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On the complexification of real-analytic polynomial mappings of $\mathbb{R}^2$

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Abstract. We give a simple algebraic condition on the leading homogeneous term of a polynomial mapping from $\mathbb{R}^2$ into $\mathbb{R}^2$ which is equivalent to the fact that the complexification of this mapping can be extended to a polynomial endomorphism of $\mathbb{CP}^2$. We also prove that this extension acts on $\mathbb{CP}^2 \setminus \mathbb{C}^2$ as a quotient of finite Blaschke products.

1. Introduction and preliminaries. The two-dimensional space $\mathbb{R}^2$ can be identified with the complex plane $\mathbb{C}$. Each real-analytic polynomial mapping on $\mathbb{R}^2$ can be written in complex coordinates as

$$Q(z) = \sum_{k=0}^{n} Q_k(z) = \sum_{k=0}^{n} \sum_{i=0}^{k} a_{ki} z^i \overline{z}^{k-i}.$$

We can now complexify the mapping $Q(z)$, in the same manner as in [Li2], [Li3], putting

$$f(z,w) = \Big(\sum_{k=0}^{n} \sum_{i=0}^{k} a_{ki} z^i w^{k-i}, \sum_{k=0}^{n} \sum_{i=0}^{k} \overline{a}_{ki} w^i z^{k-i} \Big).$$

The mapping f is a polynomial endomorphism of $\mathbb{C}^2$ preserving the completely real subset $\{(z,\overline{z})\}_{z\in\mathbb{C}}$. In [Li3] we proved that if Q is a quasiregular polynomial which has algebraic degree two, then f extends to a polynomial endomorphism of the complex projective space $\mathbb{CP}^2$.

A simple example of $Q(z) = z|z|^2 = z^2\overline{z}$ shows that this is not true in general. The aim of the present paper is to give conditions on $Q(z)$ which are equivalent to the existence of the extension of $f(z,w)$ to $\mathbb{CP}^2$ and to study the behavior of the extended map on $\mathbb{CP}^2 \setminus \mathbb{C}^2$.

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The mapping f is extendable to $\mathbb{CP}^2$ iff the mapping $\widetilde{f}:\mathbb{C}^3\to\mathbb{C}^3$ defined by

$$\widetilde{f}(z,w,t)=\Big(\sum_{k=0}^{n}\Big(\sum_{i=0}^{k}a_{ki}z^iw^{k-i}\Big)t^{n-k},\sum_{k=0}^{n}\Big(\sum_{i=0}^{k}\overline{a}_{ki}w^iz^{k-i}\Big)t^{n-k},t^n\Big),$$

vanishes only at $w=z=t=0$. This is equivalent to the fact that

$$f_n(z,w)=\Big(\sum_{i=0}^{n}a_{ni}z^iw^{n-i},\sum_{i=0}^{n}\overline{a}_{ni}w^iz^{n-i}\Big)$$

vanishes only at $w=z=0$.

The mapping

$$Q_n(z)=\sum_{i=0}^{n}a_{ni}z^i\overline{z}^{n-i}$$

can be written (for $z\neq 0$) as

$$Q_n(z)=\overline{z}^n\cdot\sum_{i=0}^{n}a_{ni}\Big(\frac{z}{\overline{z}}\Big)^i=\overline{z}^n\sum_{i=0}^{n}a_{ni}\xi^n,\quad \xi=\frac{z}{\overline{z}}.$$

Let i_0 be the greatest number for which $a_{ni}\neq 0$. The polynomial

$$P(\xi)=\sum_{i=0}^{n}a_{ni}\xi^i$$

can be written as

$$P(\xi)=a_{ni_0}\prod_{j=1}^{i_0}(\xi-p_j)$$

where the p_j are roots of $P(\xi)$. We have

$$Q_n(z)=a_{ni_0}\overline{z}^{n-i_0}\prod_{j=1}^{i_0}(z-p_j\overline{z}).$$

2. Results

THEOREM 2.1. *The complexified mapping f extends to a polynomial endomorphism of $\mathbb{CP}^2$ iff one of the following conditions holds*:

(1) $i_0=n$ *and* $p_i\overline{p}_j\neq 1$ *for each* $i,j=1,\ldots,n$;
(2) $i_0<n$, $p_i\neq 0$ *for* $i=1,\ldots,i_0$ *and* $p_i\overline{p}_j\neq 1$ *for each* i, j.

Proof. We have

$$f_n(z,w)=\Big(a_{ni_0}w^{n-i_0}\prod_{j=1}^{i_0}(z-p_jw),\overline{a}_{ni_0}z^{n-i_0}\prod_{j=1}^{i_0}(w-\overline{p}_jz)\Big).$$

Assume that (2) holds.

