ANNALES
POLONICI MATHEMATICI
106 (2012)

Hartogs type extension theorems on some domains in Kähler manifolds

by Takeo Ohsawa (Nagoya)

Abstract. Given a locally pseudoconvex bounded domain Ω, in a complex manifold M, the Hartogs type extension theorem is said to hold on Ω if there exists an arbitrarily large compact subset K of Ω such that every holomorphic function on $\Omega - K$ is extendible to a holomorphic function on Ω. It will be reported, based on still unpublished papers of the author, that the Hartogs type extension theorem holds in the following two cases: 1) M is Kähler and $\partial\Omega$ is C^2-smooth and not Levi flat; 2) M is compact Kähler and $\partial\Omega$ is the support of a divisor whose normal bundle is nonflatly semipositive.

Introduction. Analyzing the boundary behavior of holomorphic functions is one of the principal objectives of complex analysis. Accordingly, structures of the sets of singular points of analytic functions are of basic importance in function theory.

Hartogs [H] first studied analytic functions of several variables from this viewpoint, and established the pseudoconvexity of domains of holomorphy by proving that, for any polydisc $S = \{(z', z_n) \mid z' \in S',\ z_n \in \sigma\}$ ($z' = (z_1, \dots, z_{n-1})$) in $\mathbb{C}^n$, for any point $p \in S'$, for any neighborhood U of $(\{p\} \times \sigma) \cup (S' \times \partial\sigma)$, and for any holomorphic function f on U, there exists uniquely a holomorphic function $\widetilde{f}$ on S satisfying $\widetilde{f}|U = f$ (the Hartogs extension theorem). If this extendibility holds for holomorphic maps from S to a domain Ω in $\mathbb{C}^n$, then Ω is said to be *pseudoconvex in the sense of Hartogs*. Based on this, Levi [L] and Krzoska [Kr] described a geometric condition which every domain of holomorphy must satisfy at the smooth boundary points. After that, Cartan and Thullen [C–T] found that domains of holomorphy are holomorphically convex and vice versa. Exploiting the notion of holomorphic convexity, Oka [O-1,2,3], Stein [S] and Cartan [C] generalized the one-variable theorems of Mittag-Leffler, Weierstrass and Runge to the domains of holomorphy and Stein manifolds. As for the pseudoconvexity, especially in view of the works of Oka [O-4,5] and Grauert [G-1], the

2010 *Mathematics Subject Classification*: Primary 32D15.
Key words and phrases: analytic continuation, Kähler manifold, Levi form, cohomology.

DOI: 10.4064/ap106-0-19

equivalence of 1-completeness and Steinness is known for complex manifolds (solution of the Levi problem).

Let us recall that M is said to be *q-complete* (resp. *q-convex*) if it admits an exhaustion function of class C^2 whose Levi form has everywhere (resp. outside a compact subset) at most $q-1$ nonpositive eigenvalues. M is called *weakly 1-complete* if it carries a C^∞ plurisubharmonic function.

By the *Hartogs type extendibility* we shall mean that, for any compact set $K \subset \Omega$ such that $\Omega - K$ does not contain relatively compact components, holomorphic functions on $\Omega - K$ are extendible to Ω. In this sense, the assertion that $H_c^1(M, \mathcal{O})$ vanishes is equivalent to a Hartogs type extension theorem. Here $\mathcal{O}$ denotes the structure sheaf of M and $H_c^1(M, \mathcal{O})$ denotes the $\mathcal{O}$-valued first cohomology group of M with compact support. By a theorem of Andreotti and Grauert in [A–G], the above is known to hold if M is a connected $(n-1)$-complete manifold of dimension n. The notions of q-completeness and q-convexity were introduced in [A–G] to generalize the vanishing theorems and finiteness theorems for the sheaf cohomology on Stein manifolds. The theory of q-convex manifolds and subsequent works of Nakano [N] and Takayama [T] on weakly 1-complete manifolds with positive line bundles generalize also Kodaira's characterization of projective algebraic manifolds (cf. [K]), as well as Oka–Grauert's solution of the Levi problem (see also [G-2] and [G–R-2]).

It must be noted that the Hartogs type extension theorem does not generalize to arbitrary $(n-1)$-convex manifolds, because the complement of any (possibly disconnected) complex curve in $\mathbb{CP}_n$ is obviously $(n-1)$-convex. Therefore some additional condition is needed. An answer was given by a theorem of Grauert and Riemenschneider [G–R-1] which says that $H_c^1(\Omega, \mathcal{O})$ vanishes as long as Ω is a smoothly bounded relatively compact domain with hyper-$(n-1)$-convex boundary (for the definition, see §2 below) in a connected Kähler manifold of dimension n.

