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Transfinite diameter, Chebyshev constants, and capacities in $\mathbb{C}^n$

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Dedicated to Professor Józef Siciak
on the occasion of his 80th birthday

Abstract. The famous result of geometric complex analysis, due to Fekete and Szegö, states that the transfinite diameter $d(K)$, characterizing the asymptotic size of K, the Chebyshev constant $\tau(K)$, characterizing the minimal uniform deviation of a monic polynomial on K, and the capacity $c(K)$, describing the asymptotic behavior of the Green function $g_K(z)$ at infinity, coincide.

In this paper we give a survey of results on *multidimensional* notions of *transfinite diameter*, *Chebyshev constants* and *capacities*, related to these classical results and initiated by Leja's definition of transfinite diameter of a compact set $K \subset \mathbb{C}^n$ and the author's paper [Mat. Sb. 25 (1975)], where a multidimensional analog of the Fekete equality $d(K) = \tau(K)$ was first considered for any compact set in $\mathbb{C}^n$. Using some general approach, we introduce an alternative definition of transfinite diameter and show its coincidence with Fekete–Leja's transfinite diameter. In conclusion we discuss an application of the results of the author's paper mentioned above to the asymptotics of the leading coefficients of orthogonal polynomial bases in Hilbert spaces related to a given pluriregular polynomially convex compact set in $\mathbb{C}^n$.

1. Introduction. The famous result of geometric complex analysis (Fekete [24], Szegö [56], see also [25, 58]) states that three characteristics of a compact set $K \subset \mathbb{C}$, which are defined in quite different ways, coincide. These characteristics are: the *transfinite diameter* $d(K)$, measuring the asymptotic size of K (the geometric approach); the *Chebyshev constant* $\tau(K)$, characterizing the minimal uniform deviation of a monic polynomial on K (the approximation theory approach); and the *capacity* $c(K)$, describing the asymptotic behavior of the Green function $g_K(z)$ at the infinite point (the potential theory approach).

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The transfinite diameter of a compact set $K \subset \mathbb{C}$ is the number

$$d(K) := \lim_{s\to\infty} d_s(K), \tag{1.1}$$

where

$$d_s(K) := \max\{|\det(z_\nu^{\mu-1})_{\mu,\nu=1}^s|^{2/s(s-1)} : z_\nu \in K,\ \nu = 1,\dots,s\} \tag{1.2}$$

is the sth diameter of K, which can also be represented as the geometric mean of extremal distances among s points on K (if $s \geq 2$):

$$d_s(K) := \max\Big\{\Big(\prod_{\nu<\mu\le s} |z_\mu - z_\nu|\Big)^{2/s(s-1)} : z_\nu \in K\Big\}. \tag{1.3}$$

The Chebyshev constant of K is defined via

$$\tau(K) := \lim_{s\to\infty}\Big(\inf\Big\{\max_{z\in K}\Big|z^s + \sum_{j=0}^{s-1} c_j z^j\Big| : c_j \in \mathbb{C},\ j = 0,1,\dots,s-1\Big\}\Big)^{1/s}.$$

The capacity is determined by $c(K) = \exp(-\rho_K)$, where

$$\rho_K := \lim_{z\to\infty}(g_K(z) - \ln|z|)$$

is the Robin constant of K.

In this paper we give (Section 3) a survey of results on *multidimensional* notions of *transfinite diameter, Chebyshev constants* and *capacities*, related to these classical results and initiated by Leja's definition of the transfinite diameter of a compact set $K \subset \mathbb{C}^n$ and the author's paper [61], where a multidimensional analog of the Fekete equality $d(K) = \tau(K)$ was first considered for any compact set in $\mathbb{C}^n$. In Section 4 we give a general observation about interestimates between the least approximation of generalized monic "polynomials" and extremal generalized Vandermondians with respect to a given linearly independent system in a Banach space. Using this approach, we introduce (Section 5) an alternative definition of the transfinite diameter and show its coincidence with Fekete–Leja's transfinite diameter. As an application we obtain an expression of the transfinite diameter in terms of extremal "Wronskians at the infinite point", which seems to be new even in the one-dimensional case. In conclusion (Section 6) we discuss an application of the results of [61] to the asymptotics of the leading coefficients of orthonormal polynomial bases in Hilbert spaces related to a given pluriregular polynomially convex compact set in $\mathbb{C}^n$.

2. Preliminaries and notation. Given an open set $D \subset \mathbb{C}^n$ we denote by $A(D)$ the space of all analytic functions on D with the usual locally convex topology of locally uniform convergence on D. If $K \subset \mathbb{C}^n$ is a compact set then $A(K)$ is the locally convex space of all germs of analytic functions on K, endowed with the standard inductive topology: recall that a sequence

$\{\varphi_j\}$ of germs converges to a germ φ in this topology if there is an open neighborhood $G \supset K$ and functions f_j, $f \in A(G)$, representing the germs φ_j, φ respectively, such that f_j converges to f locally uniformly in G. Given a bounded positive Borel measure μ supported on K, we define the Hilbert space $AL^2(K,\mu)$ as the closure of $A(K)$ in $L^2(K,\mu)$. Moreover, $\mathcal{M}(K)$ is the space of all bounded Borel measures on K with the norm $\|\mu\| := |\mu|(K)$, $\mu \in \mathcal{M}(K)$.

The *pluripotential Green function* of a compact set $K \subset \mathbb{C}^n$ is the function ([61, 62], see also [49, 53, 55, 66, 5])

$$g_K(z) := \limsup_{\zeta\to z} g_K^\circ(\zeta), \tag{2.1}$$

$$g_K^\circ(z) := \sup\{u(z) : u|_K \le 0,\, u \in \mathcal{L}(K)\}, \tag{2.2}$$

where $\mathcal{L}$ is the Lelong class of all functions $u \in \operatorname{Psh}(\mathbb{C}^n)$ with the property that $u(\zeta) - \ln|\zeta|$ is bounded from above near the infinite point. The function $g_K(z)$ is either plurisubharmonic in $\mathbb{C}^n$ or identically $+\infty$ (the latter is equivalent to pluripolarity of K). A polynomially convex compact set K is *pluriregular* if $g_K(z) \equiv 0$ on K (then g_K is continuous in $\mathbb{C}^n$).

