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Existence and stability of solutions for semilinear Dirichlet problems

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Abstract. We provide existence and stability results for semilinear Dirichlet problems with nonlinearities satisfying some general local growth conditions. We derive a general abstract result which we then apply to prove the existence of solutions, their stability and continuous dependence on parameters for a sixth order ODE with Dirichlet type boundary data.

1. Introduction. The aim of the paper is to investigate semilinear Dirichlet problems with nonlinearities satisfying some general growth conditions. We prove both the existence of solutions and their stability, which we then apply to show that the solution depends continuously on a functional parameter. Our results may be used in investigating certain higher order Dirichlet problems governed by semilinear ODE with nonlinearities satisfying some local growth conditions.

Higher order Dirichlet problems have recently been thoroughly investigated (see for example [6], [13]). The methods applied vary from topological to variational ones—the approach via monotone operator theory as well as critical point theory may be used. Although our method is variational in spirit, i.e. it relies on minimizing a suitable action functional, it also uses some topological argument: we prove that a certain set is invariant with respect to the inverse of a suitable differential operator. Therefore we think that our research may bring about some new ideas to the study of semilinear Dirichlet problems. We also obtain some qualitative properties of the solution in the example which we provide. Such properties depend on the growth condition assumed and of course on the construction of the set on which the action functional is minimized. It is worth stressing that we prove the continuous dependence on parameters for problems which do not necessarily have

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unique solutions. A model problem, as far as the stability and existence results are concerned, which is covered by our methods is the following:

We consider the following Dirichlet problems for $k = 0, 1, 2, \ldots$

$$(1.1)\qquad \begin{gathered} -\frac{d^6}{dt^6}x + \frac{d^4}{dt^4}x - \frac{d^2}{dt^2}x - 2x = \nabla F_k(t,x), \\ x(0) = x(\pi) = \dot{x}(0) = \dot{x}(\pi) = \ddot{x}(0) = \ddot{x}(\pi) = 0, \end{gathered}$$

where we assume:

(F1) There exist numbers $d > d_0 > 0$ and $0 \le d_k \le d_0$ for all $k = 1, 2, \ldots$ such that $\nabla F_k(t, d_k), \nabla F_k(t, -d_k), \nabla F_k(t, d), \nabla F_k(t, -d) \in L^\infty(0, T)$ for all $k = 0, 1, \ldots$.

(F2) For all $k = 0, 1, \ldots,$ $F_k, \nabla F_k : [0,T] \times [-d, d]$ are Carathéodory functions, F_k is continuously differentiable and convex with respect to the second variable in $[-d, d]$ for a.e. $t \in [0,T]$ and equals $+\infty$ outside $[0,T] \times [-d, d]$ and

$$\operatorname*{ess\,sup}_{t\in[0,T]} |\nabla F_k(t,d)| \le \sqrt{\frac{12}{\pi}}\, d, \qquad \operatorname*{ess\,sup}_{t\in[0,T]} |\nabla F_k(t,-d)| \le \sqrt{\frac{12}{\pi}}\, d.$$

(F3) For all $k = 0, 1, \ldots,$ $\nabla F_k(t, 0) \neq 0$ for a.e. $t \in [0,T]$, and $t \mapsto F_k(t,0)$ and $t \mapsto F_k^*(t,0)$ are integrable.

Here F_k^* denotes the Fenchel–Young conjugate of a convex l.s.c. function F_k. With the above assumptions we may prove for example

Theorem 1.1. *Assume that* (F1)–(F3) *hold and that* $\nabla F_k(\cdot, x(\cdot)) \rightharpoonup \nabla F_k(\cdot, x(\cdot))$ *in* $L^2(0,\pi)$ *for all* $x \in H_0^2(0,\pi) \cap H^3(0,\pi)$ *such that*

$$\begin{gathered} x(t) \in [-d_0, d_0] \quad \text{a.e. on } [0,\pi], \\ \left\| \frac{d^3}{dt^3} x \right\|_{L^2}^2 \le \sqrt{\frac{12}{\pi}} d, \\ -\frac{d^6}{dt^6}x + \frac{d^4}{dt^4}x - \frac{d^2}{dt^2}x - 2x \in L^\infty(0,T). \end{gathered}$$

Then for each $k = 1, 2, \ldots$ *there exists a solution* x_k *to the problem* (1.1) *and there exists a subsequence* $\{x_{k_i}\}_{i=1}^\infty$ *of* $\{x_k\}_{k=1}^\infty$ *such that* $\lim_{i\to\infty} x_{k_i} = \overline{x}$ *strongly in* $L^2(0,\pi)$ *and*

$$-\frac{d^6}{dt^6}\overline{x}(t) + \frac{d^4}{dt^4}\overline{x}(t) - \frac{d^2}{dt^2}\overline{x}(t) - 2\overline{x}(t) = \nabla F_0(t, \overline{x}(t)).$$

We will also show some qualitative properties of $\overline{x}$ and x_k.

