Plurifine potential theory

by Jan Wiegerinck (Amsterdam)

*Dedicated to Professor Józef Siciak
on the occasion of his 80th birthday*

Abstract. We give an overview of the recent developments in plurifine pluripotential theory, i.e. the theory of plurifinely plurisubharmonic functions.

1. Introduction. The *plurifine topology* $\mathcal{F}$ on (open subsets of) $\mathbb{C}^n$ was introduced by Fuglede in [Fu86] as the weakest topology in which all plurisubharmonic functions are continuous. The concept is analogous to the H. Cartan fine topology on $\mathbb{R}^n$, in which all subharmonic functions are continuous. In particular, on $\mathbb{C} \cong \mathbb{R}^2$ the fine and plurifine topologies coincide. The plurifine topology was next employed by Bedford and Taylor [BT87] to make precise statements about the convergence of the Monge–Ampère masses associated to sequences of plurisubharmonic functions. This topology was further developed in [EMW06, EMW09].

Notions pertaining to the plurifine topology are indicated with the prefix $\mathcal{F}$ and notions pertaining to the fine topology are indicated with "f"; for instance, f-open is open in the fine topology. The fine boundary of a set V is denoted by $\partial_f V$.

In one complex variable there is a good theory of finely subharmonic, respectively finely holomorphic, functions, which was largely developed by Fuglede (cf. [Fu72, Fu74, Fu81]). Once the topology $\mathcal{F}$ is available, it is natural to try to define *plurifinely plurisubharmonic functions* and *plurifinely holomorphic functions*. It turns out (cf. [ElK, EMW09, EMW10, EFW]), that there are two reasonable ways of extending the concept of finely subharmonic, respectively finely holomorphic, functions to several complex variables: a weak concept, defined by demanding that restrictions to complex

lines are finely subharmonic, respectively holomorphic, and a strong concept, based on approximation by ordinary plurisubharmonic, respectively holomorphic, functions.

With these plurifinely plurisubharmonic functions come several questions: which properties of ordinary plurisubharmonic functions transfer to the new concepts, are the weak and strong concept the same, can we get a hold on the structure of plurifinely plurisubharmonic functions? In short: develop plurifine potential theory! In this overview (Sections 2–5) we will discuss what has been achieved so far. We will also describe in Section 6 how results on pluripolar hulls, obtained e.g. by Siciak [Si], Zwonek [Zwo], Edlund and Jöricke [EJ], and Edigarian, El Marzguioui and the author [EEW, EW03, EW03a, EW04], are clarified with the help of plurifinely plurisubharmonic functions and fine analytic continuation. In Section 6 we will also see that certain questions about the structure of pluripolar hulls and complete pluripolar sets are more naturally formulated and partially solved by means of plurifine potential theory.

In general we will refrain from giving complete proofs and refer to the literature instead. A few exceptions are made, in particular for the results in Subsections 3.3 and 4.2 that are not in the literature yet. These sections were spurred by pertinent questions by Urban Cegrell and Peter Pflug.

2. The plurifine topology. The plurifine topology $\mathcal{F} = \mathcal{F}(\Omega)$ on a Euclidean open set $\Omega \subset \mathbb{C}^n$ is the smallest topology that makes all plurisubharmonic functions on Ω, denoted by $\mathrm{PSH}(\Omega)$, continuous. Since plurisubharmonic functions are already upper semicontinuous, a *local subbasis* at any $a \in \Omega$ is given by the sets

$$(2.1) \qquad U(a, B, f) = \{z \in B : f(z) > 0\},$$

where $B \subset \Omega$ is a ball about a, and $f \in \mathrm{PSH}(B)$ with $f(a) > 0$. It is easy to see that if $\Omega_1 \subset \Omega$ are open in $\mathbb{C}^n$, then $\mathcal{F}(\Omega)$ induces on Ω_1 the topology $\mathcal{F}(\Omega_1)$. Similarly, $\mathcal{F}(\mathbb{C}^n)$ induces the topology $\mathcal{F}(\mathbb{C}^k)$ on a complex affine hyperplane $H \cong \mathbb{C}^k$.

PROPOSITION 2.1. *The sets $U(a, B, f)$ form a local* basis *for $\mathcal{F}$.*

This result was first observed in [BT87]. See [EMW06, EMW09] for a proof.

We gather some properties of the plurifine topology (cf. [Doo, Chapter 11]).

THEOREM 2.2 (Properties of the plurifine topology $\mathcal{F}$).

(1) $\mathcal{F}$ *is* quasi-Lindelöf, *that is, every arbitrary union of $\mathcal{F}$-open sets is the union of a countable subunion and a pluripolar set.*
(2) $\mathcal{F}$ *is* completely regular, *that is, for every $\mathcal{F}$-closed set $A \subset \Omega$ and $a \in \complement A$ there exists an $\mathcal{F}$-continuous function f such that $f|A = 0$ and $f(a) \neq 0$.*

(3) $\mathcal{F}$ is Baire, *that is, a countable intersection of sets that are open and dense in Ω (relative to $\mathcal{F}$).*

It is a well-known fact that the fine topology on $\mathbb{R}^n$ is connected (cf. [Fu72]), but it was for quite some time an open question whether $\mathcal{F}$ is locally connected (cf. [Fu86]). The first proof of this result was given in [EMW06]. In [EMW09] another proof gives as a by-product some extra information about the relation between plurifine topology in an open set Ω and the fine topology on complex lines in Ω. It is based on the solutions by Nevanlinna [Nev] and Beurling [Beu] to the Carleman–Milloux problem.

Recall the definition of the *extremal function* (or *harmonic measure*) of a set $E \subset \bar{D}$, where D is open in $\mathbb{C}$:

$$\omega(z, E, D) = \sup \left\{ f(z) : f \in \mathrm{SH}_-(D) \text{ and } \limsup_{D \ni w \to E} f(w) \le -1 \right\},$$

where $\mathrm{SH}_-(D)$ denotes the the space of negative subharmonic functions on D. Note that ω need not be subharmonic (as a function of z), but its upper semicontinuous regularization,

$$\omega^*(z, E, D) = \limsup_{D \ni w \to z} \omega(z, E, D),$$

will be subharmonic. Notice that $\omega \le 0$; the classical harmonic measure equals $-\omega$.

THEOREM 2.3 ([Nev, Beu]). *Let F be compact in the unit disc $\mathbb{D}$, and let $\tilde{F} = \{r : re^{i\theta} \in F \text{ for some } \theta\}$ be its circular projection. Then for $z \in \mathbb{D} \setminus F$ we have*

$$\omega(z, F, \mathbb{D}) \le \omega(-|z|, \tilde{F}, \mathbb{D}).$$

A proof may also be found in [Ran]. One needs two lemmas.

