ANNALES
POLONICI MATHEMATICI
106 (2012)

A class of maximal plurisubharmonic functions

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Dedicated to Professor Józef Siciak

Abstract. We consider a class of maximal plurisubharmonic functions and prove several properties of it. We also give a condition of maximality for unbounded plurisubharmonic functions in terms of the Monge–Ampère operator $(dd^c e^u)^n$.

1. Introduction. Complex pluripotential theory, based on plurisubharmonic (psh) functions and the Monge–Ampère operator $(dd^c u)^n$, is one of the important directions in potential theory and multidimensional complex analysis. Built in the 1980s, the theory has already found many applications in the geometrical questions of complex analysis and in the theory of psh functions. By that time the extremal Green function $\Psi(z,K)$, $K \subset\subset \mathbb{C}^n$, which was introduced by J. Siciak for the multidimensional Bernstein–Walsh theorem, the P-measure $\omega(z,K,D)$, where $K \subset D \subset \mathbb{C}^n$, the P-capacity $P(K,D)$, the condenser capacity $C(K,D)$ and other basic objects of this theory were mostly established and studied (see [S], [BT1], [BT2], [S1]–[S3], [Z]).

It is well-known that harmonic functions have a maximality property in the class of subharmonic (sh) functions: if u is harmonic in a domain $\Omega \subset \mathbb{C}$ then for every subharmonic function $v \in \mathrm{sh}(\Omega)$ such that $\liminf_{z\to\partial\Omega}(u(z)-v(z)) \geq 0$ we have $u(z) \geq v(z)$ for all $z \in \Omega$. This property of harmonic functions leads us to the definition of maximal plurisubharmonic (psh) functions in the multidimensional case of $\Omega \subset \mathbb{C}^n$.

Definition 1.1 ([S2]). We say that a function $u \in \mathrm{psh}(\Omega)$ is *maximal* in the domain Ω if the maximum principle holds, i.e. whenever $v \in \mathrm{psh}(\Omega)$ satisfies $\liminf_{z\to\partial\Omega}(u(z)-v(z)) \geq 0$, then $u(z) \geq v(z)$ for all $z \in \Omega$.

In contrast to the classical case $n = 1$, where every maximal function is harmonic, and therefore C^∞ smooth, a maximal psh function in $\mathbb{C}^n$, $n > 1$,

2010 *Mathematics Subject Classification*: 32U05, 32U15, 32U35.
Key words and phrases: plurisubharmonic function, Green function, maximal function, Monge–Ampère operator.

DOI: 10.4064/ap106-0-21

need not be C^∞. For example, the function $\ln|z_1|$, which is maximal in $\mathbb{C}^2_{z_1,z_2}$, shows that there exist maximal functions which are unbounded.

For further study of maximal functions in $\mathbb{C}^n$ we recall the following standard notation:

$$d = \partial + \overline{\partial}, \qquad d^c = i(\overline{\partial} - \partial),$$

where

$$\partial = \frac{\partial}{\partial z_1} dz_1 + \cdots + \frac{\partial}{\partial z_n} dz_n, \qquad \overline{\partial} = \frac{\partial}{\partial \overline{z}_1} d\overline{z}_1 + \cdots + \frac{\partial}{\partial \overline{z}_n} d\overline{z}_n$$

so that

$$dd^c u = 2i\partial\overline{\partial}u, \qquad du \wedge d^c u = 2i\partial u \wedge \overline{\partial}u$$

and

$$(dd^c u)^n = dd^c u \wedge \cdots \wedge dd^c u = \operatorname{const} \det\left(\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\right) dV.$$

Bremermann [B] noted that if $u \in C^2(\Omega) \cap \operatorname{psh}(\Omega)$ is maximal, then $(dd^c u)^n = 0$. Later Kerzman [K] proved that if $(dd^c u)^n = 0$ then u is maximal. For a bounded $u \in \operatorname{psh}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ Bedford and Taylor [BT1] defined the Monge–Ampère operator as a current, by the following recurrence relation:

$$(1) \quad \int (dd^c u)^k \wedge \varphi = \int u (dd^c u)^{k-1} \wedge dd^c \varphi, \quad \varphi \in D^{(n-k,n-k)},\ k = 1, \ldots, n-1.$$

