of Theorem 3.1.* In view of (3.2) and the equality $\sup_{u\in\mathcal{F}_\alpha} u = EP_{k_\alpha}$ (see the paragraph preceding Theorem 3.1) we have

$$EP_{k_\alpha} \le EL_\alpha \le E\widetilde{L}_\alpha \le k_\alpha.$$

To establish (3.3) it remains to prove that $E\widetilde{L}_\alpha \le EP_{k_\alpha}$. Equivalently, we need to show that for every continuous function $\phi : X \to \mathbb{R}$ with $\phi \ge k_\alpha$, analytic disc $h \in \mathcal{A}_X$, and number $\epsilon > 0$ there exists a disc $f \in \mathcal{A}_X$ such that $f(0) = h(0)$ and

$$\widetilde{L}_\alpha(f) = \sum_{z\in\mathbb{D}} \alpha(f(z)) \log|z| < \frac{1}{2\pi}\int_0^{2\pi} (\phi\circ h)(e^{it})\, dt + \epsilon. \tag{3.8}$$

The definition of the Poisson envelope $EP_{k_\alpha}$ of the function $k_\alpha$ shows that for every fixed $t_0 \in \mathbb{R}$ there exist an analytic disc $g_0 \in \mathcal{A}_X$ and a point $b_0 \in \mathbb{D}$ such that $g_0(0) = h(e^{it_0}) =: x_0$ and

$$\alpha(g_0(b_0)) \log|b_0| < \phi(x_0) + \frac{\epsilon}{2}.$$

We embed $g_0$ into a family (spray) of analytic discs $g_x = g(x,\cdot) \in \mathcal{A}_X$, depending holomorphically on the point $x$ in an open neighborhood $V \subset X$ of $x_0$, such that $g_0 = g(x_0,\cdot)$ is the initial disc, and for all $x \in V$ we have $g(x,0) = x$ and $g(x,b_0) = g(x_0,b_0) =: y_0$. By continuity there is a nontrivial closed arc $J \subset \mathbb{T}$ around the point $e^{it_0}$ such that $h(J) \subset V$ and

$$\alpha(y_0)\log|b_0|\cdot|J| < \int_J (\phi\circ h)(e^{it})\frac{dt}{2\pi} + \frac{\epsilon}{2}|J|.$$

Repeating this argument at other points of the circle $\mathbb{T}$ we find

- pairwise disjoint closed arcs $J_1,\ldots,J_m \subset \mathbb{T}$ with arbitrarily short gaps between them,
- points $e^{it_j} \in J_j$,
- open sets $V_1,\ldots,V_m \subset X$ with $h(J_j) \subset V_j$ for $j = 1,\ldots,m$,
- holomorphic sprays of discs $g_j : V_j \times \overline{\mathbb{D}} \to X$, and
- points $b_1,\ldots,b_m \in \mathbb{D}$,

such that:

(a) $g_j(x,0) = x$ for all $x \in V_j$ and $j = 1,\ldots,m$,
(b) the point $y_j = g_j(x,b_j) \in X$ is independent of $x \in V_j$, and

$$\sum_{j=1}^m \alpha(y_j)\log|b_j|\cdot|J_j| < \int_0^{2\pi} (\phi\circ h)(e^{it})\,\frac{dt}{2\pi} + \frac{\epsilon}{2}. \tag{3.9}$$

The integral of $\phi\circ h$ over $\mathbb{T}\setminus\bigcup_{j=1}^m J_j$ is made small by choosing the arcs

$J_j$ such that the measure of the complement is sufficiently small. We may assume that $\alpha(y_j) > 0$ for each $j$, as otherwise the corresponding term and the arc $J_j$ may be deleted.

Choose smoothly bounded simply connected sets (discs) $\Delta_1, \dots, \Delta_m$ in $\mathbb{D}$ with pairwise disjoint closures such that $b\Delta_j \cap \mathbb{T}$ contains a relative neighborhood of the arc $J_j$ in the circle $\mathbb{T} = b\mathbb{D}$. By choosing the discs $\Delta_j$ small enough we can also ensure that $h(\overline{\Delta}_j) \subset V_j$. Pick a larger arc $J'_j \subset \mathbb{T}$ such that $J_j \Subset J'_j \Subset b\Delta_j \cap \mathbb{T}$. Set $D_1 = \bigcup_{j=1}^m \Delta_j$. Let $D_0 \subset \mathbb{D}$ be a domain obtained by denting the circle $\mathbb{T}$ slightly inward along each of the arcs $J'_j$ so as to ensure that $\overline{D}_0 \cap J'_j = \emptyset$ for all $j = 1, \dots, m$, $D_0 \cup D_1 = \mathbb{D}$, and $\overline{D_0 \setminus D_1} \cap \overline{D_1 \setminus D_0} = \emptyset$. Thus $(D_0, D_1)$ is a Cartan pair in the sense of [For, §5.7]. The configuration around the disc $\Delta_j$ is shown in Figure 1.