LEMMA 2.4 ([EMW09]). *C^1-arcs are connected in the fine topology on $\mathbb{C}$.*

Proof. We can take $I = [0,1] \subset \mathbb{R}$. Let $J : [0,1] \to \mathbb{C}$ be a C^1-arc. Suppose that $J = E_1 \cup E_2$, where $E_j = U_j \cap J$ for some f-open set $U_j \subset \mathbb{C}$ ($j = 1, 2$) and $E_1 \cap E_2 = \emptyset$. Let $F_j = J^{-1}(E_j)$, and let $x \in F_j$. Then F_{1-j} is thin at x (cf. [Ran]). From Wiener's criterion one easily gets

$$(2.2) \qquad \lim_{t \to 0} \frac{\mathrm{Cap}(F_{1-j}) \cap [x-t, x+t]}{t} = 0.$$

Let l denote Lebesgue measure on $[0,1]$. Because $l(F) < 4\,\mathrm{Cap}(F)$, (2.2) remains valid with Cap replaced by l. It follows that the function

$$f(s) := \int_0^s \mathbf{1}_{F_1}(t)\, dt$$

is differentiable on $[0,1]$ and $f'|F_1 \equiv 1$, $f'|F_2 = 0$. The intermediate value theorem for differentiable functions implies that either F_1 or F_2 must be empty. ∎

Fuglede proved the following.

LEMMA 2.5 ([Fu75]). *If two points $p, q \in \mathbb{C}$ in an f-open set U are joined by a (Euclidean) continuum $K \subset U$, then there exists a polygonal path in U joining p and q.*

All this gives local connectedness of the fine and the plurifine topologies.

THEOREM 2.6 ([EMW09]). *Let $d < c < 0$ and $0 < r < 1$.*

(a) *There exists $k > 0$ such that for every $\varphi \in \mathrm{SH}_-(\mathbb{D})$ with $\varphi(0) > c$ and for every point a in the (f-open) set*
$$V = \{\varphi > c\} \cap \{|z| < r\}$$
there exists a circle $C(a, \delta_{\varphi,a})$, with $\delta_{\varphi,a} > k$, which is contained in $W = \{\varphi > d\}$.

(b) *Moreover the set $\tilde{W} = W \cap \overline{B(a, \delta_{\varphi,a})}$ is polygonally connected, and therefore f-connected.*

Sketch of proof. Part (a) follows from Theorem 2.3. For (b) one has to observe that a and $C(a, \delta_{\phi,a})$ must belong to the same component of W, because of the maximum principle. Then Lemmas 2.5 and 2.4 apply. ∎

For the plurifine topology this yields

THEOREM 2.7. *The plurifine topology $\mathcal{F}$ is weakly locally connected, hence locally connected.*

More precisely, suppose that $U = U(a, B(a, R_0), f)$ is a basic neighborhood in $\mathcal{F}$. Let $R < R_0$, $c < f(a)$, and $V = \{f \geq f(a)\} \cap \{\|z - a\| < R\}$. Then there exists a constant $r > 0$ such that for every complex line L passing through $v \in V$ the set $\{f > c\} \cap L$ contains a circle $C(v, r_{v,L})$ with $r_{v,L} \geq r$, and the set $\{f > c\} \cap L \cap \overline{B(v, r_{v,L})}$ is polygonally connected.

Now let $X_{v,L}$ denote the $\mathcal{F}$-component of v in $L \cap U$. Then $\bigcup_{v \in L} X_{v,L}$ is an $\mathcal{F}$-connected set and contains the $\mathcal{F}$-neighborhood $\{f > c\} \cap B(v, r)$ of v.

3. $\mathcal{F}$-plurisubharmonic functions. Let $\mathrm{SH}(D)$ denote the subharmonic functions on a domain $D \subset \mathbb{R}^n$, $\mathrm{SH}_-(D)$ the negative subharmonic functions on D, $\mathrm{PSH}(\Omega)$ the plurisubharmonic functions on a domain $\Omega \subset \mathbb{C}^n$, and $\mathrm{PSH}_-(\Omega)$ the negative plurisubharmonic functions on Ω. We start by defining finely subharmonic functions.

3.1. Finely subharmonic functions. Let D be a bounded domain in $\mathbb{R}^n$ and let $E \subset D$. For $u \in \mathrm{SH}_-(D)$ the reduced function R_u^E is defined by
$$R_u^E(z) = \sup\{v(z) : v \in \mathrm{SH}_-(D), v \leq u \text{ on } E\}.$$

The upper semicontinuous regularization of R_u^E is denoted by $\check{R}_u^E$. It is a subharmonic function on D. Next let $z \in D$ and δ_x the point mass at x. The

sweep or *swept out* of δ_z with respect to E is the unique Radon measure δ_z^E defined by

$$\check{R}_u^E(z) = \int u \, d\delta_z^E.$$

The key observation is that $u \mapsto \check{R}_u^E(z)$ extends as a continuous linear functional on $C(E)$, and the Riesz Representation Theorem applies. In case F is a subset of the boundary E of a domain D and $z \in D$, one sees that $\delta_z^E(F) = -\omega(z, F, D)$, the harmonic measure of $F \subset E$ relative to $z \in D$.

DEFINITION 3.1. A function f defined on a fine open set $U \subset \mathbb{R}^n$ is called *finely subharmonic* if

(1) f is finely upper semicontinuous;
(2) $f(z) \leq \int_{\partial_f V} f d\delta_z^{\complement V}$ for V in some local base of the fine topology at z;
(3) $f \not\equiv -\infty$ on every fine component of U.

Clearly, the restriction of a subharmonic function to an f-domain is finely subharmonic. A bounded finely subharmonic function on a Euclidean domain is subharmonic. In $\mathbb{R}^2$ the boundedness may be dropped. In $\mathbb{R}^n$, $n \geq 3$, there are examples of finely subharmonic functions on Euclidean domains that are not subharmonic (cf. [Fu72, Fu74]).

The finely subharmonic functions on a fine domain U will be denoted by f-SH(U).

Finely subharmonic functions share many properties with ordinary subharmonic functions; for instance we have the following result (cf. [Fu72, Lemma 10.1]).

PROPOSITION 3.2. *Let $V \subset U$ be fine open sets, $u \in$ f-SH(U), and $v \in$ f-SH(V). Suppose that f-$\limsup_{V \ni z \to w} v(z) \leq u(w)$ for all $w \in \partial_f V$. Then the function*

$$\Psi(z) = \begin{cases} u(z) & \text{if } z \in U \setminus V, \\ \max(u(z), v(z)) & \text{if } z \in V, \end{cases}$$

belongs to f-SH(U).

3.2. First properties of finely plurisubharmonic functions. For a compact set $K \subset \mathbb{C}^n$, let $S(K)$ denote the uniform limits on K of bounded continuous plurisubharmonic functions defined in (shrinking) neighborhoods of K.

DEFINITION 3.3 (Plurifinely plurisubharmonic function). Let Ω denote an $\mathcal{F}$-open (i.e., plurifinely open) subset of $\mathbb{C}^n$.

(i) A function $f : \Omega \to [-\infty, +\infty[$ is said to be *weakly $\mathcal{F}$-plurisubharmonic* if f is $\mathcal{F}$-upper semicontinuous and, for every complex line L in $\mathbb{C}^n$, the restriction of f to any $\mathcal{F}$-component of the finely open subset $L \cap \Omega$ of L is either finely subharmonic or $\equiv -\infty$.

(ii) A function $f : \Omega \to \mathbb{R}$ is said to be $\mathcal{F}$-*cpsh* if every point of Ω has a compact $\mathcal{F}$-neighborhood K in Ω such that $f|K \in S(K)$.

(iii) A function $f : \Omega \to [-\infty, +\infty[$ is said to be *strongly $\mathcal{F}$-plurisub-harmonic* if f is the pointwise limit of a decreasing net of $\mathcal{F}$-cpsh functions on Ω.