Here the space of test forms is $D^{(n-k,n-k)}$, the space of all differential forms of bi-degree $(n-k, n-k)$ with C^∞ coefficients and such that $\operatorname{supp}\varphi \subset\subset \Omega$. Later, in [BT2] it was proved that $(dd^c u)^k$ is well-defined, i.e. $(dd^c u_j)^k \to (dd^c u)^k$ for any approximation $u_j \downarrow u$. Bounded maximal functions are characterized by the Monge–Ampère equation: $u \in L^\infty_{\mathrm{loc}}(\Omega) \cap \operatorname{psh}(\Omega)$ is maximal if and only if $(dd^c u)^n = 0$. Moreover, the following comparison principle of Bedford–Taylor [BT2] is true: if $u, v \in \operatorname{psh}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ and $F = \{z \in \Omega : u(z) < v(z)\} \subset\subset \Omega$, then

$$(2) \qquad \int_F (dd^c u)^n \geq \int_F (dd^c v)^n.$$

In general, the definition of $(dd^c u)^n$ for arbitrary $u \in \operatorname{psh}(\Omega)$ is still a hard problem. Namely, first, in 1975 Shiffman and Taylor showed that there is a $u \in \operatorname{psh}(\mathbb{C}^n)$ such that $\int_B (dd^c u_j)^n \to \infty$ for a ball $B \subset\subset \mathbb{C}^n$, where $u_j \downarrow u$. Moreover Kiselman [KS] constructed the following simple example: the function

$$u(z) = (-\ln|z_1|)^{1/n}(|z_2|^2 + \cdots + |z_n|^2 - 1),$$

which is psh near the origin, has unbounded Monge–Ampère mass near $z_1 = 0$.

Secondly, U. Cegrell [C1] suggested the following example. For the psh function $u(z) = \ln|z_1|^2 + \cdots + \ln|z_n|^2$, if we take the approximation

$$u_j(z) = \ln(|z_1 \ldots z_n|^2 + 1/j) \downarrow u(z),$$

then $(dd^c u_j)^n \to 0$. On the other hand, if we take the approximation

$$v_j(z) = \ln(|z_1|^2 + 1/j) + \cdots + \ln(|z_n|^2 + 1/j) \downarrow u(z),$$

then $(dd^c u_j)^n \to n!4^n\delta_0$, where δ_0 is the Dirac measure.

This example shows that for arbitrary psh functions the operator $(dd^c u)^n$ cannot be well-defined by approximation.

The class $\mathcal{E}(\Omega)$ of psh functions, bigger than $\mathrm{psh}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$, yet with the Monge–Ampère operator $(dd^c u)^n$ well-defined on it, was introduced by Cegrell [C2]. Afterwards, Z. Błocki [B2] proved that $\mathcal{E}(\Omega)$ is the maximal class of psh functions for which the operator $(dd^c u)^n$ is well-defined, i.e. for each open set $U \subset\subset \Omega$ there exists a Borel measure μ such that for any sequence $u_j \in \mathrm{psh}(U) \cap C^\infty(U)$ with $u_j \downarrow u$ we have $(dd^c u_j)^n \to \mu$. In this case we put $(dd^c u)^n = \mu$.

For a function u in the Cegrell class $\mathcal{E}(\Omega)$ all currents $(dd^c u)^k$, $1 \le k \le n$, are also well-defined, and $u \in \mathcal{E}(\Omega)$ is maximal if and only if $(dd^c u)^n = 0$. For more details on the Cegrell class $\mathcal{E}(\Omega)$ see [C1]–[C3], [B1]–[B3], [Ko], [CGZ].

The aim of this note is to give a bigger, than $\{u \in \mathcal{E}(\Omega) : (dd^c u)^n = 0\}$, class of maximal functions in terms of $(dd^c e^u)^n$ (conditions (14) and (15) of Theorem 3.3). Every maximal psh function satisfies (14) but, unfortunately, it is unknown to the author if all maximal psh functions satisfy (15).