In order to tackle the above problem and similar ones we shall investigate the existence and stability of solutions to the family of abstract Dirichlet problems

$$(1.2)\qquad Lx = \nabla F_k(x),$$

where $k = 0, 1, 2, \ldots$, and L is defined on a separable real Hilbert space $D(L)$ with values in a separable real Hilbert space Y with scalar product $\langle \cdot, \cdot \rangle$. We assume

(A1) $D(L)$ is dense in Y; L is a selfadjoint and positive definite linear operator, i.e. there exists a constant $\alpha > 0$ such that for all $x \in D(L)$,

$$\langle Lx, x \rangle \geq \alpha \|x\|^2. \tag{1.3}$$

From (1.3) it follows that $R(L) = Y$ and the inverse operator $L^{-1} : Y \to D(L)$ is continuous and selfadjoint. By the properties of L it follows (see [5]) that there exists (exactly one) operator S, called the square root operator, which is selfadjoint and such that $S^2 = L$. The domain of S, denoted by $D(S)$, is dense in Y and $D(L)$ is dense in $D(S)$ (see [5]). Moreover $Sx \in D(S)$ for any $x \in D(L)$. Usually $D(S)$ is endowed with the graph norm which makes it into a complete space, but we shall rather use an equivalent norm

$$\|x\|_{D(S)} = \|Sx\|_Y.$$

Now we state the assumption on the right hand side of the equation considered and an additional assumption which is usually satisfied in concrete applications.

(A2) $\nabla F_k : Y \to Y$ is a gradient mapping, $F_k(0) < \infty$, $\nabla F_k(0) \neq 0$.

(A3) $D(S)$ is compactly imbedded in Y.

We do not assume yet any growth conditions and convexity conditions on F_k. These conditions are actually hidden in the definition of the set X_k below (see Section 2).

Due to the method which we will apply, we will investigate for each $k = 0, 1, 2, \ldots$ the following system:

$$Sx = p, \qquad Sp = \nabla F_k(x), \tag{1.4}$$

where $x \in D(L)$ and $p \in D(S)$. Such a pair, if it exists, will be called a solution to (1.2). Necessary conditions for the existence of a solution to (1.4) are obtained by duality results. The existence is a consequence of a modification of the well known Weierstrass theorem.

The abstract variational principle which we derive is based on a method from [9] and enables one to consider linear differential equations of even order, especially higher order equations. The main difference with some known abstract variational principles (see for example [7], [8]) is that the growth assumptions are superlinear. We do not require the gradient mapping ∇F_k to be continuous and convex on the whole space, while such assumptions must be made in [2] where a method that applies for superlinear problems and uses some similar ideas has been derived. Thus the theory developed applies to a wider class of problems and also allows one to obtain stability results (see Section 4) without using spectral theory as in [3]. This is possible since instead of perturbing the primal action functional as in [2], we perturb the dual one. In consequence, the stability of solutions follows under some mild additional assumption. As applications we exhibit growth type conditions for which we get both stability and existence of solution without assuming any additional convergence of the family of nonlinear terms, which was necessary in [3], [4], [9], [11], [12].

For sublinear Dirichlet problems the question of stability of solutions in case the solution is not unique was considered for the first time in [11], [12]. Later in [10] a dual variational method from [9] was used and some continuous dependence on parameters results were given for a specific type of nonlinearity. In case F_k satisfies quadratic growth conditions a problem similar to ours has been considered in [4], but unlike the case we investigate, L need not be positive definite there. For a superlinear problem the method from [4] does not work since in this case both the action and dual action functionals are unbounded. Thus we believe that our variational method may contribute to this research. Some results concerning stability for superlinear nonlinearities were obtained in [3] by using a variational method from [2]. But the assumptions were much more restrictive and the growth conditions were assumed to hold globally (cf. Section 5). The variational method from [2] required the gradient mappings to be continuous, and the constructions of the sets X_k were less general.