Weakly $\mathcal{F}$-plurisubharmonic functions were defined in [ElK, Section 5] (see also [EMW09, Definition 5.1])).

The concepts of strongly and weakly $\mathcal{F}$-plurisubharmonic are both $\mathcal{F}$-local ones (that is, they have the sheaf property: if a function is locally in f-SH or in $\mathcal{F}$-PSH, it is so globally).

One can fairly easily show that strongly $\mathcal{F}$-plurisubharmonic functions are weakly $\mathcal{F}$-plurisubharmonic. In the case $n = 1$ the notions are the same. A proof is indicated in Remark 3.7. Denote the class of all weakly $\mathcal{F}$-plurisubharmonic functions on an $\mathcal{F}$-open set Ω by $\mathcal{F}$-PSH(Ω). This class is a convex cone that is stable under taking the pointwise supremum of finite families. Furthermore, $\mathcal{F}$-PSH(Ω) is stable under taking the pointwise infimum for lower directed (possibly infinite) families, and is closed under $\mathcal{F}$-locally uniform convergence in view of analogous results for finely subharmonic functions (cf. [Fu72, Lemma 9.6]). Pointwise suprema of families of weakly $\mathcal{F}$-plurisubharmonic functions are discussed in Theorem 4.7 below. The restriction of $f \in \mathcal{F}$-PSH(Ω) to a complex affine subspace is of course weakly $\mathcal{F}$-plurisubharmonic.

In the following two theorems we collect some further properties of weakly $\mathcal{F}$-plurisubharmonic functions.

THEOREM 3.4 ([EMW09]). *Let f be a weakly $\mathcal{F}$-plurisubharmonic function on an $\mathcal{F}$-domain $\Omega \subset \mathbb{C}^n$ and let $E = \{z \in \Omega : f(z) = -\infty\}$.*

(a) *If $f \not\equiv -\infty$ then E has no $\mathcal{F}$-interior point.*
(b) *If $f \not\equiv -\infty$ then, for any $\mathcal{F}$-closed set $F \subset E$, $\Omega \backslash F$ is an $\mathcal{F}$-domain.*
(c) *If $f \leq 0$ then either $f < 0$ or $f \equiv 0$.*

This result was known for the case $n = 1$ (cf. [Fu72]). The case $n > 1$ is proven using the previous case and Theorem 2.7 (cf. [EMW09]).

The next theorem gives a handle on the local structure of weakly $\mathcal{F}$-plurisubharmonic functions. It is of fundamental importance in plurifine pluripotential theory. Therefore we will provide its proof here; we will follow [EMW10].

THEOREM 3.5 ([EMW10]). *Let f be a weakly $\mathcal{F}$-plurisubharmonic function on an $\mathcal{F}$-open set $\Omega \subset \mathbb{C}^n$, that maps Ω into a fixed bounded interval $]a, b[$. Every point $z_0 \in \Omega$ has an $\mathcal{F}$-open $\mathcal{F}$-neighborhood $O \subset \Omega$ on which f can be represented as the difference $f = \phi_1 - \phi_2$ between two bounded plurisubharmonic functions ϕ_1 and ϕ_2 defined on some open ball*

$B(z_0, r)$ *containing* O. *Moreover,* r, O, *and* ϕ_2 *will depend on* Ω *and* $]a, b[$, *but can be chosen independently of* f.

Proof. We may assume that $-1 < f < 0$ by scaling f if necessary. Let $V \subset \Omega$ be a compact $\mathcal{F}$-neighborhood of z_0. Since the complement $\complement V$ of V is pluri-thin at z_0, there exist $0 < r < 1$ and a plurisubharmonic function φ on $B(z_0, r)$ such that
$$\limsup_{z \to z_0, \, z \in \complement V} \varphi(z) < \varphi(z_0).$$
Without loss of generality we may suppose that φ is negative on $B(z_0, r)$ and
$$\varphi(z) = -1 \quad \text{on } B(z_0, r) \setminus V \quad \text{and} \quad \phi(z_0) = -1/2.$$
Hence

(3.1) $\qquad f(z) + \lambda\phi(z) \le -\lambda \quad$ for $z \in \Omega \cap B(z_0, r) \setminus V$ and $\lambda > 0$.

Now define a function u_λ on $B(z_0, r)$ by

(3.2) $\qquad u_\lambda(z) = \begin{cases} \max\{-\lambda, f(z) + \lambda\phi(z)\} & \text{for } z \in \Omega \cap B(z_0, r), \\ -\lambda & \text{for } z \in B(z_0, r) \setminus V. \end{cases}$

This definition makes sense because $(\Omega \cap B(z_0, r)) \cup (B(z_0, r) \setminus V) = B(z_0, r)$, and the two definitions agree on $\Omega \cap B(z_0, r) \setminus V$ in view of (3.1).

Clearly, u_λ is weakly $\mathcal{F}$-plurisubharmonic on $\Omega \cap B(z_0, r)$ and on $B(z_0, r) \setminus V$, hence on all of $B(z_0, r)$ in view of the sheaf property (cf. [EMW09]). Since u_λ is bounded on $B(z_0, r)$, it follows from [Fu72, Theorem 9.8] that u_λ is subharmonic on each complex line where it is defined. It is well known that a bounded function which is subharmonic on each complex line where it is defined, is plurisubharmonic (cf. [Lel45] or [Lel68, p. 24]). Thus, u_λ is plurisubharmonic on $B(z_0, r)$.

Since $\phi(z_0) = -1/2$, the set $O = \{z \in \Omega : \phi(z) > -3/4\}$ is an $\mathcal{F}$-neighborhood of z_0, and because $\phi = -1$ on $B(z_0, r) \setminus V$ it is clear that $O \subset V \subset \Omega$.

Observe now that $-4 \le f(z) + 4\phi(z)$ for every $z \in O$. Hence $f = \phi_1 - \phi_2$ on O, with $\phi_1 = u_4$ and $\phi_2 = 4\phi$, both plurisubharmonic on $B(z_0, r)$. ∎

COROLLARY 3.6. *Every weakly $\mathcal{F}$-plurisubharmonic function f on Ω is $\mathcal{F}$-continuous. Hence, if $f(z) > -\infty$ for some $z \in \Omega$, then there is an $\mathcal{F}$-neighborhood O where f can be written as a difference of plurisubharmonic functions defined in a neighborhood of O.*

For an unbounded f just note that, given $d < c$, the set $E = \{f < c\}$ is $\mathcal{F}$-open, and on E the function $\max\{f, d\}$ is $\mathcal{F}$-continuous, so $\{d < f < c\}$ is $\mathcal{F}$-open.

REMARK 3.7. *The extremal function of $B \setminus V$, i.e. the function $\Psi^* = (\sup\{h : h \in \mathrm{PSH}_-(B), h|(B \setminus V) \le -1\})^*$, could have been used in the*

proof of Theorem 3.5 instead of ϕ. Then one should take $\phi_2 = 4\Psi^*$. In case $n = 1$ this function is harmonic except on the boundary of V. Approximating $B \setminus V$ from the inside with compact sets $K_n \nearrow B \setminus V$, and forming the corresponding Ψ_n^*, we have Ψ_n^* harmonic in a neighborhood of z_0 and $\Psi_n^* \downarrow \Psi^*$. The Brelot property (cf. [Fu76]) states that on a suitable compact fine neighborhood K of z_0, both ϕ_1 and ϕ_2 are continuous in the Euclidean topology. Then $f = \lim \phi_1 - \Psi_n^*$, a uniform limit on K of subharmonic functions defined in a neighborhood of K, as announced after Definition 3.3. In case $n \geq 2$ this breaks down for two reasons: Ψ^* will in general not be pluriharmonic on $B \setminus \overline{V}$, and there is no Brelot property for plurisubharmonic functions.

