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Riemann mapping theorem in $\mathbb{C}^n$

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Abstract. The classical Riemann Mapping Theorem states that a nontrivial simply connected domain Ω in $\mathbb{C}$ is holomorphically homeomorphic to the open unit disc $\mathbb{D}$. We also know that "similar" one-dimensional Riemann surfaces are "almost" holomorphically equivalent.

We discuss the same problem concerning "similar" domains in $\mathbb{C}^n$ in an attempt to find a multidimensional quantitative version of the Riemann Mapping Theorem.

1. Introduction. The fact that two simply connected nontrivial domains in the complex plane $\mathbb{C}$ are holomorphically equivalent is one of the most fundamental results in complex analysis. It is also very well known that the Riemann Mapping Theorem fails miserably in $\mathbb{C}^n$ for $n > 1$. Even if we take a very simple domain like the unit ball in $\mathbb{C}^2$ and change its boundary just slightly to obtain another smooth surface, we may get a domain that is not holomorphically equivalent to the ball. The *perturbation problem* we would like to discuss here asks whether small changes of the domains in $\mathbb{C}^n$ produce new domains that are almost biholomorphically equivalent. The first two questions we will have to address are how to judge whether or not two domains are similar and whether or not they are almost holomorphically equivalent.

In the first section we review the situation in the complex plane $\mathbb{C}$, where the basic questions have already been answered and the structure of small deformations is quite clear. In the following section we discuss a much more interesting situation in $\mathbb{C}^n$; our knowledge here is however very limited. In the last section we list several open problems.

2. The one-dimensional case. A deformation of a simply connected domain in $\mathbb{C}$ is holomorphically equivalent to the original domain (as long as it remains simply connected). The *perturbation problem* is still however quite interesting and nontrivial. Let us start with an example: for $\varepsilon \geq 0$ put

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$$P_\varepsilon := \{z : 1 < |z| < 2 + \varepsilon\}. \tag{2.1}$$

Two such domains P_ε, $P_{\varepsilon'}$ look very similar, especially if $\varepsilon \approx \varepsilon'$, however they are not holomorphically equivalent unless $\varepsilon = \varepsilon'$. On the other hand we feel that they are in some sense "almost" equivalent, but we need to make this intuition much more precise. For this we need (i) a way to decide if two domains have almost the same holomorphic structures, and (ii) if they are "almost biholomorphic".

The most natural and very well established measure of how similar two Banach spaces A, B are is the Banach–Mazur distance

$$d_{\text{B-M}}(A, B) := \inf\{\|T\|\,\|T^{-1}\| : T : A \to B\}, \tag{2.2}$$

where the infimum is taken over all linear continuous isomorphisms T from A onto B. Consequently, we may define a distance between domains Ω, Ω' by

$$d_{\text{B-M}}(\Omega, \Omega') := \inf\{\|T\|\,\|T^{-1}\| : T : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega')\} \tag{2.3}$$

where $\mathcal{A}(\Omega)$ is the algebra of all analytic functions on Ω that can be continuously extended to the boundary $\partial\Omega$.

The most natural way to decide if a homeomorphism $\varphi : \Omega \to \Omega'$ is almost biholomorphic is to consider the following number associated with φ:

$$H(\varphi) := \sup_{z\in\Omega}\left\{\limsup_{r\to 0}\frac{\sup_x\{\|\varphi(x) - \varphi(z)\| : \|x - z\| \le r\}}{\inf_x\{\|\varphi(x) - \varphi(z)\| : \|x - z\| \ge r\}}\right\}.$$

The map φ is biholomorphic if and only if $H(\varphi) = 1$ (see [A, H]). We can use this concept to measure the quasiconformal or Teichmüller distance between Ω and Ω':

$$d_{\mathrm{T}}(\Omega, \Omega') := \inf\{H(\varphi) : \varphi : \Omega \to \Omega'\}, \tag{2.4}$$

where the infimum is taken over all homeomorphisms $\varphi : \Omega \to \Omega'$. It turns out that for $\Omega = P_\varepsilon$ and $\Omega' = P_{\varepsilon'}$ the two distances defined above are roughly the same. One can check this by considering the maps $T_\varepsilon : \mathcal{A}(P_0) \to \mathcal{A}(P_\varepsilon)$ defined by

$$T_\varepsilon\Big(\sum_{n=-\infty}^{\infty} a_n z^n\Big) = \sum_{n=-\infty}^{0} a_n z^n + \sum_{n=1}^{\infty} a_n\left(\frac{2}{2+\varepsilon}\right)^n z^n.$$

As $\varepsilon \to 0$ we have $\|T_\varepsilon\|\,\|T_\varepsilon^{-1}\| \to 1$ and also the Teichmüller distance between Ω and Ω' tends to 1. The above example represents the simplest case of a very deep and highly nontrivial result due to R. Rochberg; that result has been developed over a number of years culminating in the 1986 paper [R86] (page 161).

