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On convergence sets of divergent power series

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To Professor Józef Siciak on his 80th birthday

Abstract. A nonlinear generalization of convergence sets of formal power series, in the sense of Abhyankar–Moh [J. Reine Angew. Math. 241 (1970)], is introduced. Given a family $y = \varphi_s(t,x) = sb_1(x)t + b_2(x)t^2 + \cdots$ of analytic curves in $\mathbb{C} \times \mathbb{C}^n$ passing through the origin, $\mathrm{Conv}_\varphi(f)$ of a formal power series $f(y,t,x) \in \mathbb{C}[[y,t,x]]$ is defined to be the set of all $s \in \mathbb{C}$ for which the power series $f(\varphi_s(t,x),t,x)$ converges as a series in (t,x). We prove that for a subset $E \subset \mathbb{C}$ there exists a *divergent* formal power series $f(y,t,x) \in \mathbb{C}[[y,t,x]]$ such that $E = \mathrm{Conv}_\varphi(f)$ if and only if E is an F_σ set of zero capacity. This generalizes the results of P. Lelong and A. Sathaye for the linear case $\varphi_s(t,x) = st$.

We say that a formal power series $f(z) = \sum_{|\alpha|=0}^{\infty} a_\alpha z^\alpha$ is *convergent* if there exists a constant C such that $|a_\alpha| \le C^{|\alpha|}$ for all $\alpha \in \mathbb{Z}_+^n$. (Here we have used multiindex notation: $\mathbb{Z}_+^n$ denotes the set of all n-tuples $\alpha := (\alpha_1,\ldots,\alpha_n)$ of integers $\alpha_i \ge 0$ if $z = (z_1,\ldots,z_n)$ and $\alpha \in \mathbb{Z}_+^n$, then $z^\alpha = z_1^{\alpha_1}\cdots z_n^{\alpha_n}$ and $|\alpha| := \alpha_1 + \cdots + \alpha_n$ denotes the length of $\alpha \in \mathbb{Z}_+^n$.) A series f is called *divergent* if it is not convergent.

A divergent power series may still converge when restricted to a certain set of lines or planes through the origin. For example, Abhyankar and Moh [AM] considered the convergence set $\mathrm{Conv}(f)$ of a series f defined to be the set of all $s \in \mathbb{C}$ for which $f(sz_2,\ldots,z_n)$ converges as a series in $(z_2,\ldots,z_n)$. The convergence set of divergent series can be empty or an arbitrary countable set. Abhyankar and Moh proved that the one-dimensional Hausdorff measure of the convergence set of a divergent series is zero. In the case when $n = 2$, Pierre Lelong [Le] had earlier proved that if $\mathrm{Conv}(f)$ is not contained in an F_σ set of zero capacity then the series f is necessarily

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convergent, and conversely, given any set E contained in an F_σ set of zero capacity a divergent power series f can be constructed so that $E \subset \mathrm{Conv}(f)$. This result has been rediscovered, independently, by several authors (see e.g. [LM], [Ne], [Sa], and see also [FM], [FM1], [Ne1], [Ri] for other related results). The optimal result was obtained by Sathaye [Sa] who strengthened the results of Abhyankar–Moh and Lelong by proving that a necessary and sufficient condition for a set $S \subset \mathbb{C}$ to be equal to the convergence set of a divergent power series $f(z)$ is that S is an F_σ set of transfinite diameter zero, i.e. $S = \bigcup_{j=1}^{\infty} E_j$ where each E_j is a closed set of transfinite diameter zero. These results can be viewed as optimal and formal analogs of Hartogs' Theorem on separate analyticity in several complex variables.

Motivated by formal fibered diffemorphisms associated with dynamical systems, Ribón [Ri] studied holomorphic extensions of formal objects, including formal power series, formal meromorphic functions, and formal infinitesimal diffeomorphisms.

Instigated by the above mentioned results, we are interested in the following general problem: find "tight" conditions on a family $\mathcal{F}$ of formal submanifolds in $\mathbb{C}^n$ so that if the restriction of a formal series to each formal submanifold in $\mathcal{F}$ converges, then the formal power series converges. By "tight" we mean, vaguely speaking, that one does not assume too much. In case $n = 2$, Lelong, Abhyankar–Moh, and Sathaye's results can be interpreted as: for the family of lines $\{\ell_s : s \in E\}$, where $\ell_s = \{(sx, x) : x \in \mathbb{C}\}$, the tight condition is that E have positive capacity. The result of Fridman and Ma [FM] is that the tight condition on the family $\{\gamma_s : s \in E\}$, where $\gamma_s = \{(s^\sigma x, s^\tau h(x))\}$ ($h(x)$ is a fixed convergent series), is again E having positive capacity. It is natural to ask, for the family $\{\gamma_s : s \in E\}$, where $\gamma_s = \{(\varphi(s,x), \psi(s,x))\}$, whether the tight condition is that E have positive capacity. This paper deals with one of the first questions one has to address in order to understand the general problem.