Although there need not exist nonconstant holomorphic functions on $(n-1)$-convex domains, generalizations of Hartogs type extension to differential forms have significant consequences in complex geometry. For instance, a generalization of [G–R-1] was given in [Oh-1] in the framework of L^2 Hodge theory, and recently applied to study existence questions on Levi flat hypersurfaces in $\mathbb{CP}_n$ and complex tori (cf. [Oh-3,4]).

On the purely function-theoretic side, it is still anticipated that Hartogs type extension for functions holds for locally pseudoconvex bounded domains with Levi nonflat smooth boundary, because it will then strengthen a solution of the Levi problem on two-dimensional manifolds (cf. [D–Oh]). A supporting evidence for that can be seen in a potential-theoretic study of complete Kähler manifolds by Napier and Ramachandran [N–R-1, Theorem 3.5]. By the way, it is known that there is a dichotomy between $H^1_c(M, \mathcal{O}) = 0$ and the existence of proper holomorphic maps from M onto a noncompact Riemann surface if M is a weakly 1-complete Kähler manifold with precisely one end (cf. [N–R-2]).

With this background, the author has focused on the function-theoretic side of the Hartogs type extension and obtained the following two results.

Theorem 0.1 (cf. Theorem 3.2 in [Oh-5]). *Let M be a complex manifold admitting a Kähler metric and let $\Omega \subset M$ be a bounded locally pseudoconvex domain with C^2-smooth boundary. Then the Hartogs type extension theorem holds on Ω, unless $\partial\Omega$ is everywhere Levi flat. In particular, $\partial\Omega$ is connected unless it is Levi flat.*

Theorem 0.2 (cf. Theorem 0.2 in [Oh-6]). *Let M be a connected compact Kähler manifold and let D be an effective divisor on M. Assume that the line bundle associated to D has a fiber metric whose curvature form is semipositive on the Zariski tangent spaces of the support $|D|$ of D and not identically zero there. Then the Hartogs type extension theorem holds on $M - |D|$. In particular $|D|$ is connected.*

Whether or not Theorem 0.2 is a limiting case of Theorem 0.1 is an open question. In general, it is true that there exist locally pseudoconvex bounded domains in complex manifolds which are not an increasing union of locally pseudoconvex domains with smooth boundary, e.g. the complement of exceptional divisors. However, it is likely that $M - |D|$ becomes weakly 1-complete under the assumption of Theorem 0.2 as suggested by a theory of Ueda (cf. [U]). It will be shown in §4 that this is actually the case if $\dim M \leq 2$. It might be worthwhile to note that Diederich and Fornaess [D–F] found a locally pseudoconvex bounded domain Ω with smooth boundary in a non-Kähler manifold such that Ω is not weakly 1-complete. Anyway, at least at the moment, it must be understood that Theorems 0.1 and 0.2 just concern hyperbolic ends and parabolic ends, respectively. In the recent terminology of pluripotential theory, this distinction is between pluricomplex Green function and Siciak's extremal function.

The purpose of the present article is to give outlines of the proofs of these theorems following [Oh-5,6], trying to unify the presentation, and to make some remarks in §4 on the structure of $M - |D|$ as above in connection with Theorem 0.2.

1. L^2 $\overline{\partial}$-cohomology: a general tool. Let (M, g) be a connected Hermitian manifold of dimension n and let (E, h) be a Hermitian holomorphic vector bundle over M. Let $\overline{\partial}$ (resp. ∂) denote the complex exterior derivative of type $(0,1)$ (resp. $(1,0)$) and put $\partial_h = h^{-1} \circ \partial \circ h$, regarding h as a smooth section of $\mathrm{Hom}(E, \overline{E}^*)$. Let ϑ_h (resp. $\overline{\vartheta}$) be the formal adjoint of $\overline{\partial}$ (resp. ∂_h)

with respect to g and h. Let ω be the fundamental form of g and let $\varLambda$ be the adjoint of the exterior multiplication by ω. Let $\varTheta_h$ be the curvature form of h, identified with its exterior multiplication from the left. Then it is easy to see that

$$(1.1) \qquad (\overline{\partial}\vartheta_h - \partial_h\overline{\vartheta})u = \sqrt{-1}\,\varTheta_h\varLambda u$$

for any C^∞ E-valued (n,n)-form u on M.

Let (u,v) denote the inner product of C^∞ E-valued compactly supported forms u and v, and let $\|u\|^2 = (u,u)$. By (1.1),

$$(1.2) \qquad \|\vartheta_h u\|^2 - \|\overline{\vartheta}u\|^2 = (\sqrt{-1}\,\varTheta_h\varLambda u, u)$$

if u is compactly supported.