THEOREM 3.8. *Suppose that f is a weakly $\mathcal{F}$-plurisubharmonic function on an $\mathcal{F}$-domain Ω. If $f \not\equiv -\infty$ then $E = \{z \in \Omega : f(z) = -\infty\}$ is an $\mathcal{F}$-closed, pluripolar subset of $\mathbb{C}^n$.*

A few words about the proof. In an $\mathcal{F}$-neighborhood of $z_0 \in E$, f will be negative. Now keep the notation of Theorem 3.5 and write $f_n = \max(f, -n)/(4n)$ as $u_n - \Psi^*$ with Ψ^* as in Remark 3.7, and u_n defined completely analogous to (3.2):

$$(3.3) \qquad u_n(z) = \begin{cases} \max\{-1, f_n(z) + \Psi^*(z)\} & \text{for } z \in \Omega \cap B(z_0, r), \\ -1 & \text{for } z \in B(z_0, r) \setminus V. \end{cases}$$

The u_n are plurisubharmonic and will increase to Ψ^* except at points of E, which will imply that E is pluripolar, first $\mathcal{F}$-locally and then by the quasi-Lindelöf property also globally.

It is unknown whether plurisubharmonic functions have the Brelot property, but a weak version of it does hold.

THEOREM 3.9 (quasi-Brelot property, cf. [EMW09]). *Suppose that f is a weakly $\mathcal{F}$-plurisubharmonic function on an $\mathcal{F}$-domain Ω. Then there exists a pluripolar $E \subset \Omega$ such that every $z \in \Omega \setminus E$ admits an $\mathcal{F}$-neighborhood K_z such that $f|K_z$ is Euclidean continuous.*

Notice that f-subharmonic functions on Euclidean domains in $\mathbb{R}^n$ ($n \geq 3$) need not be subharmonic, but there is no difference between $\mathcal{F}$-plurisubharmonic and plurisubharmonic functions on Euclidean open sets (cf. [EFW]).

PROPOSITION 3.10. *Let Ω be a Euclidean open subset of $\mathbb{C}^n$. For a function $f : \Omega \to [-\infty, +\infty[$ the following are equivalent:*

  (i) *f is plurisubharmonic (in the ordinary sense).*
  (ii) *f is weakly $\mathcal{F}$-plurisubharmonic and not identically $-\infty$ on any component of Ω.*

3.3. Lelong characterization. When Lelong introduced the definition of plurisubharmonic functions, [Lel42], he set out from functions that are locally bounded from above and have the property that their restrictions to complex lines are subharmonic. These are indeed upper semicontinuous, and hence plurisubharmonic. We show that this also holds in the weakly $\mathcal{F}$ situation.

THEOREM 3.11. *Suppose that f is an $\mathcal{F}$-locally bounded from above function on an $\mathcal{F}$-domain D with the property that $f|D \cap L$ is finely subharmonic for every complex line L. Then f is $\mathcal{F}$-PSH on D.*

Proof. We first observe that if we can prove the result for the $\mathcal{F}$-locally bounded functions $\max\{f, -n\}$, we are done, because then f is the limit of a decreasing sequence of $\mathcal{F}$-PSH functions. Now assume that f is $\mathcal{F}$-locally bounded. By copying the proof of Theorem 3.5, we see that every point $z \in D$ admits a ball $B(z, r)$ and an $\mathcal{F}$-neighborhood K_z on which $f = \psi - \phi$, where $\phi \in \mathrm{PSH}(B(z, r))$, and ψ is defined on $B(z, r)$ and is again of the form $\max\{f + \psi, C\}$ glued to the constant C. We observe that ψ is a bounded function on $B(z, r)$ and that its restriction to complex lines is subharmonic. Therefore, by Lelong's theorem $\psi \in \mathrm{PSH}(B(z, r))$, hence f is $\mathcal{F}$-continuous on K_z, and by varying z and recalling the sheaf property, also on D. It follows that f is $\mathcal{F}$-plurisubharmonic. ∎

4. $\mathcal{F}$-plurisubharmonic functions as invariant f-subharmonic functions

4.1. Main theorem. As is well-known, a plurisubharmonic function f on a domain $\Omega \subset \mathbb{C}^n$ is subharmonic when considered as a function on $\Omega \subset \mathbb{R}^{2n}$, because the average of f over a sphere can be expressed in terms of the average of f over the circles that are intersections of the sphere with complex lines passing through the center. While this approach does not work in the fine setting, an analogous result is nevertheless valid. Indeed, a well-known characterization of plurisubharmonic functions (see [Lel68, Théorème 1 (p. 18)] or [Kl, Theorem 2.9.12]) may be adapted as follows.

THEOREM 4.1 ([EFW]). *Let Ω be $\mathcal{F}$-open in $\mathbb{C}^n$. A function f : $\Omega \to [-\infty, +\infty[$ is weakly $\mathcal{F}$-plurisubharmonic if and only if f is $\mathcal{F}$-locally bounded from above and for every $\mathbb{C}$-affine bijection h of $\mathbb{C}^n$ the function $f \circ h$ is $\mathbb{R}^{2n}$-finely subharmonic on each fine component of the $\mathcal{F}$-open set $h^{-1}(\Omega)$ on which $f \circ h \not\equiv -\infty$.*

The prefix "$\mathbb{R}^{2n}$-fine" refers to concepts relative to the Cartan fine topology on $\mathbb{C}^n \cong \mathbb{R}^{2n}$. Recall that this topology is finer than the plurifine topology $\mathcal{F}$ [Fu86]. This explains why the condition "$\mathcal{F}$-locally bounded" occurs in the statement of the theorem.

For the proof of Theorem 4.1 one needs the following

LEMMA 4.2 ([EFW]). *Let u_1, u_2 be bounded subharmonic functions on an open set $B \subset \mathbb{R}^n$, and consider the function $f = u_1 - u_2$ on B. Let U be a finely open Borel subset of B. Then $f|U$ is finely subharmonic if and only if the signed Riesz measure Δf on B has a positive restriction to U.*

Indication of the proof of the "only if" part of Theorem 4.1. Writing f $\mathcal{F}$-locally as a difference of plurisubharmonic functions on an $\mathcal{F}$-open set $U \subset \Omega$, we know that the restriction to a complex line L is f-subharmonic, hence by Lemma 4.2, it has positive Riesz mass on $L \cap U$. Then a careful application of the definition of Riesz mass in the distribution sense and Fubini's theorem lead to positivity of the Riesz mass on U. Another application of the lemma gives that f is f-subharmonic on U. We can do so in an $\mathcal{F}$-neighborhood of any point in Ω. The sheaf property ensures that f is f-subharmonic on Ω. ∎

Lemmas 4.3 and 4.4 are ingredients of the proof of the "if" part of Theorem 4.1.