2. Some properties of maximal psh functions. The next proposition is convenient in applications (see [S2, C3, Kl])

Proposition 2.1. *The following statements are equivalent:*

(i) *u is maximal in Ω;*

(ii) *for any domain $G \subset\subset \Omega$ and for any function $v \in \mathrm{psh}(G)$,*

$$\liminf_{z \to \partial G}(u(z) - v(z)) \ge 0 \quad \textit{implies} \quad u(z) \ge v(z),\ \forall z \in G;$$

(iii) *for any domain $G \subset\subset \Omega$ and for any function $v \in \mathrm{psh}(\Omega)$,*

$$u|_{\partial G} \ge v|_{\partial G} \quad \textit{implies} \quad u(z) \ge v(z),\ \forall z \in G.$$

The implication (iii)⇒(i) is clear. For the implications (i)⇒(ii)⇒(iii) we note that the function

$$w(z) = \begin{cases} \max\{u(z), v(z)\} & \text{if } z \in G, \\ u(z) & \text{if } z \in \Omega \setminus G, \end{cases} \tag{3}$$

is psh in Ω and $\liminf_{z\to\partial\Omega}(u(z) - w(z)) = 0$. Hence $u(z) \geq w(z)$ for all $z \in \Omega$ and $u(z) \geq v(z)$ for all $z \in G$. ∎

THEOREM 2.2. *If for $u \in \mathrm{psh}(\Omega)$ there exists a sequence $u_j \in \mathrm{psh}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ with $u_j \downarrow u$ and $(dd^c u_j)^n \to 0$, then u is maximal. Conversely, if u is maximal, then there exists an approximation $\{u_j\}$ such that*

$$\begin{aligned} &u_j \in \mathrm{psh}(\Omega_j) \cap L^\infty_{\mathrm{loc}}(\Omega_j), \quad (dd^c u_j)^n = 0, \quad u_j(z) \downarrow u(z), \\ &\Omega_j \subset\subset \Omega_{j+1} \subset\subset \Omega, \qquad \Omega = \bigcup_{j=1}^{\infty} \Omega_j. \end{aligned} \tag{4}$$

Theorem 2.2 was first proved in [S2] under the assumption that the sequence $\{u_j\}$ is continuous. A similar property of maximal psh functions was considered in [B1]. Cegrell [C3] has proved the following version of maximality: Let $\Omega \subset \mathbb{C}^n$ be a hyperconvex domain and let $u \in \mathrm{psh}(\Omega)$, $u < 0$. Then u is maximal if and only if there exists a sequence $\{u_j\}$, $u_j \in \mathscr{E}_0 \cap C(\overline{\Omega})$, $u_j \geq u$, which converges pointwise to u and the sequence $(dd^c u_j)^n$ converges to 0 as $j \to \infty$. Here $\mathscr{E}_0$ is the class of bounded psh functions u such that

$$\lim_{z\to\partial\Omega} u(z) = 0 \quad \text{and} \quad \int_{\Omega} (dd^c u)^n < \infty.$$

Proof of Theorem 2.2. Let $u \in \mathrm{psh}(\Omega)$ and suppose that there exists a sequence $u_j \in \mathrm{psh}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ with $u_j \downarrow u$ and $(dd^c u_j)^n \to 0$.

Suppose on the contrary that u is not maximal. Then there exists a domain $G \subset\subset \Omega$ and a function $v \in \mathrm{psh}(\Omega)$ such that $u(z) \geq v(z)$ in a neighborhood of ∂G, but $u(z^0) < v(z^0)$ for some $z^0 \in G$.

We fix an $\varepsilon > 0$ with $u(z^0) + \varepsilon < v(z^0)$ and put $\delta = \varepsilon/(2\max\{|z|^2 : z \in \overline{G}\})$. Then the function $\tilde{v} = v + \delta|z|^2$, plurisubharmonic in Ω, satisfies the conditions

$$u(z^0) + \varepsilon < \tilde{v}(z^0), \qquad u|_{\partial G} + \varepsilon > \tilde{v}|_{\partial G}. \tag{5}$$

We can choose $j_0 \in \mathbb{N}$ so large that $u_j(z^0) + \varepsilon < \tilde{v}(z^0)$ for $j \geq j_0$. Since $u_j|_{\partial G} + \varepsilon > \tilde{v}|_{\partial G}$, approximating u_j, v in a neighborhood of $\overline{G}$ by standard sequences $u_{k,j} \downarrow u_j$, $v_k \downarrow v$, $u_{k,j}, v_k \in C^\infty$, $k = 1, 2, \ldots$, and putting $\tilde{v}_k = v_k + \delta|z|^2$, by the comparison principle we have

$$\int_F (dd^c u_{k,j})^n \geq \int_F (dd^c \tilde{v}_k)^n, \qquad F = \{z \in G : u_{k,j} + \varepsilon < \tilde{v}_k\} \subset\subset G. \tag{6}$$

