On absolutely monotone set-valued functions

by Andrzej Smajdor (Kraków)

Abstract. We define absolutely monotone multifunctions and prove their analyticity on an interval \([0, b]\).

1. Let \(f : [a, b) \to \mathbb{R}\). The \(p\)th order difference \(\Delta^p_s f(t)\) of \(f\) is defined inductively as follows:
\[
\Delta^0_s f(t) = f(t), \quad \Delta^{p+1}_s f(t) = \Delta^p_s f(t + s) - \Delta^p_s f(t)
\]
for every nonnegative integer \(p, t \in [a, b)\), \(s > 0\) such that \(t + (p + 1)s < b\).

We say that the function \(f\) is absolutely monotone in the interval \([a, b)\) if \(\Delta^p_s f(t) \geq 0\) for \(a \leq t \leq t + ps < b\), \(p = 0, 1, \ldots\). The following Bernstein theorem is well known (see e.g. [3, Theorem 2.3.2]):

**Theorem.** Every absolutely monotone function \(f : [0, b) \to \mathbb{R}\) is analytic:
\[
f(t) = \sum_{n=0}^{\infty} a_n t^n
\]
in \([0, b)\) with \(a_n \geq 0, n = 0, 1, \ldots\).

2. In this paper we prove an analogue of S. Bernstein’s theorem for absolutely monotone set-valued functions. Let \(Y\) be a real normed space and let \(cc(Y)\) denote the family of all nonempty compact convex subsets of \(Y\). A set \(C \in cc(Y)\) is the Hukuhara difference of \(A \in cc(Y)\) and \(B \in cc(Y)\) if
\[
A = B + C = \{b + c : b \in B, c \in C\}
\]
(see [2]). If the difference \(C = A - B\) exists, then it is unique. This is a consequence of the following:

**Lemma 1** (cf. [5]). Let \(A, B\) and \(C\) be subsets of a real topological vector space such that
\[
A + B \subset C + B.
\]
If \(C\) is convex closed and \(B\) is nonempty bounded, then \(A \subset C\).

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Now, let $-\infty < a < b \leq \infty$ and let $H : [a, b) \to \text{cc}(Y)$. We define the $p$th differences $\Delta^p_s H(t)$ by the recurrence
$$\Delta^0_s H(t) = H(t), \quad \Delta^{p+1}_s H(t) = \Delta^p_s H(t + s) - \Delta^p_s H(t)$$
for every nonnegative integer $p, t \in [a, b)$, $s > 0$ such that $t + (p + 1)s < b$.

A set-valued function is said to be absolutely monotone if all differences $\Delta^p_s H(t)$ exist and each contains zero.

**Example.** Let $A \in \text{cc}(Y)$ be such that $0 \in A$. Suppose that $h : [a, b) \to [0, \infty)$. Then $H(t) = h(t)A$ is an absolutely monotone set-valued function if and only if $h$ is an absolutely monotone real function.

We can observe the following:

**Remark.** Let $b$ and $\alpha$ be positive numbers, $H : [0, b) \to \text{cc}(Y)$ and $G(t) = H(\alpha t)$ on $[0, b/\alpha)$. Then $G$ is absolutely monotone if and only if $H$ is absolutely monotone.

Let $G : [0, 1] \to \text{cc}(Y)$ be a given multifunction. The polynomial
$$B_n(t) = \sum_{i=0}^{n} \binom{n}{i} t^i (1 - t)^{n-i} G\left(\frac{i}{n}\right)$$
is called the $n$th Bernstein polynomial of $G$.

**Theorem 1.** If $G : [0, 1] \to \text{cc}(Y)$ is continuous (with respect to the Hausdorff metric $d$ in $\text{cc}(Y)$), then
$$d(B_n(t), G(t)) \leq \frac{3}{2} \omega\left(\frac{1}{\sqrt{n}}\right),$$
where
$$\omega(\delta) = \sup\{d(G(t''), G(t'))) : |t'' - t'| < \delta\}.$$

The proof of this theorem runs similarly to the proof of Bernstein's approximation theorem (cf. [3]).

**Lemma 2.** Let $G : [0, 1] \to \text{cc}(Y)$ be a multifunction. Then
$$B_n(t) = \sum_{i=0}^{n} \binom{n}{i} t^i \Delta^i_{1/n} G(0)$$
for positive integers $n$ and $t \in [0, 1]$.

**Proof.** Let “~” denote the Rådström equivalence relation between pairs of members of $\text{cc}(Y)$ defined by the formula
$$(A, B) \sim (C, D) \iff A + D = B + C.$$  
For any pair $(A, B)$, $[A, B]$ denotes its equivalence class. All equivalence classes form a linear space $\tilde{Y}$ with addition defined by the rule
$$[A, B] + [C, D] = [A + C, B + D]$$
and scalar multiplication
\[
\lambda[A, B] = \begin{cases} 
[\lambda A, \lambda B] & \text{for } \lambda \geq 0, \\
[-\lambda B, -\lambda A] & \text{for } \lambda < 0
\end{cases}
\]
(cf. [5]).