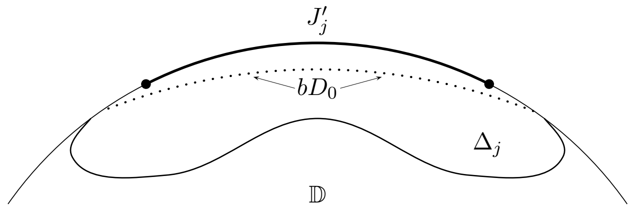


Fig. 1. The disc $\Delta_j$ and the arc $J'_j$

Pick a large constant $M > 0$ whose precise value will be fixed later. For every $j = 1, \dots, m$ let $u_j \le 0$ be a smooth real function on $\overline{\Delta}_j$, harmonic in $\Delta_j$, such that $u_j = 0$ on $J_j$, and $u_j = -M$ on $b\Delta_j \setminus J'_j$. Let $v_j$ be a harmonic conjugate of $u_j$, and define the function $a_j : \overline{\Delta}_j \to \mathbb{C}$ by

$$a_j(z) = z^k e^{u_j(z) + iv_j(z)}, \qquad z \in \overline{\Delta}_j.$$

The value of the integer $k \in \mathbb{N}$ will be fixed later. Note that $|a_j(z)| \le e^{u_j(z)}$ for every $k$, and this is uniformly as close to zero as desired outside of any neighborhood of $J'_j$ if the constant $M > 0$ is large enough.

Let $z_{j,1}, \dots, z_{j,l_j} \in \Delta_j$ be all the solutions of the equation $a_j(z) = b_j$ in the disc $\Delta_j$. (Recall that the number $b_j \in \mathbb{D}^*$ satisfies condition (b) above.) This equation can be rewritten as

$$z^k = b_j e^{-u_j(z) - iv_j(z)} =: \zeta_j(z).$$

Note that $|\zeta_j(e^{it})| = |b_j|$ for $e^{it} \in J_j$. By Lemma 3.2 we know for all large enough $k \in \mathbb{N}$ that

$$\sum_{l=1}^{l_j} \log|z_{j,l}| < \int_{J_j} \log|\zeta_j(e^{it})| \frac{dt}{2\pi} + \frac{\epsilon |J_j|}{2\alpha(y_j)} = \log|b_j| \cdot |J_j| + \frac{\epsilon |J_j|}{2\alpha(y_j)}. \tag{3.10}$$

To complete the proof we now construct a holomorphic disc $f \in \mathcal{A}_X$ with $f(0) = h(0)$ such that

$$f(z_{j,l}) = y_j, \quad l = 1, \ldots, l_j,\ j = 1, \ldots, m. \tag{3.11}$$

For such $f$, by combining (3.9) and (3.10) we obtain

$$\begin{aligned}\widetilde{L}_\alpha(f) &\le \sum_{j=1}^{m} \alpha(y_j) \sum_{l=1}^{l_j} \log|z_{j,l}| \le \sum_{j=1}^{m} \alpha(y_j) \log|b_j| \cdot |J_j| + \frac{\epsilon}{2} \\ &< \int_0^{2\pi} (\phi \circ h)(e^{it}) \frac{dt}{2\pi} + \epsilon,\end{aligned}$$

so the estimate (3.8) holds and the proof is complete.