Let ψ be any C^∞ real-valued function on M. We recall a local expression for the differential form $\sqrt{-1}\,\partial\overline{\partial}\psi\varLambda v$ for any (n,m)-form v on M. Let $\{\sigma_1,\ldots,\sigma_n\}$ be a local frame of the holomorphic cotangent bundle of M such that $\omega = \sqrt{-1}\sum_i \sigma_i\wedge\overline{\sigma}_i$ and $\partial\overline{\partial}\psi = \sum_i \lambda_i\sigma_i\wedge\overline{\sigma}_i$ for $\lambda_1\le\cdots\le\lambda_n$. Note that all λ_i are continuous on M.

Then, by letting

$$v = \sum_K v_K\sigma_*\wedge\overline{\sigma}_K,$$

where $\sigma_* = \sigma_1\wedge\cdots\wedge\sigma_n$ and $\overline{\sigma}_K = \overline{\sigma}_{k_1}\wedge\cdots\wedge\overline{\sigma}_{k_m}$ for $K=(k_1,\ldots,k_m)$, we have

$$(1.3) \qquad \sqrt{-1}\,\partial\overline{\partial}\psi\varLambda v = \sum \lambda_K v_K\sigma_*\wedge\overline{\sigma}_K,$$

where $\lambda_K = \lambda_{k_1}+\cdots+\lambda_{k_m}$. We put $\lambda^* = \lambda_{1\ldots n}$ for simplicity.

Combining (1.2) and (1.3) with $\varTheta_{he^{-\psi}} = \varTheta_h + \mathrm{Id}_E\otimes\partial\overline{\partial}\psi$, one easily deduces an inequality which implies the following by Hahn–Banach's theorem (cf. [A–V]).

THEOREM 1.1. *If (M,g) is complete, the infimum of λ^* on M is positive, and $\varTheta_h$ is bounded on M with respect to g and h, then there exists $c_0\in\mathbb{R}$ such that, for any $c>c_0$ and for any E-valued L^2 (n,n)-form v with respect to g and $h\exp(-c\psi)$, there exists an E-valued L^2 $(n,n-1)$-form u with respect to g and $h\exp(-c\psi)$ satisfying $\overline{\partial}u = v$.*

Let $\varOmega\subset M$ be a bounded domain with C^2-smooth pseudoconvex boundary. Then there exist a defining function ϱ of $\partial\varOmega$ with $\inf\varrho > -1$ such that ϱ is C^∞ on $\varOmega$, and a positive number L such that $g^* := Lg + \partial\overline{\partial}(1/\log(-\varrho))$ is a complete Hermitian metric on $\varOmega$. Since $1/\log(-\varrho)$ is bounded, the following is a corollary of Theorem 1.1.

THEOREM 1.2. *Let (M,g) and (E,h) be as above and let $\varOmega\subset M$ be a bounded domain with C^2-smooth pseudoconvex boundary. Then, with respect* *to the metrics g^* and $h|\Omega$, for any E-valued L^2 (n,n)-form v on Ω there exists an E-valued L^2 $(n,n-1)$-form u on Ω satisfying $\overline{\partial}u = v$.*

With respect to g^* and h as above, we denote by $H^{p,q}_{(2)}(\Omega, E)$ the E-valued L^2 $\overline{\partial}$-cohomology group of Ω of type (p,q). Then the conclusion of Theorem 1.2 is expressed as $H^{n,n}_{(2)}(\Omega, E) = \{0\}$.

Since g^* is complete, Theorem 1.2 implies the following by duality.

PROPOSITION 1.1. *In the situation of Theorem* 1.2, *$H^{0,1}_{(2)}(\Omega, E)$ is Hausdorff.*

(E,h) is arbitrary so far. Imposing some curvature condition on h, a vanishing theorem for $H^{0,1}_{(2)}(\Omega, E)$ will be obtained. For the moment, we shall be contented with the case where E is the trivial line bundle and $h = 1$. In this case $H^{0,1}_{(2)}(\Omega, E)$ will be denoted by $H^{0,1}_{(2)}(\Omega)$.

THEOREM 1.3. *Let M be a complex manifold admitting a Kähler metric and let Ω be a bounded domain in M with C^2-smooth pseudoconvex boundary such that $\partial\Omega$, is not everywhere Levi flat. Then $H^{0,1}_{(2)}(\Omega) = \{0\}$.*

Proof. Since $\partial\Omega$ is pseudoconvex and not Levi flat, there exists a point $x \in \partial\Omega$ and a neighborhood U of x in M such that the Levi form of ϱ has at least one positive eigenvalue everywhere on $U \cap \partial\Omega$. Let χ be a nonnegative C^∞ function on M such that $x \in \operatorname{supp}\chi \subset U$. Then one can find $\varepsilon > 0$ such that, for any C^∞ convex increasing function τ on R, the sums of $n-1$ eigenvalues of $\partial\overline{\partial}\tau(1/\log(-\varrho) + \varepsilon\chi)$ with respect to g^* are nonnegative everywhere on Ω.