$f_n(z,w)$ can vanish in the following three cases:

(a) $w=0$ and $w-\overline{p}_j z=0$ for some j,
(b) $z=0$ and $z-p_j w=0$ for some j,
(c) $z-p_i w=0$ and $w-\overline{p}_j z=0$ for some i and j.

Since $p_j \neq 0$ conditions (a) and (b) imply that $z=w=0$. In case (c) we have

$$\det \begin{bmatrix} 1 & -p_i \\ -\overline{p}_j & 1 \end{bmatrix} = 1 - p_i \overline{p}_j \neq 0$$

and again $w=z=0$.

If condition (1) holds, we only have case (c) and thus the conditions $p_i \neq 0$ are not needed.

In the opposite direction if neither (1) nor (2) is fulfilled then $f(z,w)=0$ on some one-dimensional linear subspace of $\mathbb{C}^2$. ∎

Suppose now that one of the conditions of Theorem 2.1 is valid. We have

THEOREM 2.2. *The restriction of the extended map f to $\mathbb{CP}^2 \setminus \mathbb{C}^2$ is a rational function which is equal to a quotient of two finite Blaschke products.*

Proof. We can write

$$f_n(z,w) = (f_1(z,w), f_2(z,w)).$$

The mapping f acts on $\mathbb{CP}^2 \setminus \mathbb{C}^2$ as

$$\phi\left(\frac{z}{w}\right) = \frac{f_1(z,w)}{f_2(z,w)} := \phi(\xi), \quad \xi = \frac{z}{w}.$$

If condition (1) of Theorem 2.1 is fulfilled then

$$\frac{f_1(z,w)}{f_2(z,w)} = \frac{a}{\overline{a}} \prod_{j=1}^{n} \frac{z-p_j w}{w-\overline{p}_j z} = \frac{a}{\overline{a}} \prod_{j=1}^{n} \frac{\frac{z}{w}-p_j}{1-\overline{p}_j \frac{z}{w}} = \frac{a}{\overline{a}} \prod_{j=1}^{n} \frac{\xi-p_j}{1-\overline{p}_j \xi} = \phi(\xi)$$

where $a=a_{nn}$. Since $p_i \overline{p}_j \neq 1$ for all $i,j=1,\dots,n$, none of p_j can have modulus one.

Let $p_1,\dots,p_s$ have modulus less than one and $p_{s+1},\dots,p_n$ have modulus greater than one. We now have

$$\phi(\xi) = \frac{a}{\overline{a}} \prod_{j=1}^{n} \frac{\xi-p_j}{1-\overline{p}_j \xi} = \frac{B_1(\xi)}{B_2(\xi)}$$

where

$$B_1(\xi) = \frac{a}{\overline{a}} \prod_{j=1}^{s} \frac{\xi-p_j}{1-\overline{p}_j \xi}$$

and

$$B_2(\xi) = \prod_{j=s+1}^{n} \frac{1-\overline{p}_j\xi}{\xi - p_j} = \prod_{j=s+1}^{n} \frac{\overline{p}_j}{p_j} \frac{\xi - \frac{1}{\overline{p}_j}}{1 - \frac{\xi}{p_j}}$$

are finite Blaschke products.

If condition (2) is fulfilled then

$$\phi(\xi) = \frac{a}{\overline{a}} \frac{1}{\xi^{n-i_0}} \prod_{j=1}^{i_0} \frac{\xi - p_j}{1-\overline{p}_j\xi}, \qquad a = a_{ni_0}.$$

We can again assume that $p_1, \ldots, p_s$ have modulus less than one and $p_{s+1}, \ldots, p_{i_0}$ have modulus greater than one. We have

$$\phi(\xi) = \frac{B_1(\xi)}{B_2(\xi)}$$

where

$$B_1(\xi) = \frac{a}{\overline{a}} \prod_{j=1}^{s} \frac{\xi - p_j}{1-\overline{p}_j\xi}$$

and

$$B_2(\xi) = \xi^{n-i_0} \prod_{j=s+1}^{i_0} \frac{\overline{p}_j}{p_j} \frac{\xi - \frac{1}{\overline{p}_j}}{1 - \frac{\xi}{p_j}}$$

are finite Blaschke products. ∎

Example 2.3. Let $Q(z) = |z|^2 - p\overline{z}^2$, $|p| \neq 0, 1$. Then $Q(z) = \overline{z}(z - p\overline{z})$ and condition (2) of Theorem 2.1 is fulfilled. We have

$$f(z,w) = (zw - pw^2, zw - \overline{p}z^2)$$

and

$$\phi(\xi) = \frac{1}{\xi} \frac{\xi - p}{1-\overline{p}\xi} = \frac{B_1(\xi)}{B_2(\xi)}$$

where

$$B_1(\xi) = \frac{\xi - p}{1-\overline{p}\xi}, \quad B_2(\xi) = \xi, \qquad \text{if } |p| < 1,$$

$$B_1(\xi) = 1, \qquad B_2(\xi) = \frac{\overline{p}}{p}\xi\left(\frac{\xi - \frac{1}{\overline{p}}}{1 - \frac{\xi}{p}}\right) \quad \text{if } |p| > 1.$$