2. Duality results and necessary conditions. For each $k = 1, 2, \dots$ we assume that

(A4) There exists a nonempty set $X_k \subset S(L)$ such that for each $x \in X_k$ the relation

$$L\widetilde{x} = \nabla F_k(x) \tag{2.1}$$

implies that $\widetilde{x} \in X_k$ and the sets X_k, $\nabla F_k(X_k)$ are relatively weakly compact in $D(S)$ and Y, respectively.

It follows easily that $X_k \subset L^{-1}\nabla F_k(X_k)$.

We now can make a convexity assumption:

(A5) F_k is convex and lower semicontinuous on $\operatorname{conv}(X_k)$ (the convex hull of X_k) for all $k = 0, 1, 2, \dots$.

Hence we may define a convex and l.s.c. functional on the whole space:

$$G_k(x) = \begin{cases} F_k(x) & \text{for } x \in \operatorname{conv}(X_k), \\ +\infty & \text{otherwise.} \end{cases}$$

We put

$$X_k^d = S(X_k).$$

Of course, $X_k^d \subset S(D(L))$ and X_k^d is nonempty and relatively weakly compact in Y. Since $F_k = G_k$ for all $x \in X_k$, we may consider the equation

$$Lx = \nabla G_k(x)$$

instead of (1.2) on X_k.

We assume throughout this section that (A1)–(A5) are satisfied.

Due to the above remarks, the action functional $J_k : D(S) \to \mathbb{R}$ for which (1.2) is the Euler–Lagrange equation reads

$$J_k(x) = \tfrac{1}{2}\langle Sx, Sx\rangle - G_k(x).$$

and the dual functional $J_{D_k} : D(S) \to \mathbb{R}$ is given by the formula

$$J_{D_k}(p) = G_k^*(Sp) - \tfrac{1}{2}\langle p, p\rangle,$$

where $G_k^* : Y \to \mathbb{R}$ denotes the Fenchel–Young conjugate of $G_k : Y \to \mathbb{R}$ (see [1]). J_k will be considered on X_k, and J_{D_k} on X_k^d. It is obvious that J_{D_k} is different from the functional $J_D^k(p) = F_k^*(Sp) - \frac{1}{2}\langle p, p\rangle$, but they coincide on X_k^d.

Now we relate the critical values on X_k and X_k^d to J_k and J_{D_k} respectively.

Theorem 2.1.

$$\inf_{x\in X_k} J_k(x) = \inf_{p\in X_k^d} J_{D_k}(p).$$

Proof. Define the perturbation $J_{D_{k,p}} : X_k^d \times Y \to \mathbb{R}$ of J_{D_k} by

$$J_{D_{k,p}}(p,c) = \tfrac{1}{2}\langle p+c, p+c\rangle - G_k^*(Sp).$$

Now we define a type of conjugate of $J_{D_{k,p}}$ with respect to c:

$$J^{\#}_{D_{k,p}}(p,x) = \sup_{c\in Y}\big\{\langle c, Sx\rangle - \tfrac{1}{2}\langle p+c, p+c\rangle\big\} + G_k^*(Sp).$$

By the properties of Fenchel–Young duality [1], we have

$$J^{\#}_{D_{k,p}}(p,x) = G_k^*(Sp) + \tfrac{1}{2}\langle Sx, Sx\rangle - \langle Sp, x\rangle.$$

We observe that for any $p \in X_k^d$,

$$\inf_{x\in X_k} J^{\#}_{D_{k,p}}(p,x) = J_{D_k}(p). \tag{2.2}$$

Indeed, fix $p \in X_k^d$. For a given $p \in X_k^d$ there exists $x_p \in X_k$ satisfying $Sx_p = p$. We then have $\langle x_p, Sp\rangle - \frac{1}{2}\langle Sx_p, Sx_p\rangle = \frac{1}{2}\langle p, p\rangle$ and since $X_k \subset D(L) \subset Y$,

$$\begin{aligned}\tfrac{1}{2}\langle p, p\rangle = \langle x_p, Sp\rangle - \tfrac{1}{2}\langle Sx_p, Sx_p\rangle &\le \sup_{x\in X_k}\big\{\langle x, Sp\rangle - \tfrac{1}{2}\langle Sx, Sx\rangle\big\}\\ &\le \sup_{v\in Y}\big\{\langle v, p\rangle - \tfrac{1}{2}\langle v, v\rangle\big\} \le \tfrac{1}{2}\langle p, p\rangle\end{aligned}$$

Therefore we have (2.2).