By choosing τ to be strictly increasing on $(0,\infty)$ and $\tau(t) = 0$ on $(-\infty, 0]$, one has a bounded C^∞ function $\Psi := \tau(1/\log(-\varrho) + \varepsilon\chi/2)$ on Ω such that the sums of $n-1$ eigenvalues of $\partial\overline{\partial}\Psi$ are nonnegative on Ω, and strictly positive on $V \cap \Omega$ for some neighborhood V of x.

Then, by the boundedness of Ψ, in view of Nakano's identity on Kähler manifolds, there exists a nonnegative C^∞ function c on M with $c(x) > 0$ such that

$$\|\overline{\partial}^* u\|^2 + \|\overline{\partial}u\|^2 \geq (cu, u)$$

for any C^∞ compactly supported $(0,1)$-form u on Ω. Here $|\cdot|$ denotes the L^2 norm with respect to g^*, and $\overline{\partial}^*$ the adjoint of ∂.

This implies that every element of $H^{0,1}_{(2)}(\Omega)$ is in the closure of $\{0\}$ by virtue of the unique continuation theorem of Aronszajn. On the other hand, it is known from Proposition 1.1 that $H^{0,1}_{(2)}(\Omega)$ is Hausdorff. Therefore $H^{0,1}_{(2)}(\Omega) = \{0\}$,as required.

It is clear that, for any compact subset K of Ω, there exist no L^2 holomorphic functions on $\Omega - K$ with respect to g. Hence we obtain

COROLLARY 1.1. *In the above situation,* $H^1_c(\Omega, 0) = \{0\}$. *In particular, the restriction map*

$$H^0(\Omega, \mathcal{O}) \to \varinjlim H^0(\Omega \setminus K, \mathcal{O})$$

is surjective. Here K *runs through the compact subsets of* Ω.

Proof of Theorem 0.1. If $\partial\Omega$ were neither connected nor Levi flat, one would have a locally constant but noncostant holomorphic function on Ω, which is an absurdity.

In [D–Oh], the following was proved.

THEOREM 1.4. *Let* Ω *be a relatively compact domain with smooth real-analytic boundary in a complex manifold* M. *Assume that* $\dim M = 2$ *and* $\partial\Omega$ *is connected and strongly pseudoconvex at some point. Then there is a compact analytic subset* $A \subset \Omega$, *a Stein space* Ω^*, *and a proper holomorphic map* $\varpi : \Omega \to \Omega^*$ *such that* $\Omega \setminus A$ *and* $\Omega^* \setminus \varpi(A)$ *are biholomorphic under* ϖ.

By Theorem 0.1, the connectedness assumption on $\partial\Omega$ is a consequence of the Levi nonflatness assumption if M is Kählerian.

2. q-convex and q-concave. As in §1, let $\Omega \subset M$ be a bounded domain whose boundary $\partial\Omega$ is C^2-smooth. Recall that a defining function ϱ of $\partial\Omega$ is a real-valued C^2 function defined on a neighborhood, say U, of the closure $\overline{\Omega}$ of Ω such that $\Omega = \{x \in U; \varrho(x) < 0\}$ and $d\varrho$ vanishes nowhere on $\partial\Omega$.

Let us denote by $T(\partial\Omega)$ the tangent bundle of $\partial\Omega$ which is naturally embedded in the tangent bundle of M. We put

$$T^{1,0}(\partial\Omega) = \{v \in T^{1,0}M \cap (T(\partial\Omega) \otimes \mathbb{C});\ \partial\varrho(v) = 0\}, \tag{2.1}$$

where $T^{1,0}M$ stands for the holomorphic tangent bundle of M.

By the *Levi signature* of $\partial\Omega$ at $x \in \partial\Omega$ we shall mean the signature of the Hermitian form

$$\begin{array}{ccc} T^{1,0}(\partial\Omega) \times T^{1,0}(\partial\Omega) & \longrightarrow & \mathbb{C} \\ \Cup & & \Cup \\ (v, w) & \longrightarrow & \partial\overline{\partial}\varrho(v \wedge \overline{w}) \end{array}$$

Note that $\partial\overline{\partial}\varrho$ is the $(1,1)$-part of $-d\overline{\partial}\varrho$. It is clear that the Levi signature does not depend on the choices of defining functions of Ω. If a Hermitian metric g is given on M, the eigenvalues of $\partial\overline{\partial}\varrho(v\wedge\overline{w})$ with respect to g depend on the choice of ϱ, but only up to multiplication by a positive function on $\partial\Omega$.