To find such a disc $f$ we proceed as follows. We embed the disc $h$ into a dominating spray of discs $h_w = h(w, \cdot) \in \mathcal{A}_X$, depending holomorphically on the point $w$ in a ball $\mathbb{B}$ in a Euclidean space $\mathbb{C}^N$, so that $h_0 = h$. By shrinking $\mathbb{B}$ around the origin we may assume that $h(w, z) \in V_j$ for every $w \in \mathbb{B}$ and $z \in \overline{\Delta}_j$. Over each of the discs $\overline{\Delta}_j$ we define a holomorphic spray of discs, with the parameter $w \in \mathbb{B}$, by setting

$$\widetilde{g}(w, z) = g_j\big(h(w, z), a_j(z)\big), \quad z \in \overline{\Delta}_j.$$

Since $a_j(z_{j,l}) = b_j$ and $g_j(x, b_j) = y_j$ for all $x \in V_j$, we have $\widetilde{g}(w, z_{j,l}) = y_j$ for all $w \in \mathbb{B}$, $j = 1, \ldots, m$ and $l = 1, \ldots, j_l$. Further, for points $z \in \overline{\Delta}_j \cap \overline{D}_0$ the function $|a_j(z)|$ can be made arbitrarily small by choosing the constant $M > 0$ in the above construction large enough (and this estimate is independent of the choice of the integer $k \in \mathbb{N}$). For such $z$ we have

$$\widetilde{g}(w, z) \approx g(h(w, z), 0) = h(w, z).$$

In particular, for $M > 0$ large enough (and for any $k \in \mathbb{N}$) the sprays $h(w, z)$ (over $z \in \overline{D}_0$) and $\widetilde{g}(w, z)$ (over $z \in \overline{D}_1$) can be glued into a single spray over $\overline{D}_0 \cup \overline{D}_1 = \overline{\mathbb{D}}$ by using the method from [For, §5.8–§5.9]. (A brief exposition can also be found in [DF].) Explicitly, we find maps $\beta_0 \in \mathcal{A}(D_0, \mathbb{C}^N)$ and $\beta_1 \in \mathcal{A}(D_1, \mathbb{C}^N)$, with values in $\mathbb{B}$, that are uniformly small (depending only on the uniform distance between the two sprays over $\overline{D}_0 \cap \overline{D}_1$, and hence only on the constant $M > 0$), such that $\beta_0(0) = 0$ and

$$\widetilde{g}(\beta_1(z), z) = h(\beta_0(z), z), \quad z \in \overline{D}_0 \cap \overline{D}_1.$$

The two sides define an analytic disc $f \in \mathcal{A}_X$ with center $f(0) = h(0)$ and satisfying (3.11). This completes the proof of Theorem 3.1. ∎

**4. Siciak–Zaharyuta extremal functions on affine varieties.** In this section we obtain explicit expressions for the Siciak–Zaharyuta extremal function $V_{\Omega,X}$ of an open set $\Omega$ in a locally irreducible affine algebraic variety $X \subset \mathbb{C}^n$ in terms of Green functions on complex curves $C$ in the projective closure $\overline{X} \subset \mathbb{P}^n$ of $X$, with smooth boundaries $bC$ contained in $\Omega$. Theorem 4.4 generalizes some of the results of Lempert and of Lárusson and Sigurdsson (for the case $X = \mathbb{C}^n$) mentioned below.

We begin by recalling some standard notions of pluripotential theory, referring to Klimek [Kli] for further information.

The *Lelong class* $\mathcal{L} = \mathcal{L}_{\mathbb{C}^n}$ on $\mathbb{C}^n$ is the set of all plurisubharmonic functions $v : \mathbb{C}^n \to \mathbb{R} \cup \{-\infty\}$ for which there exist constants $r > 0$ and $C \in \mathbb{R}$ (depending on $v$) such that

$$v(z) \le \log|z| + C, \qquad z \in \mathbb{C}^n,\ |z| > r.$$

Such functions are said to have at most logarithmic growth at infinity.

Given a nonempty open subset $\Omega \subset \mathbb{C}^n$, the *Siciak–Zaharyuta extremal function* $V_\Omega : \mathbb{C}^n \to \mathbb{R}$ is defined by

$$V_\Omega(z) = \sup\{v(z) : v \in \mathcal{L},\ v|_\Omega \le 0\}.$$

It is the largest function in the Lelong class $\mathcal{L}$ which is $\le 0$ on $\Omega$. By replacing any function $v \in \mathcal{L}$ in the definition of $V_\Omega$ by $\max\{v, 0\}$ we see that $V_\Omega \ge 0$ on $\mathbb{C}^n$ and $V_\Omega = 0$ on $\Omega$. If $\Omega$ has a sufficiently nice boundary then $V_\Omega$ is continuous, and hence it vanishes on $\overline{\Omega}$. (The extremal function $V_E$ can be defined for an arbitrary subset $E \subset \mathbb{C}^n$, but in general one must take its upper regularization $V_E^*$ in order to get a plurisubharmonic function. We have $V_E^* \equiv +\infty$ if and only if the set $E$ is pluripolar. Here we restrict our attention to extremal functions of open sets.)