We note that $E = \{u(z) + \varepsilon < \tilde{v}(z)\} \neq \emptyset$ by (5). Therefore, the Lebesgue measure $\mathrm{mes}\, E$ is strictly positive. Since $E = \bigcup_j E_j$, where $E_j = \{u_j + \varepsilon < \tilde{v}\}$, $E_j \subset E_{j+1}$, it follows that $\lim_{j\to\infty} \mathrm{mes}\, E_j = \mathrm{mes}\, E$. By (6) we have

$$(7)\qquad \int\limits_{\overline{G}}(dd^cu_j)^n \ge \limsup_{k\to\infty}\int\limits_{\overline{G}}(dd^cu_{k,j})^n \ge \limsup_{k\to\infty}\int\limits_{F}(dd^cu_{k,j})^n$$
$$\ge \limsup_{k\to\infty}\int\limits_{F}(dd^c\tilde v_k)^n \ge \limsup_{k\to\infty}\int\limits_{\{u_{k,j}+\varepsilon<\tilde v\}}(dd^c\tilde v_k)^n$$
$$\ge \delta^n \limsup_{k\to\infty}\int\limits_{\{u_{k,j}+\varepsilon<\tilde v\}}(dd^c|z|^2)^n = \delta^n \limsup_{k\to\infty}\operatorname{mes}\{u_{k,j}<\tilde v\}$$
$$= \delta^n \operatorname{mes} E_j;$$

on letting $j\to\infty$ this gives $\limsup_{j\to\infty}\int_{\overline{G}}(dd^cu_j)^n \ge \delta^n \operatorname{mes} E > 0$, contradicting the claim $\lim_{j\to\infty}(dd^cu_j)^n = 0$.

Let now u be a maximal function. For fixed domains $G \subset\subset \Omega$ with ∂G smooth, we fix an approximation $w_j \downarrow u$, $w_j \in \operatorname{psh}(G')\cap C^\infty(G')$, $j = 1,2,\ldots$, where $G \subset\subset G' \subset\subset \Omega$. It is well-known that the regularization $v_j^*(z) = \limsup_{w\to z} v_j(w)$ of

$$(8)\qquad v_j = \sup\{p \in \operatorname{psh}(G)\cap C(\overline{G}) : p|_{\partial G} \le w_j|_{\partial G}\}$$

is a bounded and psh function in G with vanishing Monge–Ampère operator, $(dd^cv_j^*)^n = 0$. Moreover, since u is maximal, we have $v_j^*(z)\downarrow u(z)$ for all $z \in G$.

We prove the last statement. It is clear that $\liminf_{z\to\xi} v_j^*(z) \ge w_j(\xi)$ for $\xi \in \partial G$. On the other hand, if $\tilde v_j = \sup\{p \in \operatorname{sh}(G)\cap C(\overline{G}) : p|_{\partial G} \le w_j|_{\partial G}\}$, then $\tilde v_j^*$ is a solution of the classical Dirichlet problem $\Delta \tilde v_j^* = 0$, $\tilde v_j^*|_{\partial G} = w_j|_{\partial G}$. Since $\tilde v_j^*(z) \ge v_j^*(z)$ for all $z\in G$, we have $\limsup_{z\to\xi} v_j^*(z) \le \limsup_{z\to\xi}\tilde v_j^*(z) = w_j(\xi)$ for $\xi \in \partial G$, so that $v_j^*|_{\partial G} \equiv w_j|_{\partial G}$. Hence the function

$$(9)\qquad \tilde w_j(z) = \begin{cases} v_j^*(z) & \text{if } z\in \overline{G},\\ w_j(z) & \text{if } z \in G'\setminus\overline{G},\end{cases}$$

is psh in G'. Moreover, $\tilde w_j$ is decreasing and if $\lim_{j\to\infty}\tilde w_j(z) = w(z)$, then $w(z)\in\operatorname{psh}(G')$ and $w(z)\equiv u(z)$ in $G'\setminus G$. Putting $w(z)\equiv u(z)$ for $z \in \Omega\setminus G'$ we can assume that $w(z)\in\operatorname{psh}(\Omega)$ and $w(z)\equiv u(z)$ in $\Omega\setminus G$. Since $w(z)\ge u(z)$ and u is maximal, it follows that $w(z)\equiv u(z)$ in Ω, i.e. $v_j^*(z)\downarrow u(z)$, $z\in G$.