THEOREM 2.1 (R. Rochberg, 1973–1986). *Let Ω, Ω_n be one-dimensional bordered Riemann manifolds. Then* $\lim_{n\to\infty} d_{\text{B-M}}(\Omega, \Omega_n) = 1$ *if and only if* $\lim_{n\to\infty} d_{\mathrm{T}}(\Omega, \Omega_n) = 1$.

The above theorem settles the main problem in the one-dimensional case. There are however still several open questions; we mention some of these in the last section.

3. Multidimensional case. The problem discussed in the previous section is much more interesting in the multidimensional setting, where a positive answer would provide a quantitative version of the Riemann Mapping Theorem. It is however also much more challenging and almost entirely open.

DEFINITION 3.1. We say that a domain Ω in $\mathbb{C}^n$ is *stable* if there is an $\varepsilon > 0$ such that for any Banach algebra B if $d_{\text{B-M}}(\mathcal{A}(\Omega), B) < 1 + \varepsilon$ then B must be isomorphic (as an algebra) to $\mathcal{A}(\Omega)$.

It follows from Rochberg's Theorem above that for $n = 1$ a domain Ω is stable if and only if it is simply connected. For $n > 1$ we do not know the answer for any domain! We do however have partial results for two simplest domains: polydiscs $\mathbb{D}^n$ and unit balls B_n.

THEOREM 3.2. *Assume $\Omega = \mathbb{D}^n$ or $\Omega = B_n$ and that B is a Banach algebra such that $d_{\text{B-M}}(\mathcal{A}(\Omega), B) < 1+\varepsilon$. Then, provided $\varepsilon > 0$ is sufficiently small, we may conclude that:*

(1) *B is a uniform algebra.*
(2) *The maximal ideal space of B is homeomorphic to the closure of Ω, so elements of B may be identified with continuous functions on $\overline{\Omega}$.*
(3) *Ω may be given a structure σ of an n-dimensional complex manifold such that all functions $g \in B$ are (Ω, σ)-analytic.*
(4) *We may define a new multiplication $\times$ on $\mathcal{A}(\Omega)$ such that $(\mathcal{A}(\Omega), \times)$ is a Banach algebra isomorphic (as an algebra) to B and such that*

$$\|fg - f \times g\| < k\varepsilon \|f\| \, \|g\| \quad \text{for } f, g \in \mathcal{A}(\Omega), \tag{3.1}$$

where k is an absolute constant.

For $\Omega = \mathbb{D}^n$ the proof may be found in [Ja92] and for $\Omega = B_n$ in a much more recent paper [Ja11]. Unfortunately we do not know if (Ω, σ) is biholomorphic to Ω or even if (Ω, σ) may be holomorphically embedded in $\mathbb{C}^n$.

The two proofs use some of the same techniques but also require different methods at certain crucial points. We shall discuss the main steps of the proof for the ball algebra $\mathcal{A}(B_n)$, and how some of these methods may or may not also be applied to the polydisc algebra $\mathcal{A}(\mathbb{D}^n)$ or even in the general case to an algebra $\mathcal{A}(\Omega)$ on an arbitrary domain Ω. At various points in our discussion below we use the symbol ε to denote a sufficiently small positive number. The first three steps follow from a standard description of a small deformation of a uniform algebra [Ja85, Theorem 3.1] and can be applied in the general case of $\mathcal{A}(\Omega)$.

STEP 1. Take an almost isometry $T : \mathcal{A}(B_n) \to B$ with $\|T\|\,\|T^{-1}\| < 1+\varepsilon$; it follows that B must automatically be a uniform algebra.

STEP 2. Show that without loss of generality we may assume $T\mathbf{1} = \mathbf{1}$, where $\mathbf{1}$ denotes the unit of the algebra.

STEP 3. Introduce a new multiplication $\times$ on $\mathcal{A}(B_n)$ by

$$f \times g := T^{-1}(Tf \cdot Tg) \quad \text{for } f, g \in \mathcal{A}(B_n),$$

and show that (3.1) holds true.