Now we observe that for any $x \in X_k$,

$$\inf_{p\in X_k^d} J^{\#}_{D_{k,p}}(p,x) = J_k(x). \tag{2.3}$$

Indeed, again fix $x \in X_k$. By definition of X_k^d there exists $p_x \in X_k^d$ such that $S\widetilde{x} = p_x$, where $\widetilde{x} \in X_k$ is such that $L\widetilde{x} = \nabla G_k(x)$. It follows that $Sp_x = \nabla G_k(x)$ and by the properties of the Fenchel–Young conjugate we get $G_k(x) + G_k^*(p_x) = \langle x, p_x\rangle$. In consequence

$$\langle x, Sp_x\rangle - G_k^*(Sp_x) - \tfrac{1}{2}\langle Sx, Sx\rangle = G_k(x) - \tfrac{1}{2}\langle Sx, Sx\rangle = -J_k(x).$$

and (2.3) follows.

Now by (2.2) and (2.3) we get

$$\inf_{p\in X_k^d} J_{D_k}(p) = \inf_{p\in X_k^d}\inf_{x\in X_k} J^{\#}_{D_{k,p}}(p,x) = \inf_{x\in X_k}\inf_{p\in X_k^d} J^{\#}_{D_{k,p}}(p,x) = \inf_{x\in X_k} J_k(x). \ \blacksquare$$

We use the results of duality theory to derive the variational principle providing necessary conditions for the existence of a solution to equation (1.2).

THEOREM 2.2. *Assume that* $p_k \in X_k^d$ *is such that* $J_{D_k}(p_k) = \inf_{p\in X_k^d} J_{D_k}(p)$. *Then there exists* $x_k \in X_k$ *such that*

$$Sx_k = p_k, \tag{2.4}$$

$$Sp_k = \nabla G_k(x_k). \tag{2.5}$$

Moreover,

$$\inf_{p\in X_k^d} J_{D_k}(p) = J_{D_k}(p_k) = J_k(x_k) = \inf_{x\in X_k} J_k(x). \tag{2.6}$$

Proof. For $p_k \in X_k^d$ there exists $x_k \in X_k^d$ such that $Sx_k = p_k$. So (2.4) holds. Hence and by the Fenchel–Young inequality we obtain

$$\begin{aligned}-J_{D_k}(p_k) &= \tfrac{1}{2}\langle p_k, p_k\rangle - G_k^*(Sp_k) = \langle p_k, Sx_k\rangle - \tfrac{1}{2}\langle Sx_k, Sx_k\rangle - G_k^*(Sp_k)\\ &\le -\tfrac{1}{2}\langle Sx_k, Sx_k\rangle + G_k(x_k) = -J_k(x_k).\end{aligned}$$

So $J_{D_k}(p_k) \ge J_k(x_k)$. By Theorem 2.1 it follows that $J_{D_k}(p_k) \le J_k(x_k)$. In consequence, $G_k(x_k) + G_k^*(Sp_k) - \langle Sp_k, x_k\rangle = 0$. By standard convexity arguments we now get (2.5).

Relation (2.6) follows by Theorem 2.1, since $J_{D_k}(p_k) = J_k(x_k)$. ∎

We shall show that the above results with suitable modifications are valid for minimizing sequences. These will be used in the proof of the existence theorem.

THEOREM 2.3. *Let* $\{p_k^j\}_{j=1}^\infty \subset X_k^d$ *be a minimizing sequence for* J_{D_k}. *There exists a sequence* $\{x_k^j\}_{j=1}^\infty \subset X_k$ *minimizing for* J *and such that*

$$x_k^j = S^{-1}p_k^j \tag{2.7}$$

for $j \in \mathbb{N}$. Furthermore

$$\inf_{p\in X_k^d} J_{D_k}(p) = \inf_{j\in\mathbb{N}} J_{D_k}(p_k^j) = \inf_{x\in X_k} J(x) = \inf_{j\in\mathbb{N}} J(x_k^j).$$

Moreover for any $\varepsilon > 0$ there exists j_0 such that for all $j \geq j_0$,

$$0 \leq G_k(x_k^j) + G_k^*(Sp_k^j) - \langle x_k^j, Sp_k^j\rangle \leq \varepsilon. \tag{2.8}$$