Note that for every p, $|p| \neq 0, 1$, $Q(z)$ is not quasiregular. Hence our Theorems 2.1 and 2.2 are more general than Theorem 3.1 (and remarks after it) in [Li3] even in the case of $n = 2$.

In order to show the possible use of Theorems 2.1 and 2.2 we shall give the following two propositions.

Proposition 2.4. *Assume that* $n = i_0$ *and* $|p_j| < 1$ *for each* $j = 1, \ldots, n$. *Then*:

(1) $Q_n(z) = a\prod_{j=1}^{n}(z - p_j\overline{z})$, $a \neq 0$, *is quasiregular.*
(2) *If* $Q(z) = Q_n(z)$ + *lower degree homogeneous terms then* $Q(z)$ *can be complexified and extended to a polynomial endomorphism of* $\mathbb{CP}^2$. *This extension acts on* $\mathbb{CP}^2 \setminus \mathbb{C}^2$ *as a Blaschke product of degree* n.

Proof. Part (2) follows immediately from Theorems 2.1 and 2.2 since condition (1) of Theorem 2.1 holds. By Theorem 2.2 we have

$$\phi(\xi) = \frac{a}{\overline{a}}\prod_{j=1}^{n}\frac{\xi - p_j}{1 - \overline{p}_j\xi}$$

on $\mathbb{CP}^2 \setminus \mathbb{C}^2$. Part (1) follows from

LEMMA 2.5. *Let* $Q_n(z) = a\prod_{j=1}^{n}(z - p_j\overline{z})$ *where* $a \neq 0$ *and* $p_1, \ldots, p_n \in \mathbb{C}$. *The polynomial* Q_n *is quasiregular iff for all* ξ *with* $|\xi| = 1$,

$$\sum_{j=1}^{n}\frac{1 - |p_j|^2}{|\xi - p_j|^2} > 0.$$

Proof. We have

$$Q_n(z) = a\overline{z}^n\prod_{i=1}^{n}\left(\frac{z}{\overline{z}} - p_i\right) = a\overline{z}^n w(\xi)$$

where $\xi = z/\overline{z}$. Hence

$$\frac{\frac{\partial Q_n}{\partial \overline{z}}}{\frac{\partial Q_n}{\partial z}}(z) = \frac{n\overline{z}^{n-1}w(\xi) - \overline{z}^n \cdot \frac{z}{\overline{z}^2}w'(\xi)}{\overline{z}^n \cdot \frac{1}{\overline{z}}w'(\xi)} = n\,\frac{w(\xi)}{w'(\xi)} - \xi.$$

$Q_n(z)$ is quasiregular iff

$$\left|n\,\frac{w(\xi)}{w'(\xi)} - \xi\right| < 1$$

for $|\xi| = 1$. This is equivalent to

$$\left|\frac{1}{\frac{1}{n}\frac{w'(\xi)}{w(\xi)}} - \xi\right| < 1 \;\Leftrightarrow\; \left|\frac{1}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\xi - p_i}} - \xi\right| < 1 \;\Leftrightarrow\; \left|\frac{1}{\frac{1}{n}\sum_{i=1}^{n}\frac{\xi}{\xi - p_i}} - 1\right| < 1$$

since $|\xi| = 1$. We have

$$\left|\frac{1}{\frac{1}{n}\sum_{i=1}^{n}\frac{\xi}{\xi - p_i}} - 1\right| = \left|\frac{1}{1 + \frac{1}{n}\sum\frac{p_i}{\xi - p_i}} - 1\right| = \left|\frac{\frac{1}{n}\sum_{i=1}^{n}\frac{p_i}{\xi - p_i}}{1 + \sum_{i=1}^{n}\frac{p_i}{\xi - p_i}}\right|.$$

The inequality

$$\left|\frac{\frac{1}{n}\sum_{i=1}^{n}\frac{p_i}{\xi - p_i}}{1 + \frac{1}{n}\sum_{i=1}^{n}\frac{p_i}{\xi - p_i}}\right| < 1$$

is equivalent to the inequality

$$\Re\frac{1}{n}\sum_{i=1}^{n}\frac{p_i}{\xi - p_i} > -\frac{1}{2}$$

which is equivalent to

$$\sum_{i=1}^{n}\Re\frac{\xi + p_i}{\xi - p_i} > 0 \ \Leftrightarrow\ \sum_{i=1}^{n}\frac{1-|p_i|^2}{|\xi - p_i|^2} > 0. \ ■$$

For more information on quasiregular polynomials see [Li1].