In this article, we consider "nonlinear" convergence sets of formal power series $f(y,t,x) \in \mathbb{C}[[y,t,x]]$ by restricting $f(y,t,x)$ along a one-parameter family of "tangential" perturbations of a fixed analytic curve $y = \varphi(t,x)$ through the origin. For simplicity of notation we only consider a single variable x. If x is replaced by a tuple $(x_1, \ldots, x_n)$, the theorem is still valid, and the proof goes through without difficulty.

Throughout this paper, $\varphi(t,x) := \sum_{j=1}^{\infty} b_j(x) t^j$ denotes a fixed convergent power series where $b_j(x) := \sum_{i=0}^{\infty} b_{ji} x^i$, $j = 1, 2, \ldots$, are convergent power series in x with complex coefficients. We assume that $b_{10} = 1$.

For $s \in \mathbb{C}$, we put $\varphi_s(t,x) = \varphi(s,t,x) = s b_1(x) t + \sum_{j=2}^{\infty} b_j(x) t^j$. Define the *$\varphi$-convergence set* of a series $f(y,t,x) \in \mathbb{C}[[y,t,x]]$ as follows:

$$\mathrm{Conv}_\varphi(f) := \{s \in \mathbb{C} : f(\varphi(s,t,x), t, x) \text{ converges as a series in } (t,x)\}.$$

Let K be a compact subset of $\mathbb{C}$. For a probability measure μ on the compact set K, the *logarithmic potential* of μ is

$$p_\mu(z) = \lim_{N\to\infty} \int \min\left(N, \log\frac{1}{|z-\zeta|}\right) d\mu(\zeta),$$

and the (logarithmic) *capacity* (see [Ah, Chapter 2]) of K is defined by

$$\mathfrak{c}(K) = \exp(-\min_{\mu(K)=1} \sup_{z\in\mathbb{C}} p_\mu(z)).$$

If $E = \bigcup_{n=1}^\infty K_n$, where K_n are compact sets of zero capacity, then $\mathfrak{c}(E) = 0$. A subset E of $\mathbb{C}$ is of zero capacity if and only if it is *polar*, i.e. $E \subset \{z : u(z) = -\infty\}$ for some nonconstant subharmonic function $u : \mathbb{C} \to [-\infty, \infty)$. An F_σ set E in $\mathbb{C}$ is said to have zero capacity if E is the union of a countable collection of compact sets of zero capacity.

Theorem 1. *Let $\varphi(t,x)$ be as above, and let E be a subset of $\mathbb{C}$. There exists a divergent formal power series $f(y,t,x) \in \mathbb{C}[[y,t,x]]$ such that $E = \mathrm{Conv}_\varphi(f)$ if and only if E is an F_σ set of zero capacity.*

Proof. Suppose that E is an F_σ set with $\mathfrak{c}(E) > 0$. By replacing E with a compact subset $K \subset E$ of positive capacity we can assume that E is compact. Let $f(y,t,x) := \sum_{i,j,k} a_{ijk} y^i t^j x^k \in \mathbb{C}[[y,t,x]]$ be such that $g(s;t,x) := f(\varphi(s,t,x),t,x)$ converges for each $s \in E$. We need to show that f is convergent. Rewrite f as $f(y,t,x) = \sum_{i,j} a_{ij}(x) y^i t^j$, where $a_{ij}(x) := \sum_{k=0}^\infty a_{ijk} x^k \in \mathbb{C}[[x]]$, and

$$g(s;t,x) := \sum_{q\ge 0,\, k\ge 0} \lambda_{qk}(s) t^q x^k := \sum_{p\ge 0,\, q\ge 0} d_{pq}(x) s^p t^q.$$

It is clear that $\lambda_{qk}(s)$ is a polynomial of degree at most q, and thus

$$d_{pq}(x) = 0 \quad \text{for } p > q.$$

Let $d_{pq}(x) := \sum_{k\ge 0} d_{pqk} x^k$, and write $\lambda_{qk}(s) := \sum_{p=0}^q d_{pqk} s^p$.