Now it is not difficult, applying this process, to construct a sequence of domains $\Omega_j \subset \Omega$ and approximations $u_j(z)\downarrow u(z)$, $u_j \in \operatorname{psh}(\Omega_j)\cap L^\infty_{\mathrm{loc}}(\Omega_j)$, $(dd^cu_j)^n = 0$, where $\Omega_j \subset\subset \Omega_{j+1}\subset\subset\Omega$, $\Omega = \bigcup_{j=1}^\infty \Omega_j$. ∎

Remark 2.3. If the domain $G\subset\subset\Omega$ above is strongly pseudoconvex, then the upper envelope (8) is continuous in $\overline{G}$ by the Bremermann–Walsh theorem. Since for every domain $G\subset\subset\Omega$ with smooth boundary ∂G the function (9) satisfies $\tilde w_j|_{\partial G}\equiv w_j|_{\partial G}\in C(\partial G)$ and $\tilde w_j\in\operatorname{psh}(G')$, $G'\supset\overline{G}$, the technique of Walsh allows us also to prove continuity of v_j^*, that is,

$v_j^* \in \mathrm{psh}(G) \cap C(\overline{G})$. Therefore, the functions u_j in (4) can be chosen to be continuous, $u_j \in \mathrm{psh}(\Omega_j) \cap C(\Omega_j)$.

REMARK 2.4. Theorem 2.2 shows that for a given maximal function $u \in \mathrm{psh}(\Omega)$, locally, in a fixed ball $B \subset\subset \Omega$, there exists at least one sequence $u_j \downarrow u$ with $u_j \in \mathrm{psh}(B) \cap C(B)$ and $(dd^c u_j)^n \to 0$. On the other hand, Błocki [B3] showed that the function $u(z,w) = -\sqrt{\ln|z| \cdot \ln|w|}$ is maximal in $U^2 \setminus (0,0)$, where $U^2 = \{|z| < 1, |w| < 1\}$, but for the special approximation $u_j = \max(u, -j)$ the operator $(dd^c u_j)^n$ does not tend to 0 in $U^2 \setminus (0,0)$. This counterexample shows that the special approximation $\max\{u, -j\} \downarrow u$ is not suitable for establishing criteria for maximality.

3. A class of maximal functions. Let $u \in \mathrm{psh}(\Omega)$ in a domain $\Omega \subset \mathbb{C}^n$. We put $v = e^u$ and $u_a = \ln(v + a) = \ln(e^u + a)$, $a > 0$. Then $u_a \downarrow u$ as $a \downarrow 0$ and $v \in \mathrm{psh}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$. Therefore, the operators $(dd^c v)^p$, $v(dd^c v)^p$ and $dv \wedge d^c v \wedge (dd^c v)^{p-1}$ are correctly defined. We have

$$\begin{aligned} dd^c u_a &= (v+a)^{-1}[dd^c v - (v+a)^{-1} dv \wedge d^c v], \\ (dd^c u_a)^p &= (v+a)^{-p}[(dd^c v)^p - p(v+a)^{-1} dv \wedge d^c v \wedge (dd^c v)^{p-1}] \\ &= \frac{1}{(v+a)^{p+1}}[v(dd^c v)^p - p dv \wedge d^c v \wedge (dd^c v)^{p-1}] \\ &\quad + \frac{a}{(v+a)^{p+1}}(dd^c v)^p \\ &= \omega^p_{1,a} + \omega^p_{2,a}, \quad 1 \le p \le n. \end{aligned}$$

The currents $\omega^p_{1,a}, \omega^p_{2,a}$ are positive. Indeed, this is clear for $\omega^p_{2,a}$. To prove it for $\omega^p_{1,a}$, we show that the current $\phi^p = v(dd^c v)^p - p dv \wedge d^c v \wedge (dd^c v)^{p-1}$ is positive. We take the standard approximation $u_j \downarrow u$ and put $v_j = e^{u_j}$. Then we have

$$\begin{aligned} \phi^p_j &= v_j(dd^c v_j)^p - p dv_j \wedge d^c v_j \wedge (dd^c v_j)^{p-1} \\ &= e^{(p+1)u_j}(dd^c u_j + du_j \wedge d^c u_j)^p \\ &\quad - p e^{(p+1)u_j} du_j \wedge d^c u_j \wedge (dd^c u_j + du_j \wedge d^c u_j)^{p-1} \\ &= e^{(p+1)u_j}(dd^c u_j)^p \ge 0. \end{aligned}$$

It is clear that $\phi^p_j \to \phi^p$ as $j \to \infty$. Thus $\phi^p \ge 0$.