LEMMA 4.3. *Let f be a bounded $\mathbb{R}^{2n}$-finely subharmonic function on an $\mathcal{F}$-open set $\Omega \subset \mathbb{C}^n$ and suppose that for every $\mathbb{C}$-affine bijection h of $\mathbb{C}^n$ the function $f \circ h$ is $\mathbb{R}^{2n}$-finely subharmonic on $h^{-1}(\Omega)$. Then every $z_0 \in \Omega$ admits a (compact) $\mathcal{F}$-neighborhood K_{z_0} such that f can be written as*

$$f = f_1 - f_2 \quad \text{on } K_{z_0},$$

where f_1, f_2 are plurisubharmonic functions defined on a ball $B(z_0, r) \supset K_{z_0}$.

The proof is similar to the proof of Theorem 3.5 (cf. [EFW]). One employs the fact that f_2 is plurisubharmonic, and f_1 is now subharmonic on a Euclidean ball and remains so after affine transformation.

The next lemma is a consequence of results of Bedford and Taylor on slicing of currents (cf. [BT88]). In $\mathbb{C}^n$ we will write $z = (z_1, \ldots, z_n) = (z_1, z')$; similarly $0 = (0, 0')$ and, abusing notation, $\varepsilon'^2 = \prod_{j=2}^n \varepsilon_j^2$, whereas $|z'| < \varepsilon'$ stands for $|z_j| < \varepsilon_j$, $j = 2, \ldots, n$.

LEMMA 4.4. *Let w and u be bounded plurisubharmonic functions on a bounded domain $D \subset \mathbb{C}^n$, and let $\psi = \psi(z_1)$ be in C_0^∞ on $\{z \in D : z' = 0'\}$. Then*

$$(4.1) \quad \int_{\{z_2 = 0, \ldots, z_n = 0\}} \psi(z_1) w(z_1, 0') \, dd^c u(z_1, 0')$$

$$= \lim_{\epsilon' \downarrow 0} \frac{1}{2^{n-1} \epsilon'^2} \int_{\{|z'| < \epsilon'\}} \psi(z_1) w(z) \, dd^c |z_2|^2 \wedge \cdots \wedge dd^c |z_n|^2 \wedge dd^c u.$$

Indication of proof of the "if" part of Theorem 4.1. One easily reduces the proof of the "if" part of Theorem 4.1 to the case where f is bounded. With the notation from Lemmas 4.3 and 4.4, one first observes that $dd^c f$ makes sense as a $(1,1)$-form because of Lemma 4.3, and is ≥ 0 on the compact neighborhood $K = K_{z_0}$ of z_0 provided by Lemma 4.3, because of multiple application of Lemma 4.2. Next, application of Lemma 4.4 shows that the restriction of f to any complex line passing through z_0 is finely subharmonic on a fine neighborhood of z_0. ∎

From Theorem 4.1 we derive the following two results, one about removable singularities for weakly $\mathcal{F}$-plurisubharmonic functions, and the other about the supremum of a family of such functions.

4.2. Extension over polar sets. Let Ω be a domain in $\mathbb{C}^n$ and let E be a closed polar (with respect to $\mathbb{R}^{2n}$) subset of Ω. A theorem of Lelong states that if f is a bounded plurisubharmonic function on $\Omega \setminus E$, then f extends to a plurisubharmonic function on all of Ω. The following theorem combines Lelong's idea and Theorem 4.1.

THEOREM 4.5. *Let U be an $\mathcal{F}$-open set in $\mathbb{C}^n$ and let E be a subset of U that is finely closed and (finely) polar. Suppose that f is a bounded $\mathcal{F}$-PSH function on $U \setminus E$. Then there exists a function $g \in \mathcal{F}$-PSH(U) with $f = g|U \setminus E$.*

Proof. By Theorem 4.1 the function f is finely subharmonic on $U \setminus E$, hence by [Fu72, Theorem 9.14], the function

$$
(4.2) \qquad g(z) = \begin{cases} f(z) & \text{if } z \in U \setminus E, \\ \text{f-}\limsup_{\substack{w \to z \\ w \in U \setminus E}} f(w) & \text{if } z \in E \end{cases}
$$

is finely subharmonic. For $f \circ h$, where $h : \mathbb{C}^n \to \mathbb{C}^n$ is a complex affine map, the same holds on $h^{-1}(U)$, because polarity is preserved under affine maps. Now, Theorem 4.1 in the reverse direction applies, and states that g is $\mathcal{F}$-PSH. ∎

There is a similar result about removable singularities for weakly $\mathcal{F}$-holomorphic functions (see Definition 5.2 below):

COROLLARY 4.6. *Let $h : \Omega \to \mathbb{C}$ be $\mathcal{F}$-locally bounded on Ω ($\mathcal{F}$-open in $\mathbb{C}^n$). If h is weakly $\mathcal{F}$-holomorphic on $\Omega \setminus E$ (E finely closed and $\mathbb{R}^{2n}$-polar in $\mathbb{C}^n$) then h extends uniquely to a weakly $\mathcal{F}$-holomorphic function $h^* : \Omega \to \mathbb{C}$, given by*

$$
h^*(z) = \mathcal{F}\text{-}\lim_{\substack{\zeta \to z \\ \zeta \in \Omega \setminus E}} h(\zeta), \qquad z \in \Omega.
$$

4.3. Behavior of families of $\mathcal{F}$-plurisubharmonic functions

THEOREM 4.7. *Let Ω denote an $\mathcal{F}$-open subset of $\mathbb{C}^n$. For any family of weakly $\mathcal{F}$-plurisubharmonic functions f_α on Ω such that $f := \sup_\alpha f_\alpha$ is $\mathcal{F}$-locally bounded from above, the least $\mathcal{F}$-upper semicontinuous majorant f^* of the pointwise supremum f is likewise weakly $\mathcal{F}$-plurisubharmonic on Ω, and $\{z \in \Omega : f(z) < f^*(z)\}$ is pluripolar.*

In case Ω is *Euclidean* open, we find

COROLLARY 4.8. *For any family $\{f_\alpha\}$ of ordinary plurisubharmonic functions on a Euclidean open set $\Omega \subset \mathbb{C}^n$ such that $f := \sup_\alpha f_\alpha$ is locally bounded from above, the least plurisubharmonic majorant of f exists and can be expressed as the upper semicontinuous regularization of f in the Euclidean topology on $\mathbb{C}^n$, as well as in the $\mathcal{F}$-topology and in the $\mathbb{R}^{2n}$-fine topology; that is, $\bar{f} = f^* = \check{f}$.*

The version of this involving the *Euclidean* topology is due to Lelong [Lel45]; see also [Lel68, p. 26] or [Kl, Theorem 2.9.10].

For the proof of the theorem and its corollary we refer to [EFW].

5. Biholomorphic invariance. Notions in complex analysis should remain invariant under holomorphic change of coordinates. This is indeed the case for weakly $\mathcal{F}$-plurisubharmonic (and weakly $\mathcal{F}$-holomorphic) functions. But here we do want a bit more, namely that the composition of such a function with a weakly plurifinely holomorphic map is again of the same category.

We recall the relevant notions.