Much earlier, in the context of celebrated multidimensional polynomial interpolation theory, Siciak [52] introduced the *weighted extremal functions*

$$\Phi(z,K;b) := \sup_{s\ge 1}(\sup\{|p(z)|^{1/s} : p \in \Pi_s,\, |p(\zeta)| \le \exp sb(\zeta),\, \zeta \in K\}),$$

$$\Psi(z,K;b) := \sup_{s\ge 1}(\sup\{|p(z)|^{1/s} : p \in \mathcal{H}_s,\, |p(\zeta)| \le \exp sb(\zeta),\, \zeta \in K\}),$$

where $z \in \mathbb{C}^n$ and Π_s (resp. $\mathcal{H}_s$) is the set of all polynomials (resp. homogeneous polynomials) of degree $\le s$. If $b \equiv 0$, we write $\Phi_K(z)$, $\Psi_K(z)$ instead of $\Phi(z,K;0)$, $\Psi(z,K;0)$. Set

$$\Phi_K^*(z) := \limsup_{\zeta\to z} \Phi_K(\zeta), \qquad \Psi_K^*(z) := \limsup_{\zeta\to z} \Psi_K(\zeta).$$

These functions proved to be closely related to the pluripotential Green functions. Namely, the equalities

$$g_K^\circ(z) \equiv \ln \Phi_K(z), \qquad g_K(z) \equiv \ln \Phi_K^*(z), \qquad z \in \mathbb{C}^n,$$

were shown in [62] for a pluriregular compact set K and then several different proofs were suggested for arbitrary K (see, e.g., [53, 54, 63, 22]). Siciak showed ([54, 1.3 and 2.6]) that $\ln \Psi_K^*(z)$ is equal to the logarithmically homogeneous Green function

$$h_K(z) := \limsup_{\zeta\to z} \sup\{u(\zeta) : u|_K \le 0,\, u \in \mathcal{L}_H\}, \qquad z \in \mathbb{C}^n,$$

where $\mathcal{L}_H$ consists of the functions $u \in \mathcal{L}$ that are *logarithmically homogeneous*, that is,

$$u(\lambda z) = u(z) + \ln|\lambda|, \qquad z \in \mathbb{C}^n \setminus \{0\},\ \lambda \in \mathbb{C}.$$

LEMMA 2.1. *Suppose X, Y is a pair of locally convex spaces and $J : X \to Y$ is an injective continuous linear operator with dense image. Then the adjoint operator $J^* : Y^* \to X^*$ is also injective and, if X is reflexive, the image $J^*(Y^*)$ is dense in X^*.*

Proof. Indeed, let $x^* = J^*y^*$, i.e. $x^*(x) = J^*y^*(x) = y^*(Jx)$ for all $x \in X$. Hence, if $x^* = 0$, then $y^*(Jx) = 0$ for all $x \in X$, and, since $J(X)$ is dense in Y, we see that $y^* = 0$. Thus J^* is injective. Now let X be reflexive and suppose that $J^*(Y^*)$ is not dense in X^*. Then, by the Hahn–Banach theorem, there exists $x_0 \in X^{**} = X$, $x_0 \neq 0$, such that $x_0(z^*) = z^*(x_0) = 0$ for each $z^* = J^*y^*$, $y^* \in Y^*$. Therefore $0 = J^*y^*(x_0) = y^*(Jx_0) = 0$ for any $y^* \in Y^*$, hence $Jx_0 = 0$, which contradicts the injectivity of J. ∎

REMARK 2.2. In what follows, we always treat the operator J as an identical embedding, identifying x with Jx and using the notation $X \hookrightarrow Y$ for a continuous linear embedding. In particular, we also write $Y^* \hookrightarrow X^*$ in the situation of Lemma 2.1.

NOTATION. We use the notation $|f|_E := \sup\{|f(z)| : z \in E\}$ for a function $f : E \to \mathbb{C}$. Denote by $\mathbb{Z}_+^n$ the set of all integer-valued vectors $k = (k_1, \ldots, k_n)$ with non-negative coordinates. Let $|k| := k_1 + \cdots + k_n$ be the degree of the multiindex k. Introduce an enumeration $\{k(i)\}_{i\in\mathbb{N}}$ of the set $\mathbb{Z}_+^n$ via the conditions: the sequence $s(i) := |k(i)|$ is nondecreasing and on each set $\mathcal{K}_s := \{|k(i)| = s\}$ the enumeration coincides with the lexicographic order relative to $k_1, \ldots, k_n$. Denote by $i(k)$ the number assigned to k under this ordering. Let

$$e_i(z) := z^{k(i)} := z_1^{k_1(i)} \cdots z_n^{k_n(i)}, \qquad i \in \mathbb{N}, \tag{2.3}$$

be the system of all monomials, enumerated as above. Notice that the number of multiindices of degree no larger than s is $m_s := C_{s+n}^s$ and the number of those of degree s is $N_s := m_s - m_{s-1} = C_{s+n-1}^s$, $s \geq 1$; $N_0 = 1$. Let $l_s := \sum_{q=0}^s qN_q$ for $s = 0, 1, \ldots$.

We consider the standard $(n-1)$-simplex

$$\Sigma := \Big\{\theta = (\theta_\nu) \in \mathbb{R}^n : \theta_\nu \geq 0,\ \nu = 1, \ldots, n;\ \sum_{\nu=1}^n \theta_\nu = 1\Big\} \tag{2.4}$$

and its interior (in the relative topology on the hyperplane containing Σ)

$$\Sigma^\circ := \{\theta = (\theta_\nu) \in \Sigma^\circ : \theta_\nu > 0,\ \nu = 1, \ldots, n\}.$$

For $\theta \in \Sigma$ we denote by $\mathcal{L}_\theta$ the set of all infinite sequences $L \subset \mathbb{N}$ such that $k(i)/s(i) \xrightarrow{L} \theta$. We also use the notation $k! := k_1! \cdots k_n!$, $k = (k_\nu) \in \mathbb{Z}_+^n$. We denote by Π_s (resp. $\mathcal{H}_s$) the set of all polynomials (resp. homogeneous polynomials) of degree $\leq s$, $s = 0, 1, 2, \ldots$.

We denote by $\mathfrak{N}(z)$ any norm in $\mathbb{C}^n$; the most applicable are the Euclidean and polydisc norms,

$$|z| := \Big(\sum_{\nu=1}^{n} |z_\nu|^2\Big)^{1/2}, \qquad \|z\| := \max\{|z_\nu|\}, \qquad z = (z_\nu) \in \mathbb{C}^n.$$

3. Multidimensional characteristics of sets in $\mathbb{C}^n$. A notion of transfinite diameter for $K \subset \mathbb{C}^n$ was first introduced by F. Leja [39]. Let K be a compact set in $\mathbb{C}^n$, and $\{\zeta_1, \dots, \zeta_i\}$ a finite subset of K. Consider the Vandermondians

$$V(\zeta_1, \dots, \zeta_i) := \det\,(e_\alpha(\zeta_\beta))_{\alpha,\beta=1}^{i},$$

where $e_\alpha(z) := z^{k(\alpha)}$, $\alpha \in \mathbb{N}$, and set

$$V_i := \sup\{|V(\zeta_1, \dots, \zeta_i)| : \zeta_j \in K,\, j = 1, \dots, i\}.$$

The *s-diameter* of K is the number $d_s(K) := (V_{m_s})^{1/l_s}$.