Consider the function \( g : [0, 1] \rightarrow \tilde{Y} \) defined as follows:
\[
g(t) = [G(t), \{0\}].
\]

It can be proved by induction that
\[
\Delta_s^p g(t) = [\Delta_s^p G(t), \{0\}]
\]
and
\[
\Delta_s^p g(t) = \sum_{i=0}^{p} \binom{p}{i} t^i (-1)^{p-i} g(t + is)
\]
for nonnegative integers \( p, t \in [0, 1) \) and \( s > 0 \) such that \( t + ps < 1 \).

Let \( b_n \) be Bernstein's polynomials of \( g \):
\[
b_n(t) = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} g \left( \frac{i}{n} \right).
\]

Then
\[
b_n(t) = [B_n(t), \{0\}].
\]

Using Newton's binomial formula, replacing \( j \) by \( j - i \) in the second sum below, changing the order of summation, and then making use of the identity \( \binom{n}{i} \binom{n-i}{j-i} = \binom{n}{j} \binom{j}{i} \) and equality (3) we obtain
\[
b_n(t) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \binom{n}{i} \binom{n-i}{j} (-1)^{j} t^{i+j} g \left( \frac{i}{n} \right)
\]
\[
= \sum_{i=0}^{n} \sum_{j=i}^{n} \binom{n}{i} \binom{n-i}{j-i} (-1)^{j-i} t^{j} g \left( \frac{i}{n} \right)
\]
\[
= \sum_{j=0}^{n} \sum_{i=0}^{j} \binom{n}{j} \binom{j}{i} (-1)^{j-i} t^{i} g \left( \frac{i}{n} \right) = \sum_{j=0}^{n} \binom{n}{j} t^{j} \Delta_{1/n}^{j} g(0).
\]

According to (2) we have
\[
\left[ \sum_{i=0}^{n} \binom{n}{i} t^{i} \Delta_{1/n}^{i} G(0), \{0\} \right] = \sum_{i=0}^{n} \binom{n}{i} t^{i} [\Delta_{1/n}^{i} G(0), \{0\}]
\]
\[
= \sum_{i=0}^{n} \binom{n}{i} t^{i} \Delta_{1/n}^{i} g(0) = b_n(t) = [B_n(t), \{0\}].
\]
Thus
\[
\left( \sum_{i=0}^{n} \binom{n}{i} t^i \Delta_{1/n}^i G(0), \{0\} \right) \sim (B_n(t), \{0\})
\]
and (1) holds.

**Lemma 3.** Let \(0 < c \leq b\) and \(H : [0, b) \to \text{cc}(Y)\). If \(A_i, B_i \in \text{cc}(Y), i = 0, 1, \ldots\), are such that
\[
H(t) = \sum_{n=0}^{\infty} t^n A_n \quad \text{for } t \in [0, b),
\]
\[
H(t) = \sum_{n=0}^{\infty} t^n B_n \quad \text{for } t \in [0, c),
\]
then \(A_i = B_i\) for \(i = 0, 1, \ldots\).

**Proof.** We see that \(A_0 = H(0) = B_0\). Suppose that
\[
A_0 = B_0, \ldots, A_k = B_k.
\]
Then
\[
A_{k+1} = \lim_{t \to 0^+} \frac{H(t) - \sum_{i=0}^{k} t^i A_i}{t^{k+1}} = \lim_{t \to 0^+} \frac{H(t) - \sum_{i=0}^{k} t^i B_i}{t^{k+1}} = B_{k+1}.
\]

**Theorem 2.** A set-valued function \(H : [0, b) \to \text{cc}(Y)\) is absolutely monotone if and only if there exist sets \(A_i \in \text{cc}(Y), i = 0, 1, \ldots\), containing zero such that
\[
H(t) = \sum_{n=0}^{\infty} t^n A_n \quad \text{for } t \in [0, b).
\]