If the Levi signature (s,t) of $\partial\Omega$ everywhere satisfies $s \geq n - q$ (resp. $t \geq n - q$), we say that $\partial\Omega$ is *q-convex* (resp. *q-concave*). If a Hermitian metric g is given on a neighborhood of $\overline{\Omega}$ and the sums of q eigenvalues of $\partial\overline{\partial}\varrho(v\wedge\overline{w})$ with respect to g are everywhere positive (resp. negative) on $\partial\Omega$, then we say $\partial\Omega$ is *hyper-q-convex* (resp. *hyper-q-concave*) with respect to g. We say that Ω is q-convex, q-concave and so on if so is $\partial\Omega$.

M is called a *q-complete manifold* if there exists a C^2 exhaustion function φ on M such that φ is everywhere *q-convex* in the sense that the Hermitian form

$$\begin{array}{ccc} T^{1,0}M \times T^{1,0}M & \longrightarrow & \mathbb{C} \\ \Cup & & \Cup \\ (v,w) & \longrightarrow & \partial\overline{\partial}\varphi(v \wedge \overline{w}) \end{array}$$

which is called the *Levi form* of φ, has everywhere at least $n-q+1$ positive eigenvalues. The Levi form of φ will be denoted simply by $\partial\overline{\partial}\varphi$. *Hyper-$q$-convexity* of a function is similarly defined with respect to a Hermitian metric.

It was first proved by Greene and Wu [G–W] that M is n-complete if and only if M is noncompact. The proof is based on an embedding theorem by harmonic functions. An elementary proof of Greene–Wu's theorem was given in [Oh-2]. The proof of Proposition 2.1 below, which is a convexity assertion needed for the proof of Theorem 0.2, is based on the argument of [Oh-2].

Given a compact complex submanifold $S \subset M$, the above-mentioned convexity properties of neighborhoods of S are derived from the curvature properties of the normal bundle of S. This relation naturally extends to the case of effective divisors.

Let D be an effective divisor on M such that the line bundle $[D]$ associated to D has a fiber metric b whose curvature form satisfies the assumption of Theorem 0.2. (Neither compactness nor Kählerianity is assumed on M here.)

We fix a canonical section s of $[D]$ and denote by $|s|$ the length of s with respect to b. Let $b^\wedge$ be a fiber metric of $[|D|]$, let $s^\wedge$ be a canonical section of $[|D|]$, and let $|s^\wedge|$ be the length of $s^\wedge$ with respect to $b^\wedge$.

If $|D|$ is compact, then replacing b by $b\exp(-A|s^\wedge|^2)$ for sufficiently large $A > 0$, we may assume in advance that the curvature form of b is semipositive at every point of $|D|$ and of rank ≥ 2 at some point x_0 of $|D|$.

PROPOSITION 2.1. *If $|D|$ is compact and x_0 is as above, then, for any Hermitian metric g on M and for any connected component D_0 of $|D|$ containing x_0, there exists an exhaustion function on $M - D_0$ which is hyper-$(n-1)$-convex with respect to g outside a compact subset.*

For the proof, see [Oh-6, proof of Proposition 1.2].

3. Proof of Theorem 0.2. Based on Proposition 2.1, the proof of Theorem 0.2 proceeds similarly to §1. Let us describe a condition for the vanishing of the L^2 $\overline{\partial}$-cohomology in a somewhat more general form.

Let (M, g) be a connected complete Kähler manifold of dimension n and let (E, h) be a holomorphic Hermitian vector bundle over M. The curvature form Θ_h of h is said to be *Nakano semipositive* if it induces a semipositive quadratic form on $T^{1,0}M \otimes E$ by contraction with h. (As before, for the application to Theorem 0.2, E will be the trivial line bundle and $h = 1$.)

Let ψ be a real-valued C^∞ function on M, let $x \in M$ and let $\{\sigma_1, \ldots, \sigma_n\}$ be a basis of $(T^{1,0}M)^*$, the holomorphic cotangent space of M at x, such that $\omega = \sqrt{-1}\sum_i \sigma_i \wedge \overline{\sigma}_i$ and $\partial\overline{\partial}\psi = \sum_i \lambda_i \sigma_i \wedge \overline{\sigma}_i$ $(\lambda_1 \le \cdots \le \lambda_n)$ hold at x, as before.

Then, by letting

$$u = \sum u_i \overline{\sigma}_i \quad \text{at } x$$

we have

$$\sqrt{-1}\,\Lambda\partial\overline{\partial}\psi \wedge u = \sum(\lambda^* - \lambda_i)u_i\overline{\sigma}_i \quad \text{at } x, \tag{3.1}$$

where λ^* is as before.