In the same way one defines the Lelong class $\mathcal{L}_X$, and the extremal function $V_{\Omega,X}$, when $\Omega$ is an open subset in a closed affine algebraic subvariety $X$ of a complex affine space $\mathbb{C}^n$.

We now recall the Lempert formula for the extremal function $V_\Omega$ of an open convex set $\Omega \subset \mathbb{C}^n$ (see the appendix in [Mo] and the discussion in [LS3]). Consider $\mathbb{C}^n$ as a subset of the projective space $\mathbb{P}^n$, and let $H = \mathbb{P}^n \setminus \mathbb{C}^n \cong \mathbb{P}^{n-1}$ denote the hyperplane at infinity. For every analytic disc $f \in \mathcal{A}_{\mathbb{P}^n}$ with $f(\mathbb{T}) \subset \mathbb{C}^n$ set

$$J(f) = -\sum_{f(\zeta)\in H} \log|\zeta| \ge 0, \tag{4.1}$$

where the sum is over the finitely many points $\zeta \in \mathbb{D}$ which are mapped by $f$ to $H$, counted with intersection multiplicities. (If $f(0) \in H$, we set $J(f) = +\infty$, and if $f(\overline{\mathbb{D}}) \subset \mathbb{C}^n$ then $J(f) = 0$.) We may think of $J$ as a disc functional on the set of discs in $\mathbb{P}^n$ with boundary values in $\mathbb{C}^n$. Lempert proved that, if $\Omega$ is open and convex, then for every point $z \in \mathbb{C}^n$ we have

$$V_\Omega(z) = \inf\{J(f) : f \in \mathcal{A}_{\mathbb{P}^n},\ f(0) = z,\ f(\mathbb{T}) \subset \Omega\}; \tag{4.2}$$

furthermore, one gets the same infimum over the smaller set of discs with a single point at infinity of multiplicity one.

Lempert's formula (4.2) was extended by Lárusson and Sigurdsson to the case when $\Omega$ is an arbitrary *connected* open subset of $\mathbb{C}^n$ [LS3, Theorem 3]; however, one must in general use discs with several poles.

If $\Omega$ is disconnected, then the infimum on the right hand side of (4.2) is in general larger than $V_\Omega(z)$. However, it is still possible to obtain $V_\Omega$ as follows. Let $\mathcal{B}$ be a family of analytic discs in $\mathbb{P}^n$ with boundary values in $\Omega$. Following [LS3] we introduce the following notion.

DEFINITION 4.1. A family of discs $\mathcal{B} \subset \mathcal{A}(\mathbb{D}, \mathbb{P}^n)$ with boundaries in $\Omega \subset \mathbb{C}^n$ is a *good family* (with respect to $\Omega$) if it has the following properties:

(i) for every point $z \in \mathbb{P}^n$ there is a disc $f \in \mathcal{B}$ with $f(0) = z$,
(ii) for every point $z \in \Omega$ the constant disc $\overline{\mathbb{D}} \mapsto z$ belongs to $\mathcal{B}$, and
(iii) for every point $p \in \mathbb{P}^n$ and every disc $f \in \mathcal{B}$ with $f(0) = p$ there exist a neighborhood $U \subset \mathbb{P}^n$ of $p$ and a continuous family of discs $\{f_z \in \mathcal{B} : z \in U\}$ such that $f_p = f$ and $f_z(0) = z$ for all $z \in U$.

Define a function $E_{\mathcal{B}}J : \mathbb{C}^n \to \mathbb{R}_+$ by setting

$$E_{\mathcal{B}}J(z) = \inf_{f\in\mathcal{B},\, f(0)=z} J(f), \qquad z \in \mathbb{C}^n. \tag{4.3}$$

We think of $E_{\mathcal{B}}J$ as the envelope of the disc functional $f \mapsto J(f)$ with respect to the family $\mathcal{B}$. It is easily seen that the envelope $E_{\mathcal{B}}J$ with respect to a good family of discs is an upper semicontinuous function on $\mathbb{C}^n$ which vanishes on $\Omega$ and has at most logarithmic growth at infinity (see [LS3]).