Proof. Since $p_k^j \in X_k^d$ for $j \in \mathbb{N}$, there exists $x_k^j \in X_k$ such that (2.7) holds. We will show that $\{x_k^j\}_{j=1}^\infty$ is a minimizing sequence for J_{D_k}. Reasoning as in the proof of Theorem 2.2 we get by (2.7) and the Fenchel–Young inequality, for any $j \in \mathbb{N}$,

$$J_{D_k}(p_k^j) \geq J_k(x_k^j). \tag{2.9}$$

Take arbitrary $\varepsilon > 0$. Since $-\infty < \inf_{j\in\mathbb{N}} J_{D_k}(p_k^j) = a < \infty$, there exists j_0 such that $J_{D_k}(p_k^j) < a+\varepsilon$ for all $j \geq j_0$. By (2.9) it follows that $J_k(x_k^j) < a+\varepsilon$ for $j \geq j_0$. Now Theorem 2.1 yields $\inf_{j\in\mathbb{N}} J_k(x_k^j) = a$. In consequence, $\{x_k^j\}_{j=1}^\infty$ is a minimizing sequence for J_k.

Relation (2.8) follows from Theorem 2.1. Indeed, for each $\varepsilon > 0$ there exists j_0 such that for all $j \geq j_0$,

$$J_{D_k}(p_k^j) < a + \varepsilon = \inf_{l\in\mathbb{N}} J_{D_k}(p_l) + \varepsilon = \inf_{l\in\mathbb{N}} J_k(x_l) + \varepsilon \leq J_k(x_k^j) + \varepsilon.$$

Using (2.9) we get (2.8). ∎

3. Existence of solutions

THEOREM 3.1. *There exists a pair $(x_k, p_k) \in D(L) \times S(D(L))$ such that*

$$Sx_k = p_k, \tag{3.1}$$

$$Sp_k = \nabla G_k(x_k), \tag{3.2}$$

$$\inf_{p\in X_k^d} J_{D_k}(p) = J_{D_k}(p_k) = J_k(x_k) = \inf_{x\in X_k} J_k(x). \tag{3.3}$$

Moreover, x_k is the limit (weak in $D(S)$) of a minimizing sequence $\{x_k^j\}_{j=1}^\infty$ for the restriction of J_k to X_k, and p_k is the limit (weak in $D(S)$) of a minimizing sequence $\{p_k^j\}_{j=1}^\infty$ for the restriction of J_{D_k} to X_k^d.

Proof. By the Fenchel–Young inequality and by definition of X_k^d it follows that J_{D_k} is bounded from below on X_k^d and thus we can choose a minimizing sequence p_k^j which may be assumed (by (A4) and the definition of X_k) to be weakly convergent in $D(S)$ to a certain p_k, and therefore by (A3) strongly convergent in Y. By Theorem 2.3 we can choose a minimizing sequence $\{x_k^j\}_{j=1}^\infty$ satisfying

$$Sx_k^j = p_k^j \tag{3.4}$$

and, up to a subsequence, weakly convergent in $D(S)$ and strongly convergent in Y. We denote its limit by $x_k \in D(S)$. Since S^{-1} is continuous, by (3.4) we get $\lim_{j\to\infty} S^{-1}p_k^j = S^{-1}p_k = x_k$ strongly in Y. Hence we have (3.1).

We observe that

$$J_k(p_k) = \inf_{p\in X_k^d} J_{D_k}(p). \tag{3.5}$$

Indeed, G_k^* being convex is lower semicontinuous, so $\liminf_{j\to\infty} G_k^*(Sp_k^j) \geq G_k^*(Sp_k)$. Since p_k^j is strongly convergent in Y, we get $\lim_{j\to\infty} \frac{1}{2}\langle p_k^j, p_k^j\rangle = \frac{1}{2}\langle p_k, p_k\rangle$. Hence J_{D_k} is weakly lower semicontinuous on $D(S)$ and (3.5) follows.