Definition 3.1. The (Fekete–Leja) *transfinite diameter* of a compact set $K \subset \mathbb{C}^n$ is the constant

$$d(K) := \limsup_{s\to\infty} d_s(K). \tag{3.1}$$

Leja posed the problem of *existence of the usual limit in this definition.* This problem was solved by Schiffer and Siciak in [51] for the particular case when K is the Cartesian product of plane sets. In order to solve this problem for an arbitrary compact set $K \subset \mathbb{C}^n$, Zakharyuta [61] introduced the *directional Chebyshev constants*

$$\tau(K,\theta) := \limsup_{\substack{i\to\infty\\ k(i)/|k(i)|\to\theta}} \tau_i := \sup_{L\in\mathcal{L}_\theta} \limsup_{i\in L} \tau_i, \qquad \theta \in \Sigma, \tag{3.2}$$

$$\tau_i = \tau_i(K) := (M_i)^{1/s(i)}, \qquad i \in \mathbb{N}, \tag{3.3}$$

where

$$M_i := \inf\Big\{|p|_K : p = e_i + \sum_{j=1}^{i-1} c_j e_j\Big\}, \qquad i \in \mathbb{N}, \tag{3.4}$$

is the *least uniform deviation of monic polynomials* from the identical zero on the compact set K. Any polynomial $p(z) = t_i(z)$ attaining the inf in (3.4) is called a *Chebyshev polynomial* (it always exists but may not be unique).

It is proved in [61] that the usual limit exists in (3.2) for each $\theta \in \Sigma^\circ$ and the function $\tau(K,\theta)$ is convex (hence continuous) on Σ° and bounded on Σ (Section 4 in [61]). Thus the geometric mean of the directional Chebyshev constants, called the *principal Chebyshev constant*, is well-defined:

$$\tau(K) := \exp \int_\Sigma \ln \tau(K,\theta)\, d\sigma(\theta),$$

where σ is the normalized Lebesgue measure on Σ; it is assumed that $\exp(-\infty) = 0$.

The main result of [61], which can be considered as a natural multidimensional analog of the Fekete equality [24], is

THEOREM 3.2. *The usual limit exists in* (3.1) *and*

$$d(K) = \tau(K) = \exp \int_{\Sigma} \ln \tau(K, \theta)\, d\sigma(\theta).$$

The proof of this theorem can be divided into two parts. The first one is the proof that the geometric mean of the τ_i corresponding to monomials of degree s,

$$T_s(K) := \Big(\prod_{i=m_{s-1}+1}^{m_s} \tau_i \Big)^{1/N_s} = \Big(\prod_{i=m_{s-1}+1}^{m_s} M_i \Big)^{1/sN_s}$$

converges to the principal Chebyshev constant $\tau(K)$ as $s \to \infty$ (Lemmas 5, 6 in [61]); this step is based on the existence of the directional limits in (3.2) and the properties of the function $\tau(K, \theta)$ mentioned above. The second part of the proof establishes that, if the sequence $T_s(K)$ has a limit, then $d_s(K)$ converges to the same number; it is based on the interestimates between the extremal Vandermondians and the least deviations of monic polynomials, which will be treated in some generalized form below in Section 4.

Let us give a short survey of some results related to the above assertions. Notice that the methods developed in [61] (especially Lemmas 1–6 there) proved to be applicable in more general situations. For instance, these methods were adapted by Rumely, Lau and Varley ([38, 47, 48]) in order to prove the existence of sectional capacities in arithmetic geometry.

Siciak introduced the extremal homogeneous Vandermondians of a compact set $K \subset \mathbb{C}^n$:

$$W_s = \max\{|\det (e_{m_{s-1}+\alpha}(\zeta_\beta))_{\alpha,\beta=1}^{N_s}| : \zeta_\beta \in K\}$$

and asked whether the limit exists in the definition of the *homogeneous transfinite diameter:*

$$D(K) := \lim_{s\to\infty} (W_s)^{1/sN_s}.$$

Jędrzejowski [28] gave a positive answer to this question by following arguments analogous to those used in [61].

Bloom and Levenberg [15] studied weighted characteristics of a compact set $K \subset \mathbb{C}^n$, inspired by the one-dimensional *weighted potential theory* (see, e.g., [50]) and Siciak's notion of weighted extremal functions ([52]). These characteristics are defined as follows. Suppose that a nonnegative upper semicontinuous function $\omega : K \to \mathbb{R}$ is an *admissible weight,* that is, the set $\{z \in K : \omega(z) > 0\}$ is nonpluripolar. Then the weighted directional Chebyshev constants are defined, by analogy with unweighted ones, via

$$\tau^w(K,\theta) := \limsup_{\substack{i\to\infty \\ k(i)/|k(i)|\to\theta}} \tau_i^w := \sup_{L\in\mathcal{L}_\theta} \limsup_{i\in L} \tau_i^w, \qquad \theta \in \Sigma, \tag{3.5}$$

where

$$\tau_i^w = \tau_i^w(K) := \Big(\inf\Big\{|w^{s(i)}p|_K : p = e_i + \sum_{j=1}^{i-1} c_j e_j\Big\}\Big)^{1/s(i)}, \qquad i \in \mathbb{N}. \tag{3.6}$$

Existence of the limit in (3.5), continuity of the function $\tau^w(K,\theta)$ for $\theta \in \Sigma^\circ$ and the definition of the *weighted principal Chebyshev constant*

$$\tau^w(K) := \exp \int_\Sigma \ln \tau^w(K,\theta)\, d\sigma(\theta)$$

were given in [15] by slightly modifying the arguments of [61] (the last constant was denoted by $d^w(K)$ there and named *weighted transfinite diameter*). On the other hand, also in [15] the extremal weighted Vandermondians

$$\mathcal{W}_{m_s} := \max\{|V(\zeta_1,\ldots,\zeta_{m_s})| \cdot |w(\zeta_1)\cdots w(\zeta_{m_s})|^s : \zeta_i \in K\}$$

were introduced and it was asked whether the limit

$$\delta^w(K) := \lim_{s\to\infty} (\mathcal{W}_{m_s})^{1/l_s} \tag{3.7}$$

exists and equals $\tau^w(K)$. The same authors proved in [16] that the limit (3.7) indeed exists, while the relation between the constants $\tau^w(K)$ and $\delta^w(K)$ turned out to be a harder nut. Namely, they proved the following brilliant formula:

$$\delta^w(K) = \Big(\exp\int (\ln w)(dd^c(g_K^w))^n\Big)^{1/n} \tau^w(K), \tag{3.8}$$

where

$$\begin{aligned} g_K^w(z) &= \limsup_{\zeta\to z} \ln \Phi(z,K;b) \\ &= \limsup_{\zeta\to z} \sup\{u(\zeta) : u \in \mathcal{L},\ u(\zeta) \le b(\zeta),\ \zeta \in K\} \end{aligned}$$

with $b(z) := -\ln w(z)$. An important ingredient of the proof of (3.8) is a remarkable Rumely formula, expressing the unweighted transfinite diameter $d(K)$ via the Robin function (see the end of this section).