THEOREM 4.2 ([LS3, Theorem 2]). *If $\Omega$ is a nonempty open set in $\mathbb{C}^n$ and $\mathcal{B}$ is a good family of analytic discs with respect to $\Omega$, then the Siciak–Zaharyuta function $V_\Omega$ is the Poisson envelope of the function $E_{\mathcal{B}}J$ in* (4.3):

$$V_\Omega(z) = \inf\Big\{ \int_0^{2\pi} (E_{\mathcal{B}}J)(f(e^{it}))\,\frac{dt}{2\pi} :\ f \in \mathcal{A}(\mathbb{D}, \mathbb{C}^n, z)\Big\}, \qquad z \in \mathbb{C}^n. \tag{4.4}$$

REMARK 4.3. If the set $\Omega$ is connected, then the Lempert formula (4.2) follows by applying Theorem 4.2, with $\mathcal{B}$ a suitable family of discs in projective lines (and with boundaries in $\Omega$), and then solving a Riemann–Hilbert boundary value problem. (See the last section in [LS3].) The advantage of the formula (4.2) over (4.4) is that the former expresses the extremal function $V_\Omega$ by only one application of infimum over a suitably large family of discs, while in the latter the infimum is applied twice. ∎

Lárusson and Sigurdsson proved Theorem 4.2 by lifting the problem with respect to the projection $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ and considering the plurisubharmonic function $V_\Omega \circ \pi + \log|z_0|$ on $\mathbb{C}^{n+1} \setminus \{0\}$, where the coordinates $(z_0, \ldots, z_n)$ on $\mathbb{C}^{n+1}$ are chosen such that $H = \{z_0 = 0\}$. An application of the Riesz formula on the disc leads to an auxiliary disc formula [LS3, Theorem 1] from which the result is obtained by some additional arguments.

We now give a very simple proof of Theorem 4.2 which generalizes immediately to open sets in affine varieties. The main point is to observe that the restriction of $V_\Omega$ to any complex curve $C \subset \mathbb{P}^n$ with smooth boundary contained in $\Omega \subset \mathbb{C}^n$ is bounded above by the Green function on that curve with poles at the points in $C \cap H$. (In the case at hand we can use discs, and this brings the functional $J(f)$ into the picture.) The infimum of such Green functions over sufficiently many curves yields an upper semicontinuous function $\Psi : \mathbb{C}^n \to \mathbb{R}_+$ that vanishes on $\Omega$, has at most logarithmic growth at infinity, and satisfies $V_\Omega \le \Psi$. It then follows from maximality that $V_\Omega$ is the Poisson envelope $EP_\Psi$ of $\Psi$.

*Proof of Theorem 4.2.* Let $\Omega \subset \mathbb{C}^n \subset \mathbb{P}^n$ be as in the theorem. Assume that $\bar{\Sigma}$ is a compact finite bordered Riemann surface all of whose boundary components are Jordan curves. Let $f : \bar{\Sigma} \to \mathbb{P}^n$ be a continuous map that is holomorphic in the interior, $\Sigma$, of $\bar{\Sigma}$, and with boundary $f(b\Sigma) \subset \Omega$ contained in $\Omega$. Recall that $H = \mathbb{P}^n \setminus \mathbb{C}^n$. Write $f^*(H) = \sum_{j=1}^k m_j p_j$ as a divisor, where $p_1, \ldots, p_k$ are the points in $\Sigma$ that are mapped by $f$ to the hyperplane $H$, and $m_j \in \mathbb{N}$ is the intersection multiplicity of the map $f$ with $H$ at the point $p_j$. The composition

$$V_\Omega \circ f : \bar{\Sigma} \to \mathbb{R}_+ \cup \{+\infty\}$$

is then a subharmonic function on $\bar{\Sigma} \setminus f^{-1}(H)$ that vanishes on $b\Sigma$ and has logarithmic poles at the points $p_j \in f^{-1}(H)$. More precisely, choosing a local holomorphic coordinate $\zeta$ on $\Sigma$ with $\zeta(p_j) = 0$, there is a constant $C \in \mathbb{R}$ such that $V_\Omega \circ f(\zeta) \le -m_j \log|\zeta| + C$ as $\zeta \to 0$. It follows that

$$V_\Omega \circ f \le -G_{\Sigma, f^*H} = -\sum_{j=1}^k m_j G_{\Sigma, p_j},$$

where the right hand side is the Green function on $\Sigma$ with poles determined by the divisor $f^*H$. (Precisely, $G_{\Sigma, f^*H} \le 0$ is the unique continuous function on $\bar{\Sigma} \setminus f^{-1}(H)$ that is harmonic in $\Sigma \setminus f^{-1}(H)$, vanishes on the boundary $b\Sigma$, and has a logarithmic pole with multiplicity $m_j$ at each of the points $p_j \in f^{-1}(H)$. For Green functions see any of the standard sources, or the recent book by Varolin [Var, p. 119].) The above inequality follows by observing that $V_\Omega \circ f + G_{\Sigma, f^*H}$ is a subharmonic function on $\Sigma \setminus f^{-1}(H)$ which equals zero on $b\Sigma$ and is locally bounded from above at each point $p_j \in f^{-1}(H)$; hence it extends as a subharmonic function on $\Sigma$, and the maximum principle implies that it is $\le 0$ on $\Sigma$.