We have

$$d_{pq}(x) = {\sum}' a_{ij}(x) \frac{i!}{p!m_2!m_3!\cdots} b_1(x)^p b_2(x)^{m_2} b_3(x)^{m_3} \cdots,$$

where the summation $\sum'$ is taken over all nonnegative integers $i, j, m_2, m_3, \dots$ satisfying

$$j + p + 2m_2 + 3m_3 + \cdots = q \quad \text{and} \quad p + m_2 + m_3 + \cdots = i.$$

Since

$$\begin{aligned} d_{qq}(x) &= a_{q,0}(x) b_1(x)^q, \\ d_{q-1,q}(x) &= a_{q-1,1}(x) b_1(x)^{q-1}, \\ d_{q-2,q}(x) &= a_{q-2,2}(x) b_1(x)^{q-2} + a_{q-1,0}(x)(q-1) b_1(x)^{q-2} b_2(x), \\ d_{q-k,q}(x) &= a_{q-k,k}(x) b_1(x)^{q-k} + \text{terms involving } a_{ij}(x) \text{ with } i+j<q, \end{aligned} \tag{1}$$

it follows that $a_{ij}(x)$ can be solved uniquely in terms of $d_{pq}(x)$. In particular, if $d_{pq}(x) = 0$ for p, q then $a_{ij}(x) = 0$ for all i, j.

For each $s \in E$, there is a constant C_s such that $|\lambda_{qk}(s)| \leq C_s^{q+k}$ for all $q+k \geq 1$, since the power series $\sum_{q,k} \lambda_{qk}(s) t^q x^k$ converges. For each positive integer n, set

$$E_n = \{s \in E : |\lambda_{qk}(s)| \leq n^{q+k}\ \forall q+k \geq 1\}.$$

The sets E_n are closed and $E = \bigcup_{n=1}^{\infty} E_n$. There is a positive integer N such that $E' := \bigcup_{n=1}^{N} E_n$ has positive capacity. It follows that $|\lambda_{qk}(s)| \leq N^{q+k}$ for $q + k \geq 1$ and for $s \in E'$. By the Bernstein–Walsh inequality (see [FM, Lemma 1.4]), there is a constant $C_{E'} \geq 1$ such that $|d_{pqk}| \leq C_{E'}^q N^{q+k} \leq (C_{E'} N)^{q+k}$.

For some $\tau > 0$, $g(s; t, x)$ represents a holomorphic function in $\Delta_\tau \times \Delta_\tau \times \Delta_\tau$, where $\Delta_\tau = \{z \in \mathbb{C} : |z| < \tau\}$. Shrinking τ if necessary, we may assume that

$$\min\{|b_1(x)| : x \in \mathbb{C},\ |x| \leq \tau\} \geq 1/2$$

and

$$\sum_{q,k} |b_{qk}| \tau^{q+k} < \infty, \qquad \sum_{p,q,k} |d_{pqk}| \tau^{p+k} 2^q \Big(\tau + \sum |b_{ij}| \tau^{i+j-1}\Big)^q < \infty. \tag{2}$$

The map $\psi : \mathbb{C}^{n+2} \to \mathbb{C}^{n+2}$ defined by $\psi(s, t, x) := (\varphi(s, t, x), t, x)$ is holomorphic near the origin and is injective on $Q = \{(s, t, x) \in \Delta_\tau \times \Delta_\tau \times \Delta_\tau : t \neq 0\}$. It follows that there is a holomorphic function $G(u, v, w)$ defined on $\psi(Q)$ such that $g = G \circ \psi$ on Q.

We now prove that G extends holomorphically to a neighborhood of the origin. Choose a δ, $0 < \delta < \tau/2$, sufficiently small so that the set

$$\Gamma := \{(u, v, w) \in \mathbb{C}^3 : |u| \leq \delta^2,\ |v| = \delta,\ |w| \leq \delta\}$$

is contained in $\psi(Q)$. The function G extends holomorphically to a neighborhood of the origin if for $|u_0| < \delta^2$, $|w_0| < \delta$,

$$I_k(u_0, w_0) := \frac{1}{2\pi\sqrt{-1}} \int_{|v|=\delta} v^k G(u_0, v, w_0)\, dv = 0, \qquad \forall k = 0, 1, 2, \ldots.$$

For fixed w_0 and u_0, write

$$\left(\frac{u_0 - (\varphi(t, w_0) - t b_1(w_0))}{t b_1(w_0)}\right)^p := \sum_{j=-p}^{\infty} c_{pj}(u_0, w_0) t^j.$$