Unlike the one-dimensional case, the function $g_K(z) - \ln \mathfrak{N}(z)$, in general, has plenty of partial limits as $|z| \to \infty$ and they depend on the choice of the norm $\mathfrak{N}$ in $\mathbb{C}^n$. Therefore a wide variety of multidimensional notions of capacity were suggested by many authors ([61, 62, 49, 53, 54, 1, 2, 3, 5, 33, 4]).

The earliest were the capacities ([61, 62])

$$c(K) := \exp\Big(-\limsup_{|z|\to\infty}(g_K(z) - \ln\|z\|)\Big),$$

$$C(K) := \exp\Big(-\liminf_{|z|\to\infty}(g_K(z) - \ln|z|)\Big),$$

and the capacities $c_+(K)$ and $C_+(K)$ obtained from them by replacing lim sup by lim inf. These characteristics vanish if and only if K is pluripolar. Unfortunately, none of them coincides with $d(K)$. The inequality $c(K) \le d(K)$ holds for all $K \subset \mathbb{C}^n$ [61]; it follows directly that $C(K) \le \sqrt{n}\, c(K) \le \sqrt{n}\, d(K)$. Bloom and Calvi [13] proved a stronger estimate: $C(K) \le \exp\big(\frac{1}{2}\sum_{j=2}^{n}\frac{1}{j}\big)d(K)$. On the other hand, Levenberg and Taylor [41] have shown that, if K is contained in the unit ball $\mathbb{B}^n$, then $d(K) \le AC(K)^\delta$ with some positive constants A and δ which do not depend on K. Since $C(\lambda K) = \lambda C(K)$ for each $\lambda > 0$, it follows that the transfinite diameter vanishes simultaneously with the capacity $C(K)$, that is, if and only if K is pluripolar, which answers affirmatively the problem posed in [61].

In order to get multidimensional analogs of the Szegö equality $\tau(K) = c(K)$, some authors modified the notion of Chebyshev constant or/and the notion of capacity. One way of modifying Chebyshev constants is via certain normalizations of the leading homogeneous part of a polynomial ([61, 55]):

$$\begin{aligned} T(K) &= \limsup_{s\to\infty}\Big(\inf\Big\{|p|_K : p = \sum_{|k(i)|\le s} c_i e_i \in \mathit{\Pi}_s,\ \sum_{|k(i)|=s}|c_i| = 1\Big\}\Big)^{1/s} \\ &= \limsup_{s\to\infty}\Big(\inf\Big\{|p|_K : p = \sum_{|k(i)|\le s} c_i e_i \in \mathit{\Pi}_s,\ \max_{|k(i)|=s}\{|c_i|\} = 1\Big\}\Big)^{1/s} \\ &= \limsup_{s\to\infty}\Big(\inf\Big\{|p|_K : p \in \mathit{\Pi}_s,\ \Big|\sum_{|k(i)|=s} c_i e_i\Big|_{\mathbb{B}^n} = 1\Big\}\Big)^{1/s}. \end{aligned}$$

That the usual limit exists here has been proved by Siciak [54] (there was a gap in the proof of [61]). The inequality $c(K) \le T(K)$ was proved in [61]. The equality $T(K) = C(K)$ was claimed in [49], but actually it was only proved there that $T(K) \le C(K)$ for K pluriregular; it follows from [34] that it is also true for arbitrary compact sets. Thus

$$c(K) \le T(K) \le C(K).$$

It is easy to see that $c(K) = T(K)$ for n-circular compact sets, but it *remains open whether this equality is true for an arbitrary compact set.*

J. Siciak [54] introduced and studied a wide variety of "Chebyshev constants" via normalizing restrictions on the whole polynomial on the fixed pluriregular compact set $\mathbb{X} \subset \mathbb{C}^n$ (the most interesting are the cases $\mathbb{X} = \overline{\mathbb{B}}^n$ or $\mathbb{X} = \overline{\mathbb{U}}^n$) and a related collection of capacities. In particular, he proved that the Chebyshev constant

$$\mathcal{T}_{\mathbb{X}}(K) := \lim_{s\to\infty} (\inf\{|p|_K : p \in \Pi_s,\ |p|_{\mathbb{X}} = 1\})^{1/s} \tag{3.9}$$

and its homogeneous version

$$\mathcal{T}^H_{\mathbb{X}}(K) := \lim_{s\to\infty} (\inf\{|p|_K : p \in \mathcal{H}_s,\ |p|_{\mathbb{X}} = 1\})^{1/s} \tag{3.10}$$

coincide, respectively, with the capacities

$$\mathcal{C}_{\mathbb{X}}(K) := \frac{1}{|\exp g_K(z)|_{\mathbb{X}}}, \quad \mathcal{C}^H_{\mathbb{X}}(K) := \frac{1}{|\exp h_K(z)|_{\mathbb{X}}}, \tag{3.11}$$

where $h_K(z)$ is the logarithmically homogeneous Green function (see Preliminaries). Notice that $\mathcal{C}_{\mathbb{X}}(K)$ is of interest for subsets of $\mathbb{X}$ only, since $\mathcal{C}_{\mathbb{X}}(K) = 1$ if $K \supset \mathbb{X}$.

Alexander and Taylor [3] investigated interestimates between the characteristic $\mathcal{T}_{\bar{\mathbb{B}}^n}(K)$ and the condenser capacity $C(K, \mathbb{B}^n)$, introduced by Bedford and Taylor [6].

Another way of defining capacities and Chebyshev constants is via integral normalizing restrictions instead of the uniform estimates in (3.9)–(3.11). Siciak [54] introduced the capacities

$$\mathcal{C}_\mu(K) := \exp\Big(-\int g_K(z)\, d\mu(z)\Big), \quad \mathcal{C}^H_\mu(K) := \exp\Big(-\int h_K(z)\, d\mu(z)\Big),$$

where μ is a probability measure in $\mathbb{C}^n$ satisfying some natural conditions, which hold, in particular, for the normalized Lebesgue measure σ on the sphere $\mathbb{S} = \partial\mathbb{B}^n$.

Taylor [57] proved the estimate $C(K) \le \mathcal{C}_\sigma(K)^{1/n}$.

Alexander's *projective capacity* is defined via

$$\gamma(K) := \lim_{s\to\infty} \left(\inf\left\{ |p|_K : p \in \mathcal{H}_s,\ \frac{1}{s\kappa_n} \int_{\mathbb{S}} \log|p|\, d\sigma = 1 \right\}\right)^{1/s},$$

where $\kappa_n = \int_{\mathbb{S}} \ln|z_n|\, d\sigma = -\frac{1}{2}\sum_{j=1}^{n-1} \frac{1}{j}$. This Chebyshev-type constant was originally defined on sets $K \subset \mathbb{S}$ or on subsets of the projective space $\mathbb{P}^{n-1}$ ([1, 2]), but it can be considered on sets $K \subset \mathbb{C}^n$ as well ([54, 3, 28]). Cegrell and Kołodziej [21] proved the equality

$$\gamma(K) = (\exp \kappa_n)\mathcal{C}^H_\mu(K)$$

(Siciak [53] considered earlier the case $n = 2$). Jędrzejowski [29, 30] computed this characteristic for intersections of ellipsoids in $\mathbb{C}^n$.