We put formally

$$\omega^p_1 = \lim_{a\to 0} \omega^p_{1,a} = \frac{\phi^p}{v^{p+1}}.$$

Then ω^p_1 characterizes $(dd^c u)^p$ completely outside the singular set $S = \{u(z) = -\infty\}$. If ϕ^p / v^{p+1} is locally bounded in Ω, i.e.,

$$\int\limits_{K\setminus S} \frac{\phi^p \wedge (dd^c|z|^2)^{n-p}}{v^{p+1}} < \infty \quad \forall K \subset\subset \Omega,$$

then $\omega_1^p = \phi^p/v^{p+1}$ represents a current in Ω which we call the *regular part* (the part outside S) of $(dd^c u)^p$. However, for Kiselman's example $u(z) = (-\ln|z_1|)^{1/n}(|z_2|^2 + \cdots + |z_n|^2 - 1)$ the measure $\omega_1^n = \phi^n/v^{n+1}$ is not bounded near $z_1 = 0$. It follows that for some psh functions, ω_1^p may be unbounded near the singular set S. In this case it is not possible to define of $(dd^c u)^p$ as a current, i.e., $(dd^c u)^p$ is undefinable.

DEFINITION 3.1. We say that $(dd^c u)^p$ is *definable* at a point $o \in \Omega$ if there exists a neighborhood U of o such that ω_1^p bounded in U (then it is a current) and as $a \to 0$ the $\omega_{2,a}^p$ weakly tends to some current ω_2^p, $\lim_{a\to 0} \omega_{2,a}^p = \omega_2^p$.

We note that if $(dd^c u)^p$ is definable at a point $o \in \Omega$, then $\operatorname{supp} \omega_2^p \subset S$. We will now study the current $\omega_{2,a}^p$ and its limit.

Fix $\alpha \in C^\infty(\Omega)$ with $B = \operatorname{supp} \alpha \subset\subset \Omega$. We can assume that $u < 0$ in B. Let

$$B_t = \{v < t\} \cap B \quad \text{and} \quad \mu_\alpha(t) = \int\limits_{B_t} (dd^c v)^p \wedge (dd^c|z|^2)^{n-p}\alpha(z), \quad t > 0.$$

(We note that $v = e^u \in \operatorname{psh}(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$ and $(dd^c v)^p \wedge (dd^c|z|^2)^{n-p}$ is a Borel measure.)

We want to find

$$\lim_{a\to 0} \omega_{2,a}^p(\alpha) = \lim_{a\to 0} \int\limits_{B_1} \frac{a}{(v+a)^{p+1}} (dd^c v)^p \wedge (dd^c|z|^2)^{n-p}\alpha(z). \tag{10}$$

For a C^2 smooth function v the integral in (10) is equal to (see [F])

$$\int\limits_{B_1} \frac{a}{(v+a)^{p+1}} (dd^c v)^p \wedge (dd^c|z|^2)^{n-p}\alpha(z) = \int\limits_0^1 \frac{a}{(t+a)^{p+1}}\, d\mu_\alpha(t).$$

Integrating by parts we have

$$\begin{aligned} &\int\limits_{B_1} \frac{a}{(v+a)^{p+1}} (dd^c v)^p \wedge (dd^c|z|^2)^{n-p}\alpha(z) \\ &\quad = \int\limits_0^1 \frac{a}{(t+a)^{p+1}}\, d\mu_\alpha(t) = \frac{a\mu_\alpha(1)}{(1+a)^{p+1}} + a(p+1)\int\limits_0^1 \frac{\mu_\alpha(t)}{(t+a)^{p+2}}\, dt. \end{aligned} \tag{11}$$

For an arbitrary plurisubharmonic function $u \in \operatorname{psh}(\Omega)$ formula (11) also holds. Indeed, one can find an approximating sequence $u_j \downarrow u$ with $u_j \in \operatorname{psh}(G) \cap C^\infty(G)$, G being some fixed neighborhood of $\overline{B}$. Then (11) follows from the weak convergence $dd^c e^{u_j} \to dd^c e^u$.