By making use of (2), and substituting the above series expansion into the integrand in $I_k(u_0, w_0)$, we obtain

$$\begin{aligned} I_k(u_0, w_0) &= \frac{1}{2\pi\sqrt{-1}} \int_{|t|=\delta} t^k g\left(\frac{u_0 - (\varphi(t, w_0) - t b_1(w_0))}{t b_1(w_0)}; t, w_0\right) dt \\ &= \sum d_{pq}(w_0) c_{pj}(u_0, w_0), \end{aligned}$$

where the sum is over all p, q, j with $q + j = -k - 1$ and $j \geq -p$. Since $q + j = -k - 1$ and $j \geq -p$ imply $q = -j - k - 1 < -j \leq p$, and since $d_{pq} = 0$ for $q < p$, we see that $I_k(u_0, w_0) = 0$ for $k = 0, 1, 2, \ldots$ and for all (u_0, w_0) with $|u_0| < \delta^2$, $|w_0| < \delta$. Therefore, G extends holomorphically to a neighborhood of the origin.

Now $g = G \circ \psi$ on $\Delta_\tau \times \Delta_\tau \times \Delta_\tau$. Hence $g = G \circ \psi$ as a formal power series. Since $g = f \circ \psi$, we see that $\hat{f} \circ \psi := \sum \hat{d}_{pq}(x) s^p t^q = 0$, where $\hat{f} := \sum_{i,j} \hat{a}_{ij}(x) y^i t^j = f - G$. It follows that all $\hat{d}_{pq}(x)$, and hence $\hat{a}_{ij}(x)$ are all 0. This proves f is convergent as $\hat{f} = 0$ and $f \equiv G$.

Conversely, suppose E is an F_σ set with $\mathfrak{c}(E) = 0$. We construct a divergent power series $f(y, t, x)$ such that $\mathrm{Conv}_\varphi(f) = E$.

By Theorem 6.1 of [Ri], there exists an increasing sequence $\{q_j\}$ of positive integers and a sequence of polynomials $\{P_j(s)\}$ with $\deg(P_j) \leq q_j$, for all $j = 1, 2, \ldots$, such that the series $\psi_s(t) = \sum_j P_j(s) t^{q_j}$ converges for each $s \in E$, and diverges for each $s \notin E$. Set $g(s; t, x) := \psi_s(t) := \sum_{p,q} d_{pq}(x) s^p t^q$. We solve (1) for $a_{ij}(x)$ in terms of $d_{pq}(x)$, and set $f(y, t, x) = \sum a_{ij}(x) y^i t^j$. Then, $f(\varphi(s, t, x), t, x) = g(s; t, x)$. Therefore $f(y, t, x)$ diverges and $\mathrm{Conv}_\varphi(f) = E$. ∎

Corollary 2. *For any $f(y, t, x) \in \mathbb{C}[[y, t, x]]$, either $c(\mathrm{Conv}_\varphi(f)) = 0$ or $\mathrm{Conv}_\varphi(f) = C$.*

We point out that basic properties of convergence sets follow directly from the corresponding properties of polar sets. In particular, a finite or a countable set is a convergence set.

Remark 3. The closure of a φ-convergence set is not necessarily a φ-convergence set. For example, the countable set $\mathbb{Q}$ is a φ-convergence set of divergent series but its closure $\mathbb{R}$, being nonpolar, cannot be a φ-convergence set of a divergent series.

Remark 4. The situation is quite different, as one would expect, when restrictions of functions are considered. For example, for any positive integer k, it is elementary to construct a function $f : \mathbb{R}^n \to \mathbb{R}$ that is exactly k-times differentiable but whose restriction to every line in $\mathbb{R}^n$ is real-analytic. The function

$$f(x, y) := \begin{cases} (x^2 + y^2) \exp\left(-\dfrac{y^2}{x^2 + y^4}\right) & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

is in $C^\infty(\mathbb{R}^2)$ and for all $m \neq 0$ the single-variable function $f(mt, t)$ is real-analytic. However, f is not a real-analytic function as it fails to be real-analytic along the y-axis. Is there a C^∞ function $f : \mathbb{R}^n \to \mathbb{R}$ which is not real-analytic but whose restriction to every line is real-analytic? The answer is negative, as it was shown by J. Bochnak [Bo] and J. Siciak [Si] that if a C^∞ function $f : \mathbb{R}^n \to \mathbb{R}$ is real-analytic on every line segment through a point x_0, then f is real-analytic in a neighborhood of x_0. Bierstone, Milman and Parusiński [BMP] provided an example of a discontinuous function whose restriction to every analytic arc is analytic. (See [Ne] and [Ne1] for C^∞-analogs of the Bochnak–Siciak theorem.)

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