It seems that the most natural way to a multidimensional analog of the Szegö equality is via the notion of *Robin function* of K [57, 5]. The projective version of the Robin function is defined via

$$\tilde{\rho}_K([z]) := \limsup_{|\lambda|\to\infty} (g_K(\lambda z) - \ln|\lambda z|), \qquad [z] \in \mathbb{P}^{n-1},\ z \in \mathbb{C}^n \setminus \{0\},$$

where $[z]$ is the point of $\mathbb{P}^{n-1}$ determined by z; its modification

$$\rho_K(z) := \tilde{\rho}_K([z]) + \ln|z| = \limsup_{|\lambda|\to\infty}(g_K(\lambda z) - \ln|\lambda|), \qquad z \in \mathbb{C}^n,$$

is a plurisubharmonic function in $\mathbb{C}^n$ (we set $\rho_K(0) = -\infty$), which is logarithmically homogeneous.

Nivoche [43] introduced a directional Chebyshev constant via

$$m(K,\zeta) := \lim_{s\to\infty} m_s(K,\zeta)^{1/s} = \sup\{m_s(K,\zeta)^{1/s} : s \in \mathbb{N}\},$$
$$m_s(K,\zeta) := \inf\{|p|_K : p \in \varPi_s,\ \hat{p}(\zeta) = 1\}, \qquad \zeta \in \mathbb{S},$$

and proved that it coincides with the directional capacity determined by the Robin function:

$$m(K,\zeta) = C_K(\zeta) := \exp(-\rho_K(\zeta)), \qquad \zeta \in \mathbb{S}.$$

Bloom and Levenberg [15] proved a weighted version of this result.

Bloom and Calvi [13] proved that the transfinite diameter $d(K)$ is determined uniquely by the Robin function ρ_K (this result is based on a polynomial approximation theorem of Bloom [11], see also [55]). Recently Rumely [46] has established an impressive integro-differential formula expressing the transfinite diameter of a compact set in $\mathbb{C}^n$ via its Robin function ρ_K, which can be considered as a *genuine analog of the classical Szegö equality* $d(K) = c(K)$ (see, also, [23, 17, 16, 40, 7, 8, 9, 10]).

Numerous aspects concerned with the capacity characteristics remained untouched here: extension of set characteristics to arbitrary sets in $\mathbb{C}^n$, Choquet's axioms, further applications to polynomial and rational approximation, convergence of measures generated by extremal Fekete points etc. They can be found in [4, 5, 6, 12, 18, 19, 22, 32, 33, 35, 36, 37, 49, 53, 54, 64, 67].

4. Some general observations. An important step in the proof of the equality $\tau(K) = d(K)$ are interestimates between the least deviation M_i and the extremal Vandermondian V_i (see Lemma 4 and Corollary 5 in [61] and Corollary 4.3 at the end of this section). Here we show that this relation is of a quite general nature.

Let X be an infinite-dimensional complex Banach space. Given a sequence $\{h_i\}_{i=1}^\infty$ in X set

$$\varDelta_i = \varDelta_{i,X} := \inf\Big\{\Big\|h_i + \sum_{j=1}^{i-1} c_j h_j\Big\|_X : c_j \in \mathbb{C}\Big\}, \qquad i \in \mathbb{N}, \tag{4.1}$$

which is a natural generalization of the least deviation of monic polynomials from zero. It is a commonplace that there exists a "Chebyshev polynomial" (maybe nonunique) $T_i = h_i + \sum_{j=1}^{i-1} t_{ij} h_j$ such that $\varDelta_i = \|T_i\|$.

Let X^* be the dual space of X. Consider the determinant (Vandermondian)

$$(4.2)\qquad \mathcal{V}(f_1,\ldots,f_i) := \det\,(f_\beta(h_\alpha))_{\alpha,\beta=1}^{i},$$

with $f_\beta \in X^*$, $\beta = 1,\ldots,i$. Suppose that $A \subset \mathbb{B}_{X^*}$ is a norming set, that is,

$$\|x\| = \sup\{|f(x)| : f \in A\}.$$

Introduce a sequence

$$(4.3)\qquad \mathcal{V}_i = \mathcal{V}_i^{X,A} := \sup\{|\mathcal{V}(f_1,\ldots,f_i)| : f_\beta \in A,\ \beta = 1,\ldots,i\}.$$

If $A = \mathbb{B}_{X^*}$, we denote this characteristic by $\mathcal{V}_i^X$; if X has a predual space $Y = X_*$ and $A = \mathbb{B}_{X_*}$, we use the notation $\widetilde{\mathcal{V}}_i^X$. If K is a compact set in $\mathbb{C}^n$, $X = C(K)$, h_i is the sequence of monomials (2.3), and $A = \{\delta_z : z \in K\}$, where $\delta_z(x) := x(z)$, then $\mathcal{V}_i^{X,A}$ coincides with the Leja sequence V_i. The following statement shows that the sequence $\mathcal{V}_i$ characterizes the linear independence of the sequence h_i.

Lemma 4.1. *The following statements are equivalent:*

(i) *the sequence $\{h_j\}$ is linearly independent in X;*
(ii) *the numbers Δ_i are all nonzero;*
(iii) *the numbers $\mathcal{V}_i = \mathcal{V}_i^{X,A}$ are all nonzero.*

Proof. (i)⇔(ii) is trivial. To prove (iii)⇒(i) suppose that there is j such that $\sum_{\alpha=1}^{j} c_\alpha h_\alpha = 0$ with some nontrivial coefficients. Then all determinants (4.2) with $i = j$ vanish for any sequence $\{f_\beta\}_{\beta=1}^{j} \subset X^*$, since their rows are not linearly independent. Thus $\mathcal{V}_j = 0$. The implication (i)⇒(iii) is included in the proof of the lemma below. ∎

Lemma 4.2. *Let $\{h_j\}$ be a linearly independent sequence in X. Then $\mathcal{V}_j = \mathcal{V}_j^{X,A} > 0$ for each j, and*

$$(4.4)\qquad \Delta_j \le \frac{\mathcal{V}_j}{\mathcal{V}_{j-1}} \le j\Delta_j, \qquad j = 2,3,\ldots.$$