We now restrict attention to the case when $\Sigma = \mathbb{D}$ is the disc. The Green function with pole at the point $a \in \mathbb{D}$ equals

$$G_a(\zeta) = \log\left|\frac{\zeta - a}{1 - \bar{a}\zeta}\right|.$$

For $a \neq 0$ we get $G_a(0) = \log|a|$. Given a disc $f \in \mathcal{A}_{\mathbb{P}^n}$ with $f(\mathbb{T}) \subset \Omega$ and $f^*H = \sum_{j=1}^k m_j a_j$, where $a_j \in \mathbb{D}$ and $m_j \in \mathbb{N}$, we thus have

$$V_\Omega(f(0)) \le -\sum_{j=1}^k m_j \log|a_j| = J(f).$$

Therefore we have for each point $z \in \mathbb{C}^n$ the estimate

$$V_\Omega(z) \le \Psi(z) := \inf_{f\in\mathcal{B},\ f(0)=z} J(f).$$

If the family $\mathcal{B}$ is good in the sense of Definition 4.1, then the function $\Psi : \mathbb{C}^n \to \mathbb{R}_+$ is upper semicontinuous, $\Psi|_\Omega = 0$, and $\Psi$ has at most logarithmic growth at infinity. Poletsky's theorem (Theorem 2.1 for $X = \mathbb{C}^n$) implies that the Poisson envelope $EP_\Psi$ is the extremal plurisubharmonic minorant of $\Psi$. Since $\Psi$ grows logarithmically, we infer that $EP_\Psi$ belongs to the Lelong class $\mathcal{L}$. Finally, as $V_\Omega \le \Psi$, we have $V_\Omega \le EP_\Psi$, and hence $V_\Omega = EP_\Psi$ by maximality of $V_\Omega$. This proves Theorem 4.2. ∎

The above proof generalizes immediately to the following situation. Let $\Omega$ be a nonempty open subset in an affine algebraic variety $X \subset \mathbb{C}^n$. The Lelong class $\mathcal{L}_X$, and the extremal function $V_{\Omega,X}$, are defined in essentially the same way as in the case $X = \mathbb{C}^n$. Assume now that $X$ is irreducible, and let $k = \dim_{\mathbb{C}} X \in \{1,\dots,n-1\}$. Denote by $\overline{X}$ the closure of $X$ in $\mathbb{P}^n$ (an algebraic subvariety of $\mathbb{P}^n$). Complex curves $C \subset \overline{X}$ whose boundaries $bC$ are smooth and contained in the open subset $\Omega \subset X$ fill the entire projective variety $\overline{X}$. (We consider only curves that have no isolated points in their boundaries.) An explicit way to obtain such curves is to take the intersection of $\overline{X}$ with a generic projective linear subspace $\Lambda \subset \mathbb{P}^n$ of dimension $n-k+1$ such that $\Lambda \cap \Omega \neq \emptyset$, and then remove from the closed projective curve $\Lambda \cap \overline{X}$ finitely many smoothly bounded discs lying in $\Omega$. If $\overline{X}$ is smooth (without singularities), then a generic such intersection will be a smooth embedded complex curve with boundary, but in general we cannot expect it to be a disc. In fact, the degree of a generic curve $\Lambda \cap \overline{X}$ equals the degree of $X$.