STEP 4. Show that the Shilov boundary of B is homeomorphic to the topological boundary ∂B_n of the unit ball and that

$$|Tf(\boldsymbol{z}) - f(\boldsymbol{z})| \le \varepsilon\|f\| \quad \text{for } \boldsymbol{z} \in \partial B^n \text{ and } f \in \mathcal{A}(B_n), \tag{3.2}$$

where we use the same symbol $\boldsymbol{z} = (z_1, \ldots, z_n)$ for a point in ∂B^n and for the corresponding point in the homeomorphic copy of ∂B^n in the maximal ideal space of B. This step also follows from a general description of deformations of uniform algebras [Ja85, Theorem 3.1] provided the Choquet boundary of the algebra coincides with the Shilov boundary, as is the case for both $\mathcal{A}(B_n)$ and $\mathcal{A}(\mathbb{D}^n)$, but not for arbitrary $\mathcal{A}(\Omega)$.

STEP 5. Show that

$$\varphi(\mathbf{0}) := \Big\{\sum_{k=1}^{n} T(Z_k) \cdot g_k : g_k \in B\Big\}$$

is a codimension one ideal in B; here Z_k is a function in $\mathcal{A}(B^n)$ defined by $Z_k(z_1, \ldots, z_n) = z_k$. The proof of this step involves a detailed analysis of the solution of the Gleason Problem. In the early sixties Gleason asked if

$$\Big\{\sum_{k=1}^{n} Z_k \cdot f_k : f_k \in \mathcal{A}(B_n)\Big\} = \{f \in \mathcal{A}(B_n) : f(\mathbf{0}) = 0\}.$$

That original question was answered by Leibenzon [Kh]. A large number of papers were subsequently published on that topic—we now know that the answer is positive for the ball algebra and the polydisc algebra and many other domains, but negative in general; some related questions still remain open. For our purpose we not only need to know that the answer is positive for a particular domain Ω but also we need to control the way the functions $(f_1, \ldots, f_n)$ are generated by the function f. For a general domain Ω we need to replace the point $\mathbf{0}$ by an arbitrary point $\boldsymbol{w}_0 \in \Omega$; even if the answer to the Gleason Problem at such a point is positive the methods used for the ball algebra may not work if $\boldsymbol{w}_0$ is very close (in comparison with ε) to the boundary of Ω.

STEP 6. Show that

$$|Tf(\varphi(\mathbf{0})) - f(\mathbf{0})| \le \varepsilon\|f\| \quad \text{for } f \in \mathcal{A}(B_n). \tag{3.3}$$

STEP 7. Fix $\boldsymbol{w} = (w_1, \ldots, w_n) \in B^n$ and let $\Psi_{\boldsymbol{w}}$ be the holomorphic automorphism of the unit ball that maps $\boldsymbol{w}$ to $\mathbf{0}$. Put

$$\varphi(\boldsymbol{w}) := \Big\{\sum_{k=1}^{n} T(Z_k \circ \Psi_{\boldsymbol{w}}) \cdot g_k : g_k \in B\Big\};$$

deduce from the previous two steps that $\varphi(\boldsymbol{w})$ is a maximal ideal in B and that

$$|Tf(\varphi(\boldsymbol{w})) - f(\boldsymbol{w})| \le \varepsilon\|f\| \quad \text{for } f \in \mathcal{A}(B_n). \tag{3.4}$$

This step is quite routine but it depends on the fact that for both $\mathbb{D}^n$ and B_n the group of holomorphic automorphisms is transitive; this does not hold for most other domains Ω.

STEP 8. Extend the function φ to $\overline{B^n}$ by putting $\varphi(\boldsymbol{z}) = \boldsymbol{z}$ for $\boldsymbol{z} \in \partial B_n$ and show that the map φ is continuous and one-to-one. The continuity follows from the continuity of $\boldsymbol{w} \mapsto \Psi_{\boldsymbol{w}}$ and the control we established in Step 5 of the way the functions $(f_1, \ldots, f_n)$ are generated by f. The fact that φ is injective depends on certain topological properties of the unit ball; it should work for many other domains Ω but not for all of them.

STEP 9. Take a linear and multiplicative functional F on B and define a functional on $\mathcal{A}(B_n)$ by $G = F \circ T$; notice that G is almost multiplicative (see Definition 4.2 below). By [Ja97] for any almost multiplicative functional G on $\mathcal{A}(B_n)$ there is a multiplicative functional $\widetilde{G}$ with $\|\widetilde{G} - G\| \le \varepsilon$. The same result holds true for the polydisc algebra; it is however false for uniform algebras in general [SS]; whether all algebras $\mathcal{A}(\Omega)$ have this property is an open problem.

STEP 10. Show that the function φ is a surjection from $\overline{B^n}$ onto the maximal ideal space of B. The proof depends on the previous step.

STEP 11. Show that we can introduce an analytic structure on B_n such that all functions from $T^{-1}(B)$ are analytic. The proof of this step is based on a straightforward application of the Gleason Theorem [G] since we already demonstrated that the ideals in the maximal ideal space of B outside the Shilov boundary are finitely generated.