Proof. It is clear that $\mathcal{V}_1 = \sup\{|f_1(h_1)| : f_1 \in A\} = \|h_1\| = \Delta_1 > 0$. Supposing that it has been proved that $\mathcal{V}_j > 0$ for $i \le j-1$, we will prove that $\mathcal{V}_i > 0$. To this end it suffices to prove the left inequality in (4.4) for $j = i$, assuming that $\mathcal{V}_{i-1} > 0$. Given ε with $0 < \varepsilon < \mathcal{V}_{i-1}$, choose $f_\nu \in A$, $\nu = 1,\ldots,i-1$, so that

$$0 < \mathcal{V}_{i-1} - \varepsilon \le V(f_1,\ldots,f_{i-1}).$$

For an arbitrary $f_i \in A$ consider a "polynomial"

$$h := \sum_{j=1}^{i} \frac{A_{ij}}{V(f_1,\ldots,f_{i-1})}\ h_j,$$

where A_{ij} is the cofactor of $f_i(h_j)$ in the matrix $(f_\beta(h_\alpha))_{\alpha,\beta=1}^i$. Since $A_{ii} = \mathcal{V}(f_1, \dots, f_{i-1})$, the leading coefficient of h is 1. Therefore

$$\|h\| \geq \Delta_i.$$

On the other hand,

$$f_i(h) = \sum_{j=1}^{i} \frac{A_{ij}}{V(f_1, \dots, f_{i-1})} f_i(h_j) = \frac{\mathcal{V}(f_1, \dots, f_{i-1}, f_i)}{\mathcal{V}(f_1, \dots, f_{i-1})},$$

hence

$$\|h\| = \sup\{|f_i(h)| : f_i \in A\} \leq \frac{\mathcal{V}_i}{\mathcal{V}_{i-1} - \varepsilon}.$$

Since $\varepsilon > 0$ is arbitrary, we obtain the estimate from below in (4.4).

In order to get the estimate from above in (4.4), we take arbitrary elements $f_\nu \in A$, $\nu = 1, \dots, i$. Let $T_i = h_i + \sum_{j=1}^{i-1} t_{ij}\, h_j$ be such that $\Delta_i = \|T_i\|$. Then the value of the determinant $\det\,(f_\beta(h_\alpha))_{\alpha,\beta=1}^i$ remains the same if its last row $f_\beta(h_i)$, $\beta = 1, \dots, i$, is replaced with $f_\beta(T_i)$. Expanding the modified determinant along the last row, we obtain

$$|\mathcal{V}(f_1, \dots, f_i)| \leq \sum_{\beta=1}^{i} |f_\beta(T_i)|\, |V(f_1, \dots, f_{\beta-1}, f_{\beta+1}, \dots, f_i)| \leq i\Delta_i \mathcal{V}_{i-1}.$$

Now, taking the least upper bound of the left side over $f_\nu \in A$, $\nu = 1, \dots, i$, we obtain the right estimate in (4.4). ∎

Corollary 4.3. *Let K be an infinite compact set in $\mathbb{C}^n$. Then in the notation of Section 3,*

$$(T_s)^{sNs} \leq \frac{V_{m_s}}{V_{m_{s-1}}} \leq \frac{m_s!}{m_{s-1}!}(T_s)^{sNs}, \qquad s = 2, 3, \dots. \tag{4.5}$$

The following statement has been proved implicitly in [61, proof of Theorem 1]. Here we give a direct slightly modified proof.

Lemma 4.4. *If $\lim_{s\to\infty} T_s(K)$ exists, then*

$$d(K) = \lim_{s\to\infty} d_s(K) = \lim_{s\to\infty} T_s(K). \tag{4.6}$$

Proof. Let $\tau = \lim_{s\to\infty} T_s(K)$. Since $\frac{\ln m_s!}{sN_s} \to 1$, due to (4.5) we have the asymptotic formula

$$\ln V_{m_s} - \ln V_{m_{s-1}} \sim sN_s \ln \tau \qquad \text{as } s \to \infty.$$

Summing from 1 to s we derive the asymptotic formula (see, e.g., [20])

$$\ln V_{m_s} \sim \ln V_{m_s} - \ln V_{m_0} \sim \sum_{q=1}^{s} qN_q \ln \tau = l_s \ln \tau \qquad \text{as } s \to \infty.$$

So (4.6) is proved. ∎

5. Equivalent definition of transfinite diameter in $\mathbb{C}^n$. Let K be a pluriregular compact set in $\mathbb{C}^n$ with $\hat{K} = K$. In the context of Section 3, take the space $X = C(K)$ and let h_i be the system of monomials $e_i(z) = z^{k(i)}$. Then V_i does not exceed

$$\mathcal{V}_i = \mathcal{V}_i^X = \sup\{|\det(f_\beta^*(e_\alpha))_{\alpha,\beta=1}^i| : f_\beta^* \in \mathbb{B}_{C(K)^*},\ \beta = 1, \ldots, i\} \tag{5.1}$$

for all $i \in \mathbb{N}$, since the supremum in the definition of V_i is taken only over the evaluation functionals $f_\zeta^*(h) = h(\zeta)$, $\zeta \in K$. Hence,

$$\limsup_{s\to\infty} (\mathcal{V}_{m_s})^{1/l_s} \geq d(K). \tag{5.2}$$

THEOREM 5.1. *Under the above assumptions the relation* (5.2) *holds with equality and with the usual limit, and we have the following representation of the transfinite diameter:*

$$d(K) = \lim_{s\to\infty} \sup\Big\{\Big|\det\Big(\int_K e_\alpha\, d\mu_\beta\Big)_{\alpha,\beta=1}^i\Big|^{1/l_{s(i)}}\Big\}, \tag{5.3}$$

where the supremum is taken over the set $\{(\mu_\beta) \in \mathcal{M}(K)^i : |\mu_\beta|(K) \leq 1,$ $\beta = 1, \ldots, i\}$.

Proof. Analyzing the proof of Theorem 3.2, one can see that, in order to prove Theorem 5.1, all arguments can be left untouched, except that Lemma 4.2 has to be applied instead of Corollary 5 of [61]. Then, remembering that, by the Riesz Theorem, $C(K)^*$ is represented as the space $\mathcal{M}(K)$ of all complex Borel measures on K, we obtain (5.3). ∎

The dual space $A(\mathbb{C}^n)^*$ can be realized as the space $A_0(\{\infty^n\})$ of all germs of analytic functions φ at $\infty^n := (\infty, \ldots, \infty) \in \bar{\mathbb{C}}^n$, having an expansion

$$\varphi(z) = \sum_{k\in\mathbb{Z}_+^n} \frac{a_k(\varphi)}{z^{k+I}}, \tag{5.4}$$

converging uniformly on

$$\overline{\mathbb{U}_r(\infty^n)} := \{z = (z_\nu) \in \bar{\mathbb{C}}^n : |z_\nu| \geq r\},$$

with $r = r(\varphi)$. Namely, there is a natural isomorphism $T : A(\mathbb{C}^n)^* \to A_0(\{\infty^n\})$ such that if $\varphi = Tf^*$, then

$$f^*(f) = \langle f, \varphi\rangle := \Big(\frac{1}{2\pi i}\Big)^n \int_{\mathbb{T}_R} f(z)\varphi(z)\, dz,$$

where $\mathbb{T}_R = \{z = (z_\nu) \in \bar{\mathbb{C}}^n : |z_\nu| = R\}$, $R > r$. An element $f^* \in A(\mathbb{C}^n)^*$ is said to be an *analytic functional in* $\mathbb{C}^n$ (see, e.g., [27]) and the series (5.4) can be considered as its *Taylor expansion at* ∞^n.