Let $H^{p,q}_\psi(M, E)$ denote the L^2 $\overline{\partial}$-cohomology group of type (p, q) with respect to g and $h\exp\psi$. Since (M, g) is complete, combining (3.1) with $O_{he^\psi} = \Theta_h - \mathrm{Id}_E \otimes \partial\overline{\partial}\psi$, from (3.1) one deduces the following similarly to Theorem 1.3.

THEOREM 3.1. *Assume that the dual bundle of (E, h) is Nakano semipositive,* $\inf \lambda^*$ *is positive for some compact set $B \subset M$, and $\lambda^* - \lambda_n$ is everywhere nonnegative and somewhere positive. Then $H^{0,0}_\psi(M, E) = \{0\}$ and $H^{0,0}_\psi(M, E) = \{0\}$.*

Proof of Theorem 0.2 (outline). Let M and D be as in Theorem 0.2. Let D' be the union of connected components of $|D|$ along which the curvature form of h is tangentially identically zero. By Proposition 2.1, one can find an exhaustion function Ψ on the complement of $|D| - D'$ which is hyper-$(n-1)$-convex near $|D| - D'$ with respect to a Kähler metric g on M. Let σ denote the fundamental form of g.

We fix $c \in \mathbb{R}$ in such a way that the sums of $n-1$ eigenvalues of $\partial\overline{\partial}\Psi$ are positive on the set $V(c) = \{x;\, \Psi(x) > c\}$. We may assume that $V(c-1) \cap D' = \emptyset$. Then we put

$$\gamma = \inf\{\log|s(x)|;\, x \in \partial V(c)\}$$

and

$$\omega_\varepsilon = \begin{cases} \omega - \varepsilon\sqrt{-1}\,\partial\overline{\partial}\lambda(\Psi - c) & \text{on } V(c), \\ \omega - \varepsilon\sqrt{-1}\,\partial\overline{\partial}\xi(-\log|s| + \gamma) & \text{on } M - V(c) - D', \end{cases}$$

where ε is a positive number, λ is a real-valued C^∞ function on $\mathbb{R}$ satisfying $\lambda(t) = 0$ if $t < 1$ and $\lambda(t) = \log t$ if $t \ge 2$, and $\xi : \mathbb{R} \to \mathbb{R}$ is a C^∞ function with $\operatorname{supp}\xi \subset (1, \infty)$ such that $\xi(t) = t^2$ if $t \ge 2$. It is easy to see that ω_ε is the fundamental form of a complete Kähler metric, say g_ε, on $M - |D|$ if ε is sufficiently small.

Denoting by $\varkappa(x)$ the minimum of the sums of $n-1$ eigenvalues of $\partial\overline{\partial}(\Psi - c)^2$ at $x \in M - |D|$ with respect to g_ε, it is easy to see that $\inf\{\varkappa(x); \Psi(x) > c\}$ diverges to ∞ as $c \to \infty$. Hence, by composing a convex increasing function with $(\Psi - c)^2$ we obtain a function, say φ, satisfying the conditions of ψ in Theorem 3.1 with respect to g_ε.

Then, by Theorem 3.1, for any neighborhood $V \supset |D|$, for any holomorphic function f on $V - |D|$, and for any C^2 function $\widehat{f}$ on $M - |D|$ which coincides with f outside a compact subset, the equation $\overline{\partial} u = \overline{\partial}\widehat{f}$ has a solution u on $M - |D|$ which is square integrable with respect to the measure defined as the product of the volume form of g_ε and $\exp\varphi$. From the L^2 condition, it is easy to see that u is extendible holomorphically across D'.

It is also clear that $|s(x)|^\nu \exp\varphi(x)$ diverges as $x \to |D| - D'$ for any $\nu \in \mathbb{N}$. Therefore u must vanish outside a compact subset of $M - (|D| - D')$. On the other hand, it is easy to see by the maximum principle that u is locally constant on a neighborhood of D' (cf. [Oh-6, Proposition 1.5]). Combining this with the infiniteness of the volume of g_ε around D', we conclude that u is zero on a neighborhood of D'. Thus $\widehat{f} - u$ is the desired extension of f.

Remark 3.1. Combining the last paragraph of the above proof with a vanishing theorem of Demailly–Peternell [Dm–P], one gets a shorter proof of Theorem 0.2 (cf. [Oh-6]).