In the case of the ball algebra discussed above the maximal ideal space consists of two disjoint parts: the interior of the ball with an n-dimensional analytic structure and the Shilov boundary which coincides with the topological boundary of the ball. For the polydisc algebra $\mathcal{A}(\mathbb{D}^n)$ the maximal ideal space consists of three parts: the open polydisc with an n-dimensional analytic structure, the Shilov boundary which is equal to the Cartesian product of the unit circles, and the remaining part of the topological boundary of $\mathbb{D}^n$. That third part which is not present in the ball algebra requires a few extra steps in the proof. However, overall the proof for the polydisc algebra is simpler. Because the polydisc algebra is an injective tensor product of n copies of the disc algebra, and because we understand quite well small deformations of the disc algebra, we can simplify some of the other steps of the proof.

4. Related results and open problems. The concept of small perturbations discussed above for $\mathcal{A}(\Omega)$ algebras applies also to arbitrary Banach algebras and is especially interesting for uniform algebras, that is, for closed subalgebras of $C(X)$. For uniform algebras many natural definitions of perturbations coincide.

THEOREM 4.1. *Let A be a uniform algebra and let B_n be Banach algebras with units. Then $\lim_{n\to\infty} d_{\text{B-M}}(A, B_n) = 1$ if and only if there is a sequence ε_n of positive numbers such that $\lim_{n\to\infty} \varepsilon_n = 0$ and a sequence of linear invertible maps $T_n : A \to B_n$ with $T_n\mathbf{1} = \mathbf{1}$ such that*

$$\|T_n(fg) - T_n(f)T_n(g)\| \le \varepsilon_n \|f\|\,\|g\| \quad \textit{for all } f, g \in A.$$

The proof of this and several related results may be found in [Ja85]. The theory of small perturbations of Banach algebras directly involves almost multiplicative operators and functionals.

DEFINITION 4.2. A linear functional F defined on a Banach algebra A is called *δ-multiplicative* if

$$|F(fg) - F(f)F(g)| \le \delta \|f\|\,\|g\| \quad \text{for } f, g \in A.$$

One can easily construct an almost multiplicative functional by adding a small perturbation to a multiplicative functional; for some algebras this is the only way to construct such a functional but for some others there may be almost multiplicative functionals very far from the multiplicative ones. An interested reader may consult papers by B. E. Johnson [Jo77, Jo86, Jo88], R. Rochberg [R79a, R79b, R86], K. Jarosz [Ja85, Ja92, Ja97], S. J. Sidney [SS], and others for many interesting results. For example we know that a small perturbation of the disc algebra or of the algebra $H^\infty(\mathbb{D})$ of all bounded analytic functions defined on the unit disc is automatically identical with the original algebra; we know that this property is false in general; we know that an almost multiplicative functional on the disc algebra must be close to a multiplicative one; and we know that this property is false in general for uniform algebras. Surprisingly there are still a large number of very natural open questions. Below we list just a few of them.

PROBLEM 1. Assume F is a δ-multiplicative functional on $H^\infty(\mathbb{D})$. Is there a multiplicative functional G on $H^\infty(\mathbb{D})$ such that $\|F - G\| \to 0$ as $\delta \to 0$? We are asking here if the algebra $H^\infty(\mathbb{D})$ has an almost corona, that is, a set of almost multiplicative functionals far from the unit disc.

PROBLEM 2. Let Ω be a domain of holomorphy in $\mathbb{C}^n$, for $n > 1$. Assume $d_{\text{B-M}}(\mathcal{A}(\Omega), B) < \varepsilon$. Does it follow that $\mathcal{A}(\Omega) = B$ (as algebras)? The question is open for all domains in $\mathbb{C}^n$.

PROBLEM 3. Assume that A is a uniform algebra and $\varepsilon > 0$. Is it possible that the set

$$\{B : d_{\text{B-M}}(A, B) < 1 + \varepsilon\}$$

consists of countably many nonisomorphic algebras? In all known cases such a set consists of a single algebra (stability) or of uncountably many different and nonisomorphic algebras.

PROBLEM 4. It is known that the Choquet boundaries of uniform algebras A, B must be homeomorphic provided $d_{\text{B-M}}(A, B)$ is small enough. For a large class of algebras the Choquet boundary coincides with the Shilov boundary of that algebra. It is not known if the n-tuple Shilov boundaries must also be preserved. Since the n-tuple Shilov boundaries are directly related to the existence of a multidimensional analytic structure in the spectrum of the algebra (see [T]), any such result may advance our knowledge about the main question we discuss here: small deformations of $\mathcal{A}(\Omega)$ algebras.

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