Let $AC(K)$ be the completion of the space of all polynomials in $C(K)$. Then $AC(K)^* = \mathcal{M}(K)/AC(K)^\perp$, where

$$AC(K)^\perp := \Big\{\mu \in \mathcal{M}(K) : \int_K f\, d\mu = 0,\ f \in AC(K)\Big\}.$$

The restriction operator $R : A(\mathbb{C}^n) \to AC(K)$ is injective, so it can be treated as a linear continuous and dense embedding $A(\mathbb{C}^n) \hookrightarrow AC(K)$. Then, by Lemma 2.1, $AC(K)^* \hookrightarrow A(\mathbb{C}^n)^*$, hence $AC(K)^*$ can be realized as $AC(K)' = T(AC(K)^*) \hookrightarrow A(\{\infty^n\})$, so that $\|\varphi\|_{AC(K)'} := \|f^*\|_{AC(K)^*}$. By the Hahn–Banach theorem, we obtain the same number $\mathcal{V}_i$ if the sup in (5.1) is taken over the set

$$\{f^*_\beta \in AC(K)^* : \|f^*_\beta\|_{AC(K)^*} \le 1,\ \beta = 1, \dots, i\}.$$

If $\varphi = Tf^* \in AC(K)'$, then

$$f^*(e_i) = \langle e_i, \varphi\rangle = a_{k(i)}(\varphi).$$

Therefore we obtain the following representation of the transfinite diameter in terms of the Taylor expansions of the analytic functionals f^*_β at ∞^n:

$$d(K) = \lim_{s\to\infty} \sup\{|\det\,(a_{k(\alpha)}(\varphi_\beta))^i_{\alpha,\beta=1}|^{1/l_{s(i)}}\}, \tag{5.5}$$

where the sup is taken over the set

$$\{(\varphi_\beta)^i_{\beta=1} : \varphi_\beta \in \mathbb{B}_{AC(K)'},\ \beta = 1, \dots, i\}. \tag{5.6}$$

The mapping $S : A(\{\infty^n\}) \to A(\{0^n\})$ defined by

$$g = S\varphi, \qquad g(\zeta) := \frac{\varphi(1/\zeta_1, \dots, 1/\zeta_n)}{\zeta_1 \cdots \zeta_n} = \sum_{k\in\mathbb{Z}^n_+} a_k(\varphi)\zeta^k$$

is an isomorphism. Therefore, due to (5.5), one can represent the transfinite diameter in terms of extremal multivariate Wronskians. Let

$$\mathcal{W}(g_1, \dots, g_i) := \det\,(g_\beta^{(k(\alpha))}(0))^i_{\alpha,\beta=1}$$

and

$$\mathcal{W}_i := \sup\{|\mathcal{W}(g_1, \dots, g_i)|\}, \tag{5.7}$$

where the supremum is taken over all $(g_\beta)^i_{\beta=1}$ such that $(\varphi_\beta) = (S^{-1}g_\beta)$ runs over the set (5.6). Notice first that

$$l_s \sim \lambda_s := \frac{s^{n+1}}{(n-1)!(n+1)} \quad \text{as } s \to \infty. \tag{5.8}$$

Theorem 5.2. *In the above notation*

$$d(K) = \lim_{i\to\infty}\bigg(\frac{\mathcal{W}_i}{\prod_{\alpha=1}^i k(\alpha)!}\bigg)^{1/l_{s(i)}} = \bigg(\exp\sum_{\nu=1}^{n+1}\frac{1}{\nu}\bigg)\cdot \lim_{i\to\infty}\frac{(\mathcal{W}_i)^{1/\lambda_{s(i)}}}{s(i)}. \tag{5.9}$$

Proof. The first equality follows from (5.5), because $a_k(\varphi) = g^{(k)}(0)/k!$ if $g = S_\varphi$. Since $l_{s(i)}$ can be replaced by $\lambda_{s(i)}$ in the middle term of (5.9), we need to prove the asymptotics

$$A_s := \frac{1}{\lambda_s} \sum_{|k| \le s} \ln k! = \ln s - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n+1}\right) + o(1) \tag{5.10}$$

as $s \to \infty$. By Stirling's formula, we have

$$A_s = \frac{1}{\lambda_s} \sum_{|k| \le s} \sum_{\nu=1}^{n} k_\nu (\ln k_\nu - 1) + o(1) \quad \text{as } s \to \infty.$$

Since $\sum_{|k| \le s} \sum_{\nu=1}^{n} k_\nu = l_s$, due to (5.8) we have $A_s = B_s - 1 + o(1)$ with

$$B_s := \frac{1}{\lambda_s} \sum_{|k| \le s} \sum_{\nu=1}^{n} k_\nu \ln k_\nu. \tag{5.11}$$

Further

$$\begin{aligned} B_s &= \frac{1}{\lambda_s}\Big(\int\limits_{V_s} \Big(\sum_{\nu=1}^{n} x_\nu \ln x_\nu\Big)\, dx_1 \dots dx_n \ + O(s^n \ln s)\Big) \\ &= \frac{n}{\lambda_s} \int\limits_{V_s} x_1 \ln x_1 \ dx_1 \dots dx_n + o(1) \quad \text{as } s \to \infty, \end{aligned}$$

where $V_s = \{(x_\nu) \in \mathbb{R}^n : \sum_{\nu=1}^{n} x_\nu \le s;\ x_\nu \ge 1,\ \nu = 1, \dots, n\}$. Then, integrating $n-1$ times, we have

$$\begin{aligned} B_s &= \frac{n}{(n-1)!\lambda_s} \int\limits_1^s \xi (s-\xi)^{n-1} \ln \xi \, d\xi + o(1) \\ &= \frac{n}{(n-1)!\lambda_s} \sum_{j=0}^{n-1} (-1)^j C_{n-1}^j s^{n-j-1} \int\limits_1^s \xi^{j+1} \ln \xi \, d\xi + o(1) \end{aligned}$$

as $s \to \infty$. By elementary computations we obtain

$$B_s = \frac{n s^{n+1}}{(n-1)!\lambda_s} (a \ln s - b) + o(1) = n(n+1)(a \ln s - b) + o(1)$$

with (see, e.g., [45, 4.2.2.44, 4.2.2.57])

$$\begin{aligned} a &= \sum_{j=0}^{n-1} \frac{(-1)^j C_{n-1}^j}{j+2} = \frac{1}{n(n+1)}, \\ b &= \sum_{j=0}^{n-1} \frac{(-1)^j C_{n-1}^j}{(j+2)^2} = \frac{\frac{1}{2} + \cdots + \frac{1}{n+1}}{n(n+1)}. \end{aligned}$$

Finally, combining all the relations obtained, we get (5.10). ∎

Notice that, applying Euler's summation formula, one can obtain essentially stronger asymptotics for sums like (5.11) (similarly to [26, p. 595], or [31]), but the above rough asymptotics is sufficient for the present purposes.