4. Notes and remarks. If $\dim M \le 2$, Theorem 0.2 can be strengthened as follows.

Theorem 4.1. *Let M be a connected compact complex manifold of dimension 2 and let D be an effective divisor on M. If $D^2 > 0$, then there exists a connected component D^* of $|D|$ such that $M - D^*$ is 1-convex.*

Proof. Replacing D, if necessary by an effective divisor D_0 such that $D_0^2 > 0$, $D - D_0$ is effective, $|D - D_0| \cap |D_0| = \emptyset$ and $|D_0|$ is connected, we may assume that D is connected. Let $D = \sum_{i=1}^m n_i D_i$, where D_i are irreducible and $n_i \in \mathbb{N}$. Then $D^2 > 0$ means that the matrix $\Delta = (\Delta_{ij})$ defined by $\Delta_{ij} = D_i \cdot D_j$ satisfies

$$\sum_{1 \le i,j \le m} \Delta_{ij} n_i n_j > 0. \tag{4.1}$$

From this property of Δ it is easy to deduce that the image of the set $(0,\infty)^m$ under the linear map from $\mathbb{R}^m$ to itself defined by the matrix Δ must have a nonempty intersection with $(0,\infty)^m$. Hence one can find an element ν of $\mathbb{N}^m$ whose image under Δ does not have nonpositive components.

Thus, replacing the coefficients of D by the components of ν, one has an effective divisor D° satisfying $|D^\circ| = |D|$ and $D^\circ \cdot D > 0$ for all j. Hence $M - |D|$ $(= M - |D^\circ|)$ is 1-convex by Proposition 2.1.

Remark 4.1. There exists a compact complex manifold M of dimension 3 which admits a nonsingular divisor D with semipositive normal bundle such that $D^3 > 0$ holds but $M - |D|$ is not holomorphically convex (cf. [G-3]).

Remark 4.2. One can prove Theorem 4.1 also by combining Grothendieck's lemma asserting that $H^0(M, [\mu D]) \neq \{0\}$ for sufficiently large μ, which follows from Riemann–Roch's theorem and $D^2 > 0$, and Simha's theorem [Sm] asserting that the complement of any closed complex curve in a Stein surface is Stein. However, an advantage of the above proof is that it naturally extends to pseudoholomorphic divisors in almost complex surfaces. Since the notions of pseudoconvexity and Levi flatness are naturally carried over to real hypersurfaces in almost complex manifolds, extension of Theorems 0.1 and 0.2 to almost complex symplectic manifolds might be interesting as a question of topology.

References

[A–G] A. Andreotti et H. Grauert, *Théorème de finitude pour la cohomologie des espaces complexes*, Bull. Soc. Math. France 90 (1962), 193–259.

[A–V] A. Andreotti and E. Vesentini, *Carleman estimates for the Laplace–Beltrami equation on complex manifolds*, Inst. Hautes Études Sci. Publ. Math. 25 (1965), 81–130; Erratum, ibid. 27 (1965), 153–155.

[C] H. Cartan, *Problèmes globaux dans la théorie des fonctions analytiques de plusieurs variables complexes*, in: Proc. Int. Congress of Mathematicians (Cambridge, MA, 1950), Vol. 1, Amer. Math. Soc., Providence, RI, 1952, 152–164.

[C–T] H. Cartan und P. Thullen, *Zur Theorie der Singularitäten der Funktionen mehrerer komplexen Veränderlichen: Regularitäts- und Konvergenzbereiche*, Math. Ann. 106 (1932), 617–647.

[Dm–P] J.-P. Demailly and T. Peternell, *A Kawamata–Viehweg vanishing theorem on compact Kähler manifolds*, J. Differential Geom. 63 (2003), 231–277.

[D–F] K. Diederich and J.-E. Fornæss, *A smooth pseudoconvex domain without pseudoconvex exhaustion*, Manuscripta Math. 39 (1982), 119–123.

[D–Oh] K. Diederich and T. Ohsawa, *A Levi problem on two-dimensional complex manifolds*, Math. Ann. 261 (1982), 255–261.

[G-1] H. Grauert, *On Levi's problem and the imbedding of real-analytic manifolds*, Ann. of Math. (2) 68 (1958), 460–472.

[G-2] H. Grauert, *Über Modifikationen und exzeptionelle analytische Mengen*, Math. Ann. 146 (1962), 331–368.

[G-3] H. Grauert, *Bemerkenswerte pseudokonvexe Mannigfaltigkeiten*, Math. Z. 81 (1963), 377–391.

[G–R-1] H. Grauert und O. Riemenschneider, *Kählersche Mannigfaltigkeiten mit hyper-q-konvexem Rand*, in: Problems in Analysis (Lectures Sympos. in Honor of Salomon Bochner, Princeton, 1969), Princeton Univ. Press, Princeton, NJ, 1970, 61–79.

[G–R-2] H. Grauert und O. Riemenschneider, *Verschwindungssätze für analytische Kohomologiegruppen auf komplexen Räumen*, Invent. Math. 11 (1970), 263–292.

[G–W] R. E. Greene and H. Wu, *Embedding of open Riemannian manifolds by harmonic functions*, Ann. Inst. Fourier (Grenoble) 25 (1975), 215–235.