**Proof.** 1. Suppose that \(H : [0, b) \to \text{cc}(Y)\) is of the form (4) and that \(0 \in A_n \in \text{cc}(Y)\). We see that
\[
H(t + s) = \sum_{n=0}^{\infty} (t + s)^n A_n = H(t) + \sum_{n=0}^{\infty} ((t + s)^n - t^n) A_n,
\]
therefore
\[
\Delta_1^s H(t) = \sum_{n=0}^{\infty} \Delta_1^s t^n A_n.
\]
By induction it may be shown that
\[
\Delta_p^s H(t) = \sum_{n=0}^{\infty} \Delta_p^s t^n A_n.
\]
Thus all differences \(\Delta_p^s H(t)\) exist. As they contain zero, \(H\) is an absolutely monotone multifunction.
2. Now, suppose that $H : [0, b) \to \text{cc}(Y)$ is an absolutely monotone multifunction. The differences $\Delta^1_s H(t)$ and $\Delta^2_s H(t)$ exist and contain zero, therefore

$$H(t) \subset H(t) + \Delta^1_s H(t) = H(t + s)$$

and

$$2H(t + s) \subset 2H(t + s) + \Delta^2_s H(t) = H(t + s) + H(t) + \Delta^1_s H(t) + \Delta^2_s H(t) = H(t + s) + H(t) + \Delta^1_s H(t + s) = H(t) + H(t + 2s).$$

Thus $H$ is increasing and midconcave in $[0, b)$.

Fix a number $c \in (0, b)$. The function $H$, being midconcave and bounded on $[0, c]$, is continuous, according to Theorem 4.4 in [4]. Define $G(t) = H(ct)$ for $t \in [0, 1]$. Then $G$ is continuous and by Theorem 1 it is the uniform limit of the sequence of its Bernstein polynomials $B_n(t)$. By Lemma 2 we have

$$B_n(t) = \sum_{i=0}^{n} \binom{n}{i} t^i \Delta^i_{1/n} G(0) = \sum_{i=0}^{n} t^i A^n_i,$$

where $A^n_i = \binom{n}{i} \Delta^i_{1/n} G(0)$. We note that $0 \in A^n_0$ and

$$A^n_0 \subset B_n(1) = G(1).$$

Since $G(1)$ is compact, the family of all closed subsets of $G(1)$ is compact (see [1, p. 41]). By (5) there exists a strictly increasing sequence $(n^0_k)$ and $A_0(c) \in \text{cc}(Y)$ such that

$$A^n_0 \to A_0(c).$$

Similarly, since

$$A^n_1 \subset B_n(1) = G(1),$$

there exists a strictly increasing subsequence $(n^1_k)$ of $(n^0_k)$ and $A_1 \in \text{cc}(Y)$ such that

$$A^n_1 \to A_1(c)$$

and so on. Applying the diagonalization procedure to the sequences $(A^n_0)$, $(A^n_1)$, \ldots we obtain a strictly increasing sequence $(n_k)$ such that

$$A^n_0 \to A_0(c), \ A^n_1 \to A_1(c), \ldots$$

Fix $t \in [0, 1)$, $\varepsilon > 0$ and define

$$S_n(t) = \sum_{i=0}^{n} t^i A_i(c) \quad \text{for } n = 0, 1, \ldots.$$ 

Choose a positive integer $k$ so large that $2 \|G(1)\| t^k (1 - t)^{-1} < \varepsilon/3$, where $\|G(1)\| = \sup\{|y| : y \in G(1)\}$, and then choose $L$ large enough to get $d(B_{n_l}(t), G(t)) < \varepsilon/3$ and $\sum_{j=0}^{k-1} d(A^n_j, A_j) < \varepsilon/3$ for $l \geq L$. Then
\[ d(S_{n_l}(t), G(t)) \leq d(S_{n_l}(t), B_{n_l}(t)) + d(B_{n_l}(t), G(t)) \]
\[ \leq 2 \varepsilon/3 + \sum_{i=k+1}^{n_l} t^i d(A_i^{n_l}, A_i) \leq (2/3) \varepsilon + 2\|G(1)\| \frac{t^{k+1}}{1-t} < \varepsilon. \]

Thus
\[ \lim_{l \to \infty} S_{n_l}(t) = G(t) \]
and according to Theorem II-2 in [1],
\[ G(t) = \bigcup_{l=1}^{\infty} S_{n_l}(t). \]

Using the monotonicity of the sequence \((S_n(t))\) we get
\[ S_l(t) \subset S_{n_l}(t) \subset G(t) \quad \text{for } l = 0, 1, \ldots. \]
Therefore the sequence \(d(G(t), S_l(t))\) is decreasing. By (6),
\[ \lim_{l \to \infty} d(G(t), S_l(t)) = \lim_{l \to \infty} d(G(t), S_{n_l}(t)) = 0. \]

Consequently,
\[ G(t) = \lim_{n \to \infty} S_n(t) = \sum_{i=0}^{\infty} t^i A_i \quad \text{for } t \in [0, 1). \]

The definition of \(G\) leads to
\[ H(t) = G(t/c) = \sum_{i=0}^{\infty} t^i c^{-i} A_i \quad \text{for } t \in [0, c). \]

Now (4) follows from Lemma 3.

References


Pedagogical University
Podchorążych 2
30-084 Kraków, Poland
E-mail: asmajdor@ap.krakow.pl

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