REMARK 5.3. The relation (5.9) can be considered as an asymptotic expression for extremal Wronskians through the transfinite diameter. It must be emphasized that $\lambda_{s(i)}$ cannot be replaced by $l_{s(i)}$ in the right member of (5.9).

6. Asymptotics of leading coefficients of orthonormal polynomial bases. Let K be a polynomially convex compact set in $\mathbb{C}^n$. We say that a Banach space $X \hookleftarrow A(K)$ is *adherent to the space* $A(K)$ (has the *Bernstein–Markov property relative to* K) if for for every $q > 1$ there exists an open set $G \supset K$ and a constant M such that

$$|p|_G \le M q^s \|p\|_X \tag{6.1}$$

for each polynomial p of degree $s \in \mathbb{N}$. If K is pluriregular, then there exist adherent Banach spaces; for instance, the spaces $AC(K)$ and $AL^2(K, \mu)$, where $\mu^\circ = (dd^c g_K)^n$, are adherent to $A(K)$ (see, e.g., [62, 53, 63, 66]). For the condition (6.1) in a general frame of interpolation properties of locally convex spaces, see [59, 42, 64, 65].

Let K be a pluriregular polynomially convex compact set in $\mathbb{C}^n$ and $H \hookleftarrow A(K)$ be any Hilbert space adherent to $A(K)$. Let

$$p_i = \sum_{j=1}^{i} a_{ji} e_j, \qquad i \in \mathbb{N}, \tag{6.2}$$

be the orthonormal system in H obtained by the Gram–Schmidt procedure from the system of monomials $e_i(z) = z^{k(i)}$. Denote by H_R the Hilbert scale defined via

$$H_R := \Big\{ x = \sum_{i=1}^{\infty} \xi_i p_i \in H : \|x\|_{H_R} := \Big(\sum_{i=1}^{\infty} |\xi_i|^2 R^{2s(i)} \Big)^{1/2} < \infty \Big\} \tag{6.3}$$

with $R > 1$. Consider two one-parameter families of sublevel sets of the pluripotential Green function:

$$D_R = \{z \in \mathbb{C}^n : g_K(z) < \ln R\}, \ K_R = \{z \in \mathbb{C}^n : g_K(z) \le \ln R\}, \ R > 1.$$

The system $\{p_i\}$ is a common basis in the spaces $A(K)$, $A(\mathbb{C}^n)$, $A(D_R)$, $A(K_R)$, and the scale (6.3) satisfies the following embeddings ([62, 63, 66, 64]):

$$A(K_R) \hookrightarrow H_R \hookrightarrow A(D_R), \qquad R > 1. \tag{6.4}$$

The following assertion is an easy consequence of the above considerations and the results of [61] (see Section 3 above).

THEOREM 6.1 ([63, 66, 64]). *The asymptotic behavior of the leading coefficients in* (6.2) *is determined by the directional Chebyshev constants of K:*

$$\lim_{k(i)/|k(i)|\to\theta} |a_{ii}|^{1/s(i)} = \frac{1}{\tau(K,\theta)}, \qquad \theta \in \Sigma^\circ,$$

which means that the same limit exists for any subsequence $L \in \mathcal{L}_\theta$. Moreover, the asymptotics of the geometric mean of the leading coefficients of degree s is controlled by the principal Chebyshev constant of K:

$$\lim_{s\to\infty} \Big| \prod_{s(i)=s} a_{ii} \Big|^{1/sN_s} = \frac{1}{\tau(K)} = \frac{1}{d(K)}. \tag{6.5}$$

Proof. Given any Hilbert space H adherent to $A(K)$, consider the characteristic $\Delta_{i,H}$ defined by (4.1) with $X = H$ and $h_i = e_i$. Then we have

$$\Delta_{i,H} = \Big\| \frac{p_i}{|a_{ii}|} \Big\|_H = \frac{1}{|a_{ii}|}, \qquad \Delta_{i,H_R} = \frac{R^{s(i)}}{|a_{ii}|}, \qquad i \in \mathbb{N}.$$

Let $G = AL^2(K,\mu^\circ)$ with $\mu^\circ := (dd^c g_K(z))^n$ and let G_R be the scale (6.3) relative to G instead of H. Let $1 < r < R$. Taking into account the embeddings (6.4), we have

$$G_R \hookrightarrow A(D_R) \hookrightarrow A(K_r) \hookrightarrow H_r \hookrightarrow A(K) \hookrightarrow AC(K) \hookrightarrow G,$$

hence there are positive constants C_1, C_2, C_3 such that

$$\Delta_{i,G} \le C_1 M_i \le C_2 \Delta_{i,H_r} = C_2 r^{s(i)} \Delta_{i,H} \le C_3 \Delta_{i,G_R} = C_3 R^{s(i)} \Delta_{i,G} \tag{6.6}$$

for all $i \in \mathbb{N}$. Take any sequence $L \in \mathcal{L}_\theta$, $\theta \in \Sigma^\circ$. Since the usual limit exists in (3.2), (6.6) implies

$$\begin{aligned}\limsup_{i\in L} (\Delta_{i,G})^{1/s(i)} &\le r^{s(i)} \lim_{i\in L} (\Delta_{i,G})^{1/s(i)} \le \lim_{i\in L} (M_i)^{1/s(i)} \\ &= \tau(K,\theta) \le R \liminf_{i\in L} (\Delta_{i,G})^{1/s(i)}.\end{aligned}$$

By arbitrariness of r, R, we derive from this and (6.6) that

$$\lim_{i\in L} \Big(\frac{1}{|a_{ii}|}\Big)^{1/s(i)} = \lim_{i\in L} (\Delta_{i,H})^{1/s(i)} = \lim_{i\in L} (\Delta_{i,G})^{1/s(i)} = \tau(K,\theta), \qquad \theta \in \Sigma^\circ.$$

To prove (6.5) we repeat all arguments from [61] (proofs of Lemmas 5 and 6 there) applied to $\tau_i(H) := (\Delta_{i,H})^{1/s(i)}$ instead of $\tau_i = (M_i)^{1/s(i)}$. ■

Now we give an alternative proof of [13, Corollary 1 on p. 296)].

COROLLARY 6.2. *For $R > 1$, we have $\tau(K_R, \theta)a = R\tau(K, \theta)$ and $d(K_R) = Rd(K)$.*

Proof. Let $R > 1$, $0 < \varepsilon < R - 1$ and

$$M_{i,R} = \inf\Big\{ |p|_{K_R} : p = e_i + \sum_{j=1}^{i-1} c_j\ e_j \Big\}, \qquad i \in \mathbb{N}.$$

Then, by (6.4), $H_{R+\varepsilon} \hookrightarrow AC(K_R) \hookrightarrow H_{R-\varepsilon}$, hence

$$c(R-\varepsilon)^{s(i)} \Delta_{i,H} \le M_{i,R} \le C(R+\varepsilon)^{s(i)} \Delta_{i,H}.$$

It remains to apply Theorems 6.1 and 3.2. ∎

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