DEFINITION 5.1 (Finely holomorphic function, [Fu81, Fu88]). Let Ω be a fine domain in $\mathbb{C}$. A function $f : \Omega \to \mathbb{C}$ is called *finely holomorphic* if for every $z \in \Omega$ there exists a (compact) fine neighborhood K_z of z and a smooth function ϕ defined on a Euclidean neighborhood of K_z such that $\phi = f$ on K_z and $\bar{\partial}\phi = 0$ on K_z.

DEFINITION 5.2. An $\mathcal{F}$-continuous function f on an $\mathcal{F}$-domain Ω in $\mathbb{C}^n$ is called *weakly $\mathcal{F}$-holomorphic* if its restriction to $\Omega \cap L$ is finely holomorphic for every complex line L that meets Ω.

It is called *strongly $\mathcal{F}$-holomorphic* if for every $z \in \Omega$ there exists a (compact) $\mathcal{F}$-neighborhood K_z on which it is the uniform limit of holomorphic functions defined on Euclidean neighborhoods of K_z.

In fact, we could replace $\mathcal{F}$-continuous in the definition of weak holomorphy by $\mathcal{F}$-locally bounded: the real and imaginary parts of f would be weakly $\mathcal{F}$-plurisubharmonic, hence $\mathcal{F}$-continuous. In the case $n = 1$ it is known that weak and strong fine holomorphy are the same (cf. [Fu81]).

DEFINITION 5.3 (Plurifinely biholomorphic map, [EFW]). A *strongly $\mathcal{F}$-biholomorphic map h* from an $\mathcal{F}$-open set $U \subset \mathbb{C}^n$ onto its image in $\mathbb{C}^n$ is an $\mathcal{F}$-homeomorphism with the property that for every $z \in U$ there exists a compact $\mathcal{F}$-neighborhood K_z of z in U and a C^∞-diffeomorphism Φ_z from an open neighborhood of K_z to its image in $\mathbb{C}^n$ such that $\Phi_z|_{K_z} = h|_{K_z}$ and that $\Phi_z|_{K_z}$ is a C^2-limit of biholomorphic maps defined on open sets containing K_z.

Finely holomorphic functions of one variable are in fact locally strongly finely biholomorphic at points were they are locally injective. They can be approximated $\mathcal{F}$-locally uniformly by holomorphic functions at any point of their domain.

DEFINITION 5.4. We call an n-tuple $(h_1, \dots, h_n)$ of strongly/weakly $\mathcal{F}$-holomorphic functions $h_j : U \to \mathbb{C}$ defined on some $\mathcal{F}$-open $U \subset \mathbb{C}^m$ a *strongly/weakly $\mathcal{F}$-holomorphic map* (or *curve* if $m = 1$).

We now have

THEOREM 5.5 ([EFW]). *Let $h : U \to \Omega$ be a weakly $\mathcal{F}$-holomorphic map from an $\mathcal{F}$-open $U \subset \mathbb{C}^m$ into an $\mathcal{F}$-open $\Omega \subset \mathbb{C}^n$. The composition $f \circ h$ of a weakly $\mathcal{F}$-plurisubharmonic (resp. weakly $\mathcal{F}$-holomorphic) function f on Ω with h is weakly $\mathcal{F}$-plurisubharmonic* (*resp. weakly $\mathcal{F}$-holomorphic*) *on U.*

As for a sketch of the proof, by a fairly easy change of coordinates one can give a proof if f is holomorphic. To pass merely to strongly $\mathcal{F}$-biholomorphic maps f requires more effort. By the results of Section 6 it is sufficient to show that $h \circ f$ is f-subharmonic. To employ the approximation property, one resorts to the description of f-subharmonic functions in terms of the Dirichlet spaces of Beppo Levi and Deny [DeLi], that was studied in [Fu82]. The case of a weakly $\mathcal{F}$-holomorphic map f reduces to that of a weakly $\mathcal{F}$-holomorphic curve. Such curves are locally injective except for a countable set of points. At a point where the curve is injective, it is the restriction to a complex line of a strongly $\mathcal{F}$-biholomorphic map.

We refer to [EFW] for details.

6. Applications to pluripolar hulls. In this section we review some results concerning pluripolar hulls of graphs.

DEFINITION 6.1. Let E be a pluripolar subset of an open set $\Omega \subset \mathbb{C}^n$. The *pluripolar hull E_Ω^** of E with respect to Ω is the set

$$E_\Omega^* = \{z \in \Omega : \forall h \in \mathrm{PSH}(\Omega) \text{ if } h|E = -\infty \text{ then } h(z) = -\infty\}.$$

We will write E^* for $E_{\mathbb{C}^n}^*$.

The notion was introduced by Zeriahi in [Zer]. In case E is an analytic variety, in particular if E is the graph of a holomorphic function on a domain in $\mathbb{C}$, interesting results were obtained.

Sadullaev [Sa] and Levenberg, Martin and Poletsky [LMP] showed that for certain holomorphic functions defined by a lacunary series on the unit disc in $\mathbb{C}$, the graph Γ_f equals Γ_f^*. Answering questions of Sadullaev (cf. [Sa]) Levenberg and Poletsky [LePo] showed that if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ then $\Gamma_{z^\alpha}^* = \Gamma_{z^\alpha}$. Here, abusing the notation, $\Gamma_{z^\alpha} = \{(z, w) : |w| = |z|^\alpha, \arg w \in \{\alpha(\arg z + 2k\pi) : k \in \mathbb{Z}\}\}$, the graph of the complete analytic function z^α. The author showed that if f is a holomorphic function except for isolated singularities on a domain $\Omega \subset \mathbb{C}$, then $(\Gamma_f)_\Omega^* = \Gamma_f$ [Wi00, Wi00a]. These results support a conjecture by Levenberg, Martin and Poletsky, [LMP] stating that if E is an analytic set that admits no analytic extension, then $E = E^*$. However, Edigarian and the author gave counterexamples (cf. [EW03, EW03a]), which were followed by many others (cf. [Zwo, Si, PW]). Eventually Edlund and Jöricke [EJ] made the connection with fine holomorphy, observing that in all the available counterexamples the set E under consideration admits no analytic extension, but it does admit so-called *fine analytic extension*. Their results were extended in [EEW, EMW06].

For any set $E \subset \mathbb{C}^m$, $m \in \mathbb{N}$, and any function $h : E \to \mathbb{C}$ we denote by $\Gamma_h(E) = \{(z, h(z)) : z \in E\}$ the graph of $h|E$ and by $\Gamma_h(E)_{\mathbb{C}^{m+1}}^*$ the pluripolar hull of $\Gamma_h(E)$.

PROPOSITION 6.2 ([EFW]). *Let h be a weakly $\mathcal{F}$-holomorphic function on an $\mathcal{F}$-domain $U \subset \mathbb{C}^m$.*

(a) *If $h \not\equiv 0$ then the set $h^{-1}(0)$ of zeros of h is pluripolar in $\mathbb{C}^m$. Also, the graph $\Gamma_h(U)$ of h is pluripolar in $\mathbb{C}^{m+1}$.*

(b) *If E is a non-pluripolar subset of U then $\Gamma_h(E) \subset \Gamma_h(U)$ is pluripolar, and $\Gamma_h(U) \subset \Gamma_h(E)_{\mathbb{C}^{m+1}}^*$.*

With h supposed *strongly* $\mathcal{F}$-holomorphic on U, Proposition 6.2 was obtained in [EMW10, Corollary 4.4 and Theorem 4.5], extending [EMW09, Theorem 6.4], and [EEW, Theorem 3.5].