Given a curve $C \subset \overline{X}$ with smooth boundary $bC \subset \Omega$, let $f : \overline{\Sigma} \to \overline{C}$ be a normalization of $C$ by a finite bordered Riemann surface, $\Sigma$, with smooth boundary. The normalization map extends smoothly to the boundary. Let $f^*H = \sum_{j=1}^d m_j p_j$ denote the intersection divisor of the map $g$ with the hyperplane at infinity $H = \mathbb{P}^n \setminus \mathbb{C}^n$. As before, let $G_{\Sigma,f^*H}$ be the Green function on $\Sigma$ with logarithmic poles determined by the divisor $f^*H$. Choose a family $\mathcal{B}$ of such normalized curves $(\Sigma, f, C)$ in $X$ whose images $C = f(\Sigma)$ fill $\overline{X}$, and define the function $\Psi_{\mathcal{B}} : X \to \mathbb{R}_+$ by

$$\Psi_{\mathcal{B}}(x) = \inf -G_{\Sigma,f^*H}(\zeta) = -\sup G_{\Sigma,f^*H}(\zeta), \qquad x \in X, \tag{4.5}$$

where the infimum is over all $(\Sigma, f, C) \in \mathcal{B}$ and points $\zeta \in \Sigma$ with $f(\zeta) = x$.

By including in $\mathcal{B}$ all the constant discs $\overline{\mathbb{D}} \mapsto z \in \Omega$, and by assuming the existence of continuous local families of curves in $\mathcal{B}$ (in analogy to property (iii) in Def. 4.1), we ensure as before that $\Psi_{\mathcal{B}}$ is an upper semicontinuous function on $X$ that equals zero on $\Omega$ and grows at most logarithmically at infinity. (The continuity of a family of curves with respect to some parameter $t$ can be made precise by fixing a smooth oriented real surface with boundary, $\overline{\Sigma}$, and choosing a continuous family of almost complex structures $J_t$ on $\overline{\Sigma}$, and a continuous family of holomorphic maps $f_t : (\overline{\Sigma}, J_t) \to X$.)

The argument in the proof of Theorem 4.2 shows that for each complex curve $f : \overline{\Sigma} \to \overline{C} \subset \overline{X}$ with $f(b\Sigma) \subset \Omega$ we have $V_{\Omega,X} \circ f \leq -G_{\Sigma,f^*H}$, and hence $V_{X,\Omega} \leq \Psi_{\mathcal{B}}$. By applying the general case of Theorem 2.1 we thus obtain the following expression for the extremal function $V_{\Omega,X}$.

THEOREM 4.4. *Let $X$ be an irreducible and locally irreducible algebraic subvariety of $\mathbb{C}^n$, and let $\Omega$ be a nonempty open set in $X$. Assume that $\mathcal{B}$ is a good family of complex curves in $X$ with boundaries in $\Omega$, and let $\Psi_{\mathcal{B}} : X \to \mathbb{R}_+$ denote the associated function* (4.5). *Then the Siciak–Zaharyuta function $V_{\Omega,X}$ is the envelope of the Poisson functional $P_\Psi$:*

$$V_{\Omega,X}(x) = \inf\Big\{ \int_0^{2\pi} \Psi_{\mathcal{B}}(f(e^{it}))\, \frac{dt}{2\pi} : f \in \mathcal{O}(\overline{\mathbb{D}}, X, x)\Big\}, \qquad x \in X.$$

EXAMPLE 4.5. An explicit example of a good family $\mathcal{B}$ that one can use in this theorem is obtained by taking all constant discs in $\Omega$, together with all transverse intersections $L = \Lambda \cap \overline{X}$ that intersect $\Omega$, where $\Lambda \cong \mathbb{P}^{n-k+1}$ is a projective linear subspace of $\mathbb{P}^n$ of dimension $n - \dim X + 1$, and removing from each such closed curve $L$ a finite family of pairwise disjoint, closed, smoothly bounded discs $\overline{\Delta}_1, \ldots, \overline{\Delta}_m \subset \Omega$ whose boundaries $b\Delta_j$ belong to the regular locus $L_{\rm reg}$ of $L$. The difference $C = L \backslash \bigcup_{j=1}^m \overline{\Delta}_j$ is then a complex curve in $X$ with smooth boundary $bC = \bigcup_{j=1}^m b\Delta_j$ contained in $\Omega$. ∎

PROBLEM 4.6. *Assume that $\Omega \subset X$ are as in Theorem* 4.4. *Let $\mathcal{B}$ consist of all complex curves $f : \overline{\Sigma} \to \overline{C} \subset X$, where $\overline{\Sigma}$ is a finite bordered Riemann surface and $f$ is a holomorphic map such that $f(b\Sigma) \subset \Omega$. Is the extremal function $V_{\Omega,X}$ then given by the Lempert type formula*

$$V_{\Omega,X}(x) = \inf\{-G_{\Sigma,f^*H}(\zeta) : (\overline{\Sigma}, f) \in \mathcal{B},\ f(\zeta) = x \in X\}\,?$$

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## References

[AHV] J. M. Aroca, H. Hironaka and J. L. Vicente, *Desingularization theorems*, Mem. Math. Inst. Jorge Juan 30, Consejo Sup. Invest. Cient., Madrid, 1977.