REMARK 2.6. For $n = 1$ or $n = 2$, if Q_n is quasiregular then all p_j must have modulus less than one. This is not true for $n > 2$.

If Q is a polynomial mapping such that Q^{-1} exists and is also a polynomial map then we say that Q is a *polynomial automorphism.*

We have the following

PROPOSITION 2.7. *The complexification of a polynomial automorphism can be extended to a polynomial endomorphism of* $\mathbb{CP}^2$ *iff it is an affine mapping.*

Proof. The Jung–van der Kulk theorem ([J, K]) shows that each polynomial automorphism of $\mathbb{R}^2$ is a finite superposition of a nondegenerate affine map and so-called shears, i.e., mappings

$$(x,y) \mapsto (x, y + h(x))$$

where $h(x)$ is a polynomial of one variable. The complexification of a nondegenerate affine map extends to an automorphism of $\mathbb{CP}^2$. Hence it suffices to prove our proposition for maps of the type

$$g_n \circ f_{n-1} \circ \cdots \circ f_2 \circ g_2 \circ f_1 \circ g_1$$

where the g_i are shears and the f_j are affine. If some of the g_i are not affine then the leading homogeneous term of

$$Q = g_n \circ f_{n-1} \circ \cdots \circ f_1 \circ g_1$$

has the form ax^m, $m > 1$. In complex coordinates it has the form

$$Q_n(z) = a\left(\frac{z+\overline{z}}{z}\right)^m.$$

Hence neither condition (1) nor condition (2) of Theorem 2.1 can be fulfilled and Q is not extendable to $\mathbb{CP}^2$.

Of course, we can give another proof of Proposition 2.7 by complexifying Q and Q^{-1}. Since $\{(z,\overline{z})_{z\in\mathbb{C}}\}$ is a uniqueness set for holomorphic functions, we conclude that the complexified map f is a polynomial automorphism of $\mathbb{C}^2$. Thus one can use the Jung–van der Kulk theorem to show that it is not extendable to $\mathbb{CP}^2$ unless it is affine. ■

3. Further remarks

REMARK 3.1. If the assumptions of Proposition 2.4 are fulfilled then, similarly to [Li3], we can use the vast knowledge of the dynamics of finite Blaschke products (see [C-G, H, Sh-Su]) to study the dynamics of Q_n and try to generalize Theorem 4.2 and Proposition 4.1 of [Li3].

REMARK 3.2. If condition (2) of Theorem 2.1 holds or some p_j have modulus less than one and some greater than one then, in general, nothing is known about the dynamics of ϕ on $\mathbb{CP}^2 \setminus \mathbb{C}^2$. However, one can still try to use the results of Hubbard–Papadopol [H-P] in the homogeneous case.

Nothing is known when Q is nonhomogeneous except the last theorem of [Li3].

References

[C-G] L. Carleson and T. W. Gamelin, *Complex Dynamics*, Springer, New York, 1993.

[H] D. H. Hamilton, *Absolutely continuous conjugation of Blaschke products*, (I), Adv. Math. 121 (1996), 1–20; (II), J. London Math. Soc. (2) 51 (1995), 279–285; (III), J. Anal. Math. 63 (1994), 333–343.

[H-P] J. Hubbard and J. Papadopol, *Superattractive fixed points in* $\mathbb{C}^n$, Indiana Univ. Math. J. 43 (1994), 321–365.

[J] H. W. E. Jung, *Über ganze birationale Transformationen der Ebene*, J. Reine Angew. Math. 184 (1942), 161–174.

[K] W. van der Kulk, *On polynomial rings in two variables*, Nieuw. Arch. Wisk. 1 (1953), 33–41.

[Li1] E. Ligocka, *On quasiregular polynomial mappings*, Colloq. Math. 75 (1998), 79–89.

[Li2] —, *On fixed points of holomorphic type*, ibid. 94 (2002), 61–66.

[Li3] —, *On complexification and iteration of quasiregular polynomials which have algebraic degree two*, Fund. Math. 186 (2005), 269–285.

[Sh-Su] M. Shub and D. P. Sullivan, *Expanding isomorphisms of the circle revisited*, Ergodic Theory Dynam. Systems 5 (1985), 285–289.

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