Now we need the following lemma:

Lemma 3.2. *If the limit*

$$\lim_{t\to 0} \frac{\mu_\alpha(t)}{t^p} = A \tag{12}$$

exists, then the limit

$$\lim_{a\to 0} \int_0^1 \frac{a\mu_\alpha(t)}{(t+a)^{p+2}}\, dt,$$

and consequently (10)*, also exists.*

Proof. It is clear that the limit

$$\lim_{a\to 0} \int_0^1 \frac{at^p}{(t+a)^{p+2}} dt = C = \text{const}$$

exists. Hence

$$\lim_{a\to 0} \int_0^1 \frac{a\mu_\alpha(t)}{(t+a)^{p+2}}\, dt = \lim_{a\to 0} \int_0^1 \frac{at^p}{(t+a)^{p+2}} (A + O(t))\, dt = AC,$$

because it is not hard to see that $\int_0^1 \frac{at^p}{(t+a)^{p+2}} O(t) dt = O(a)$. ∎

We note that if the limit

$$\lim_{t\to 0} \frac{1}{t^p} \int_{B_t} (dd^c e^u)^p \wedge (dd^c|z|^2)^{n-p} \tag{13}$$

exists for any $B \subset\subset \Omega$, then (12) exists for every $\alpha \in C^\infty(\Omega)$ with $\operatorname{supp}\alpha \subset\subset \Omega$. So we obtain the following result:

Theorem 3.3. *If the psh function u satisfies condition* (13) *and ω_1^p is a locally bounded current in Ω, then $(dd^c u)^p$ is definable.*

Theorems 2.2 and 3.3 give the following class of maximal functions:

Theorem 3.4. *Let $u \in \operatorname{psh}(\Omega)$ satisfy the following conditions:*

$$\phi^n = e^u (dd^c e^u)^n - n de^u \wedge d^c e^u \wedge (dd^c e^u)^{n-1} = 0, \tag{14}$$

$$\lim_{t\to 0} \frac{1}{t^n} \int_{B_t} (dd^c e^u)^n = 0. \tag{15}$$

Then u is maximal.

We note that

$$\lim_{t\to 0} \frac{1}{t^{n-\varepsilon}} \int_{B_t} (dd^c e^u)^n = 0$$

for any fixed $\varepsilon > 0$.

REMARK 3.5. If u is maximal then $\phi^n = 0$. In fact, for $u \in \mathrm{psh}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ formula (15) is satisfied automatically, i.e. $\omega_2^n = 0$. In this case

$$\phi^n = v(dd^c v)^n - n dv \wedge d^c v \wedge (dd^c v)^{n-1} = e^{(n+1)u}(dd^c u)^n$$

and $\phi^n = 0$ if u is maximal.

For any maximal function $u \in \mathrm{psh}(\Omega)$ we take by Theorem 2.2 an approximation $u_j \in \mathrm{psh}(\Omega_j) \cap L^\infty_{\mathrm{loc}}(\Omega_j)$ with $(dd^c u_j)^n = 0$, $\Omega_j \subset\subset \Omega_{j+1} \subset\subset \Omega$, $\Omega = \bigcup_{j=1}^\infty \Omega_j$, $u_j(z) \downarrow u(z)$. Then

$$\phi_j^n = v_j(dd^c v_j)^n - n dv_j \wedge d^c v_j \wedge (dd^c v_j)^{n-1} = 0,$$

where $v_j = e^{u_j}$, and $\phi_j^n \to \phi^n$ as $j \to \infty$. It follows that $\phi^n = 0$. ■

EXAMPLE 3.6. Let $u(z) = \ln(|f_1(z)|^2 + \cdots + |f_k(z)|^2)$ be a psh function in the domain $\Omega \subset \mathbb{C}^n$, where $f_1, \ldots, f_k$, $1 \le k < n$, are holomorphic in Ω, such that the analytic set $\{z \in \Omega : f_1(z) = \cdots = f_k(z) = 0\}$ is not empty. Then $u \notin \mathscr{E}$, but u satisfies conditions (14), (15): $(dd^c e^u)^n = 0$. Therefore it is a maximal function in Ω.

Acknowledgements. I am extremely grateful to the referee for remarks and a lot of corrections in the previous version of this paper.

This research was partially supported by Grant 1-024 for fundamental research of Khorezm Mamun Academy.

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Received 2.11.2011
and in final form 6.1.2012 (2609)