[BM] E. Bierstone and P. D. Milman, *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant*, Invent. Math. 128 (1997), 207–302.

[BS] S. Q. Bu and W. Schachermayer, *Approximation of Jensen measures by image measures under holomorphic functions and applications*, Trans. Amer. Math. Soc. 331 (1992), 585–608.

[Chi] E. M. Chirka, *Complex Analytic Sets*, Math. Appl. (Soviet Ser.) 46, Kluwer, Dordrecht, 1989.

[Dem] J.-P. Demailly, *Mesures de Monge–Ampère et caractérisation géométrique des variétés algébriques affines*, Mém. Soc. Math. France (N.S.) 19 (1985).

[DF] B. Drinovec Drnovšek and F. Forstnerič, *The Poletsky–Rosay theorem on singular complex spaces*, Indiana Univ. Math. J., in press.

[Ed1] A. Edigarian, *On definitions of the pluricomplex Green function*, Ann. Polon. Math. 67 (1997), 233–246.

[Ed2] A. Edigarian, *A note on Lárusson–Sigurdsson's paper*, Math. Scand. 92 (2003), 309–319.

[Ed3] A. Edigarian, *A note on Rosay's paper*, Ann. Polon. Math. 80 (2003), 125–132.

[For] F. Forstnerič, *Stein Manifolds and Holomorphic Mappings*, Ergeb. Math. Grenzgeb. (3) 56, Springer, Berlin, 2011.

[Hir] H. Hironaka, *Desingularization of complex-analytic varieties*, in: Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2, Gauthier-Villars, Paris, 1971, 627–631.

[Kis] C. O. Kiselman, *The partial Legendre transformation for plurisubharmonic functions*, Invent. Math. 49 (1978), 137–148.

[Kli] M. Klimek, *Pluripotential Theory*, London Math. Soc. Monogr. (N.S.) 6, Oxford Univ. Press, New York, 1991.

[LS1] F. Lárusson and R. Sigurdsson, *Plurisubharmonic functions and analytic discs on manifolds*, J. Reine Angew. Math. 501 (1998), 1–39.

[LS2] F. Lárusson and R. Sigurdsson, *Plurisubharmonicity of envelopes of disc functionals on manifolds*, J. Reine Angew. Math. 555 (2003), 27–38.

[LS3] F. Lárusson and R. Sigurdsson, *The Siciak–Zahariuta extremal function as the envelope of disc functionals*, Ann. Polon. Math. 86 (2005), 177–192.

[LS4] F. Lárusson and R. Sigurdsson, *Siciak–Zahariuta extremal functions and polynomial hulls*, Ann. Polon. Math. 91 (2007), 235–239.

[LS5] F. Lárusson and R. Sigurdsson, *Siciak–Zahariuta extremal functions, analytic discs and polynomial hulls*, Math. Ann. 345 (2009), 159–174.

[Mo] S. Momm, *An extremal plurisubharmonic function associated to a convex pluricomplex Green function with pole at infinity*, J. Reine Angew. Math. 471 (1996), 139–163.

[Po1] E. A. Poletsky, *Plurisubharmonic functions as solutions of variational problems*, in: Several Complex Variables and Complex Geometry, Part 1 (Santa Cruz, CA, 1989), Proc. Sympos. Pure Math. 52, Part 1, Amer. Math. Soc., Providence, RI, 1991, 163–171.

[Po2] E. A. Poletsky, *Holomorphic currents*, Indiana Univ. Math. J. 42 (1993), 85–144.

[Po3] E. A. Poletsky, *The minimum principle*, Indiana Univ. Math. J. 51 (2002), 269–303.

[Ro1] J.-P. Rosay, *Poletsky theory of disks on holomorphic manifolds*, Indiana Univ. Math. J. 52 (2003), 157–169.

[Ro2] J.-P. Rosay, *Approximation of non-holomorphic maps, and Poletsky theory of discs*, J. Korean Math. Soc. 40 (2003), 423–434.

[Var] D. Varolin, *Riemann Surfaces by Way of Complex Analytic Geometry*, Grad. Stud. Math. 125, Amer. Math. Soc., Providence, RI, 2011.

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