Sketch of proof of Proposition 6.2. (a) The function $\log |h|$ is weakly $\mathcal{F}$-plurisubharmonic on U. Since $\log |h(z)| = -\infty$ for $z \in h^{-1}(0)$, but $\log |h| \not\equiv -\infty$, it follows from Theorem 3.8 that the set $h^{-1}(0)$ is pluripolar.

Apply this result to the function $(z, w) \mapsto w - h(z)$, which is weakly $\mathcal{F}$-plurisubharmonic and $\not\equiv -\infty$ on $U \times \mathbb{C}$. Since $\log |w - h(z)|$ equals $-\infty$ on $\Gamma_h(U)$ we conclude that $\Gamma_h(U)$ is pluripolar.

(b) Now suppose that we have a plurisubharmonic function f on $\mathbb{C}^{m+1}$ such that $g(z) = f(z, h(z)) = -\infty$ for every $z \in E$. As g is $\mathcal{F}$-plurisubharmonic on U by Theorem 5.5 and E is not pluripolar, hence by Theorem 3.8 also not $\mathcal{F}$-pluripolar in U, it follows that $f(z, h(z)) = -\infty$ for $z \in U$, and therefore $\Gamma_h(U) \subset \Gamma_h(E)_{\mathbb{C}^{m+1}}^*$. ∎

In [EW04] the following result was proved.

THEOREM 6.3. *Let D be an open set in $\mathbb{C}$ and let A be a closed polar subset of D. Suppose that f is holomorphic on $D \setminus A$ and that $z_0 \in A$. Assume that $U \subset \mathbb{C}$ is an open set. Then $(\Gamma_f \cap (D \times U))^*_{D \times U} \subset (\Gamma_f \cap (D \times U)) \cup (\mathbb{C} \times A)$. Furthermore, the following conditions are equivalent:*

(1) $(\{z_0\} \times \mathbb{C}) \cap (\Gamma_f \cap (D \times U))^*_{D \times U} = \emptyset$.
(2) *There exists a sequence of open sets $V_1 \subset V_2 \subset \cdots \Subset U$ such that $\bigcup_j V_j = U$ and the set $\{z \in D \setminus A : f(z) \in U \setminus V_j\}$ is not thin at z_0 for any $j \geq 1$.*
(3) *For any open set $V \Subset U$ the set $\{z \in D \setminus A : f(z) \in U \setminus V\}$ is not thin at z_0.*

Moreover, if the set $\{z \in D \setminus A : f(z) \notin V\}$ is thin at z_0 for some open set $V \Subset U$, then there exists a $w_0 \in \overline{V}$ such that $(z_0, w_0) \in (\Gamma_f \cap D \times U)^*_{D \times U}$.

The formulation in the language of fine holomorphy is much more transparent:

THEOREM 6.4 (cf. [EEW]). *Let D be an open set in $\mathbb{C}$ and let A be a closed polar subset of D. Suppose that f is holomorphic on $D \setminus A$ and that $z_0 \in A$. Assume that $U \subset \mathbb{C}$ is an open set. Then $(\Gamma_f \cap (D \times U))^*_{D \times U} \subset (\Gamma_f \cap (D \times U)) \cup (\mathbb{C} \times A)$. Furthermore, the following conditions are equivalent:*

(1) $(\{z_0\} \times \mathbb{C}) \cap (\Gamma_f \cap (D \times U))^*_{D \times U} \neq \emptyset$.
(2) *f admits a finely holomorphic continuation $\tilde{f}$ to a fine neighborhood of z_0 and $\tilde{f}(z_0) \in U$.*

Moreover, if this is the case, then

$$(z_0, \tilde{f}(z_0)) = (\{z_0\} \times \mathbb{C}) \cap (\Gamma_f \cap (D \times U))^*_{D \times U}.$$

Some parts are now easy to see: by Proposition 6.2 we have (2)⇒(1), and also $(z_0, \tilde{f}(z_0)) \in (\Gamma_f \cap (D \times U))^*_{D \times U}$. To show that $(\Gamma_f \cap (D \times U))^*_{D \times U} \subset (\Gamma_f \cap (D \times U)) \cup (\mathbb{C} \times A)$ remains difficult.

More pluripolar hulls of graphs. Here we briefly depict some of the examples mentioned in connection with non-trivial pluripolar hulls. In all these examples there is a holomorphic function f which admits no analytic continuation, but the graph Γ_f has a non-trivial pluripolar hull.

Suppose that $D_1 \subset D_2$ are two domains such that $D_2 \setminus D_1$ has a point of density $z_0 \in D_2$. Then there exists a holomorphic function on D_1 which cannot be analytically extended so that $(\Gamma_f^*)_{D_2 \times \mathbb{C}} \neq \Gamma_f$. This was shown in [EW03]. In hindsight the function f is a finely holomorphic function on a fine domain in D_2 that contains $D_1 \cup \{z_0\}$.

Let $\mathbb{D}$ be the unit disc. Siciak gave an example of an $f \in A^\infty(\mathbb{D})$ which does not admit even a pseudocontinuation in the sense of Ross and Shapiro, but Γ_f^* contains the graph of a meromorphic function on $|z| > 1$ (cf. [Si]).

We now consider Blaschke products B on $\mathbb{D}$ and note that B also defines a meromorphic function on $\mathbb{C} \setminus \overline{\mathbb{D}}$. Zwonek [Zwo] constructed Blaschke products B that do not admit analytic continuation, and have the property that the pluripolar hull of the graph Γ_B contains the graph of B over $\mathbb{C} \setminus S$, where S is the closure of the set of poles of B. Multiplying such a B with multiple-valued holomorphic functions, he obtained examples of non-extendable holomorphic functions on the unit disc with graphs having a pluripolar hull consisting of several sheets.

In the same vein is the example given in [PW], where a Cantor type set E is constructed and a non-extendible holomorphic function f on $\mathbb{C} \setminus E$ with the property that Γ_f^* contains two sheets over $\mathbb{C} \setminus E$.

The point we wish to make is that in all these examples it can be seen that the function f does admit a finely holomorphic continuation and the graph of the maximal finely holomorphic continuation is contained in the pluripolar hull. So far no other points in the pluripolar hull have been found. This leads us to state a modified Levenberg–Martin–Poletsky conjecture.

CONJECTURE 6.5. *Suppose that f is a finely holomorphic function on a fine domain U in $\mathbb{C}$. Then Γ_f^* equals the graph of the maximal finely holomorphic continuation of f.*

It is here understood that such a maximal finely holomorphic continuation may be multiple-valued. The theory developed in the present paper shows that Γ_f^* contains the graph of the maximal finely holomorphic continuation of f. However, it does not clarify why equality would hold.

References

[BT82]   E. Bedford and B. A. Taylor, *A new capacity for plurisubharmonic functions*, Acta Math. 149 (1982), 1–40.

[BT87]   E. Bedford and B. A. Taylor, *Fine topology, Šilov boundary, and $(dd^c)^n$*, J. Funct. Anal. 72 (1987), 225–251.

[BT88]   E. Bedford and B. A. Taylor, *Plurisubharmonic functions with logarithmic singularities*, Ann. Inst. Fourier (Grenoble) 38 (1988), no. 4, 133–171.