[H] F. Hartogs, *Zur Theorie der analytischen Funktionen mehrerer unabhängiger Veränderlichen, insbesondere über die Darstellung derselben durch Reihen, welche nach Potenzen einer Veränderlichen fortschreiten*, Math. Ann. 62 (1906), 1–88.

[K] K. Kodaira, *On Kähler varieties of restricted type* (*an intrinsic characterization of algebraic varieties*), Ann. of Math. (2) 60 (1954), 28–48.

[Kr] J. Krzoska, *Über die naturlichen Grenzen der analytischen Funktionen mehrerer Veränderlichen*, Dissertation, Greifswald, 1933.

[L] E. E. Levi, *Sulle ipersuperficie dello spazio a 4 dimensioni che possono essere frontiera del campo di esistenza di una funzione analitica di due variabili complesse*, Ann. Mat. Pura Appl. 18 (1911), 69–79.

[N] S. Nakano, *Vanishing theorems for weakly 1-complete manifolds*, in: Number Theory, Algebraic Geometry and Commutative Algebra, in Honor of Yasuo Akizuki, Kinokuniya, Tokyo, 1973, 169–179.

[N–R-1] T. Napier and M. Ramachandran, *Structure theorems for complete Kähler manifolds and applications to Lefschetz type theorems*, Geom. Funct. Anal. 5 (1995), 809–851.

[N–R-2] T. Napier and M. Ramachandran, *The Bochner–Hartogs dichotomy for weakly 1-complete Kähler manifolds*, Ann. Inst. Fourier (Grenoble) 47 (1997), 1345–1365.

[Oh-1] T. Ohsawa, *A reduction theorem for cohomology groups of very strongly q-convex Kähler manifolds*, Invent. Math. 63 (1981), 335–354.

[Oh-2] T. Ohsawa, *Completeness of noncompact analytic spaces*, Publ. Res. Inst. Math. Sci. 20 (1984), 683–692.

[Oh-3] T. Ohsawa, *On the complement of Levi-flats in Kähler manifolds of dimension* ≥ 3, Nagoya Math. J. 185 (2007), 161–169.

[Oh-4] T. Ohsawa, *A remark on pseudoconvex domains with analytic complements in compact Kähler manifolds*, J. Math. Kyoto Univ. 47 (2007), 115–119.

[Oh-5] T. Ohsawa, L^2 *vanishing for certam* $\bar{\partial}$*-cohomology of pseudoconvex Levi nonflat domains and Hartogs type extension*, preprint.

[Oh-6] T. Ohsawa, *On the complement of effective divisors with semipositive normal bundle*, Kyoto J. Math. 52 (2012), 503–515.

[O-1] K. Oka, *Sur les fonctions analytiques de plusieurs variables. I. Domaines convexes par rapport aux fonctions rationnelles*, J. Sci. Hiroshima Univ. Ser. A 6 (1936), 245–255.

[O-2] K. Oka, *Sur les fonctions analytiques de plusieurs variables. II. Domaines d'holomorphie*, J. Sci. Hiroshima Univ. Ser. A 7 (1937), 115–130.

[O-3] K. Oka, *Sur les fonctions analytiques de plusieurs variables. III. Deuxième problème de Cousin*, J. Sci. Hiroshima Univ. Ser. A 9 (1939), 7–19.

[O-4] K. Oka, *Sur les fonctions analytiques de plusieurs variables. VI. Domaines pseudoconvexes*, Tôhoku Math. J. 49 (1942), 15–52.

[O-5] K. Oka, *Sur les fonctions analytiques de plusieurs variables. IX. Domaines finis sans point critique intérieur*, Japan. J. Math. 27 (1953), 97–155.

[Sm] R. R. Simha, *On the complement of a curve on a Stein space of dimension two*, Math. Z. 82 (1963), 63–66.

[S] K. Stein, *Topologische Bedingungen für die Existenz analytischer Funktionen komplexer Veränderlichen zu vorgegebenen Nullstellenflächen*, Math. Ann. 117 (1941), 727–757.

[T] S. Takayama, *Adjoint linear series on weakly 1-complete Kähler manifolds. I. Global projective embedding*, Math. Ann. 311 (1998), 501–531.

[U] T. Ueda, *On the neighborhood of a compact complex curve with topologically trivial normal bundle*, J. Math. Kyoto Univ. 22 (1982/83), 583–607.

Takeo Ohsawa
Graduate School of Mathematics
Nagoya University
Chikusaku Furocho 464-8602 Nagoya, Japan
E-mail: ohsawa@math.nagoya-u.ac.jp

Received 2.8.2011
and in final form 21.11.2011 (2601)