[Beu]   A. Beurling, *Études sur un problème de majorisation*, thesis, Uppsala, 1933.

[DeLi]   J. Deny et J.-L. Lions, *Les espaces du type de Beppo Levi*, Ann. Inst. Fourier (Grenoble) 5 (1953-54), 305–370.

[Doo]   J. L. Doob, *Classical Potential Theory and Its Probabilistic Counterpart*, Grundlehren Math. Wiss. 262, Springer, New York, 1984.

[EEW]   A. Edigarian, S. El Marzguioui, and J. Wiegerinck, *The image of a finely holomorphic map is pluripolar*, Ann. Polon. Math. 97 (2010), 137–149.

[EW03]   A. Edigarian and J. Wiegerinck, *Graphs that are not complete pluripolar*, Proc. Amer. Math. Soc. 131 (2003), 2459–2465.

[EW03a] A. Edigarian and J. Wiegerinck, *The pluripolar hull of the graph of a holomorphic function with polar singularities*, Indiana Univ. Math. J. 52 (2003), 1663–1680.

[EW04]   A. Edigarian and J. Wiegerinck, *Determination of the pluripolar hull of graphs of certain holomorphic functions*, Ann. Inst. Fourier (Grenoble) 54 (2004), 2085–2104.

[EJ]   T. Edlund and B. Jöricke, *The pluripolar hull of a graph and fine analytic continuation*, Ark. Mat. 44 (2006), 39–60.

[ElK]   M. El Kadiri, *Fonctions finement plurisousharmoniques et topologie plurifine*, Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. (5) 27 (2003), 77–88.

[EFW]   M. El Kadiri, B. Fuglede, and J. Wiegerinck, *Plurisubharmonic and holomorphic functions relative to the plurifine topology*, J. Math. Anal. Appl. 381 (2011), 706–723.

[EMW06] S. El Marzguioui and J. Wiegerinck, *The pluri-fine topology is locally connected*, Potential Anal. 25 (2006), 283–288.

[EMW09] S. El Marzguioui and J. Wiegerinck, *Connectedness in the pluri-fine topology*, in: Functional Analysis and Complex Analysis, Contemp. Math. 481, Amer. Math. Soc., Providence, RI, 2009, 105–115.

[EMW10] S. El Marzguioui and J. Wiegerinck, *Continuity properties of finely plurisubharmonic functions and pluripolarity*, Indiana Univ. Math. J. 59 (2010), 1793–1800.

[Fu72]   B. Fuglede, *Finely Harmonic Functions*, Lecture Notes in Math. 289, Springer, Berlin, 1972.

[Fu74]   B. Fuglede, *Fonctions harmoniques et fonctions finement harmoniques*, Ann. Inst. Fourier (Grenoble) 24 (1974), no. 4, 77–91.

[Fu75]   B. Fuglede, *Asymptotic paths for subharmonic functions*, Math. Ann. 213 (1975), 261–274.

[Fu76]   B. Fuglede, *Finely harmonic mappings and finely holomorphic functions*, Ann. Acad. Sci. 2 (1976), 113–127.

[Fu81]   B. Fuglede, *Sur les fonctions finement holomorphes*, Ann. Inst. Fourier (Grenoble) 31 (1981), no. 4, 57–88.

[Fu82]   B. Fuglede, *Fonctions BLD et fonctions finement surharmoniques*, in: Séminaire de Théorie du Potentiel No. 6, Lecture Notes in Math. 906, Springer, Berlin, 1982, 126–157.

[Fu86]   B. Fuglede, *Fonctions finement holomorphes de plusieurs variables—un essai*, in: Séminaire d'Analyse P. Lelong–P. Dolbeault–H. Skoda, 1983/85, Lecture Notes in Math. 1198, Springer, Berlin, 1986, 133–145.

[Fu88]   B. Fuglede, *Finely holomorphic functions. A survey*, Rev. Roumaine Math. Pures Appl. 33 (1988), 283–295.

[Jo]   B. Josefson, *On the equivalence between locally polar and globally polar sets for plurisubharmonic functions on $\mathbb{C}^n$*, Ark. Mat. 16 (1978), 109–115.

[Kl]   M. Klimek, *Pluripotential Theory*, Clarendon Press, Oxford, 1991.

[La]   N. S. Landkof, *Foundations of Modern Potential Theory*, Grundlehren Math. Wiss. 180, Springer, Berlin, 1972.

[Lel42]   P. Lelong, *Définition des fonctions plurisousharmoniques*, C. R. Acad. Sci. Paris 215 (1942), 398–400.

[Lel45]   P. Lelong, *Les fonctions plurisousharmoniques*, Ann. Sci. École Norm. Sup. 62 (1945), 301–338.

[Lel56]   P. Lelong, *Prolongement d'une fonction plurisous-harmonique sur certains ensembles de capacité nulle*, C. R. Acad. Sci. Paris 242 (1956), 55–57.

[Lel68]   P. Lelong, *Fonctions plurisousharmoniques et formes différentielles positives*, Gordon & Breach, New York, 1968.

[LMP]   N. Levenberg, G. Martin, and E. A. Poletsky, *Analytic disks and pluripolar sets*, Indiana Univ. Math. J. 41 (1992), 515–532.

[LePo]   N. Levenberg and E. A. Poletsky, *Pluripolar hulls*, Michigan Math. J. 46 (1999), 151–162.

[Nev]   R. Nevanlinna, *Über eine Minimumaufgabe in der Theorie der konformen Abbildung*, Nachr. Ges. Wiss. Göttingen Math.-Phys. Kl. I 37 (1933), 103–115.

[PW]   E. Poletsky and J. Wiegerinck, *Graphs with multiple sheeted pluripolar hulls*, Ann. Polon. Math. 88 (2006), 161–171.

[Ran]   T. Ransford, *Potential Theory in the Complex Plane*, Cambridge Univ. Press, Cambridge, 1995.

[Sa]   A. Sadullaev, *Plurisubharmonic measures and capacities on complex manifolds*, Uspekhi Mat. Nauk 36 (1981), no. 4, 53–105 (in Russian).

[Si]   J. Siciak, *Pluripolar sets and pseudocontinuation*, in: Complex Analysis and Dynamical Systems II (Nahariya, 2003), Contemp. Math. 382, Amer. Math. Soc., Providence, RI, 2005, 385–394.

[Wi00]   J. Wiegerinck, *The pluripolar hull of $\{w = e^{-1/z}\}$*, Ark. Mat. 38 (2000), 201–208.

[Wi00a] J. Wiegerinck, *Graphs of holomorphic functions with isolated singularities are complete pluripolar*, Michigan Math. J. 47 (2000), 191–197.

[Zer]   A. Zeriahi, *Ensembles pluripolaires exceptionnels pour la croissance partielle des fonctions holomorphes*, Ann. Polon. Math. 50 (1989), 81–91.

[Zwo]   W. Zwonek, *A note on pluripolar hulls of graphs of Blaschke products*, Potential Anal. 22 (2005), 195–206.

Jan Wiegerinck
KdV Institute for Mathematics
University of Amsterdam
Science Park 904
P.O. Box 94248
1090 GE Amsterdam, The Netherlands
E-mail: j.o.o.wiegerinck@uva.nl