We now show that (p_k, x_k) also satisfies (3.2). By Theorem 2.3 (relation (2.8)), there exists a numerical sequence $\{\varepsilon_n\}_{n=1}^{\infty}$, $\varepsilon_n > 0$, $\varepsilon_n \to 0$, having the property: for each ε_n there exists j_n such that $0 \leq G_k(x_k^j) + G_k^*(Sp_k^j) - \langle x_k^j, Sp_k^j\rangle \leq \varepsilon$ for all $j \geq j_n$. We may assume that $j_n \to \infty$ as $n \to \infty$. Therefore we obtain

$$\begin{aligned} 0 &\geq \liminf_{j\to\infty}(G_k(x_k^j) + G_k^*(Sp_k^j) - \langle x_k^j, Sp_k^j\rangle) \\ &\geq \liminf_{j\to\infty} G_k(x_k^j) + \liminf_{j\to\infty} G_k^*(Sp_k^j) - \lim_{j\to\infty} \langle x_k^j, Sp_k^j\rangle \\ &\geq G_k(x_k) + G_k^*(Sp_k) - \langle x_k, Sp_k\rangle. \end{aligned}$$

From the above and the Fenchel–Young inequality we obtain $G_k(x_k) + G_k^*(Sp_k) - \langle x_k, Sp_k\rangle = 0$. Hence (3.2) follows by convexity arguments. By (3.5) and by Theorem 2.1, relation (3.3) follows. ∎

Now we get the following

COROLLARY 3.2. *Let X_k be weakly compact. There exists a pair $(x_k, p_k) \in X_k \times X_k^d$ such that*

$$Sx_k = p_k, \qquad Sp_k = \nabla G_k(x_k),$$
$$\inf_{p\in X_k^d} J_{D_k}(p) = J_{D_k}(p_k) = J_k(x_k) = \inf_{x\in X_k} J_k(x).$$

Moreover, x_k is the limit (weak in $D(S)$) of a minimizing sequence $\{x_k^j\}_{j=1}^{\infty}$ for the restriction of J_k to X_k, and p_k is the limit (weak in $D(S)$) of a minimizing sequence $\{p_k^j\}_{j=1}^{\infty}$ for the restriction of J_{D_k} to X_k^d.

4. Stability result. We assume (A1)–(A4) and

(A6) There exists a weakly compact convex set $B \subset Y$ such that $X_k \subset B$ and ∇G_k is uniformly bounded on B.

(A7) F_k is convex and lower semicontinuous on B for each $k = 0, 1, 2, \ldots$.d

Since all sets X_k are now relatively weakly compact, this assumption is not very restrictive. We efine as before a convex and l.s.c. functional on the whole space for $k = 0, 1, 2, \dots$ by

$$G_k(x) = \begin{cases} F_k(x) & \text{for } x \in B, \\ +\infty & \text{otherwise.} \end{cases}$$

By *stability* we mean conditions under which from a sequence $\{x_k\}_{k=1}^{\infty}$, where x_k for $k = 1, 2, \dots$ is a solution to (1.2), one may choose a subsequence converging weakly to a certain $\overline{x}$ which is a solution to the problem $L\overline{x} = \nabla G_0(\overline{x})$. Here we mean that $\lim_{k\to\infty} x_k = \overline{x}$ weakly in $D(S)$ and $\lim_{k\to\infty} \nabla G_k(x) = \nabla G_0(x)$ weakly in Y for any $x \in B$, up to subsequences.

THEOREM 4.1. *Assume* (A1)–(A4), (A6), (A7) *and that for any $x \in B$ there is a subsequence k_j such that*

$$\lim_{j\to\infty} \nabla G_{k_j}(x) = \nabla G_0(x)$$

weakly in Y. Then for each $k = 0, 1, 2, \dots$ there exists a solution x_k to (1.2), *and there exists a subsequence $\{x_{k_i}\}_{i=1}^{\infty}$ of $\{x_k\}_{k=1}^{\infty}$ and $\overline{x} \in D(L)$ such that*

$$\lim_{i\to\infty} x_{k_i} = \overline{x} \quad \textit{weakly in } D(S), \textit{ strongly in } Y.$$

Moreover

$$L\overline{x} = \nabla G_0(\overline{x}).$$

Proof. From Theorem 3.1 it follows that for each $k = 1, 2, \dots$ there exists a pair $(x_k, p_k) \in D(L) \times S(D(L))$ such that

$$Sx_k = p_k, \quad Sp_k = \nabla G_k(x_k). \tag{4.1}$$

Due to assumption (A6) we may choose from $\{x_k\}_{k=1}^{\infty}$ a subsequence weakly converging in Y which we still denote by $\{x_k\}_{k=1}^{\infty}$. Now by (4.1) and by the boundedness of ∇G_k (see (A7)) it follows that $\{x_k\}_{k=1}^{\infty}$ is, up to subsequence, strongly convergent in $D(S)$ to a certain $\overline{x}$ by (A3). The sequence $\{p_k\}_{k=1}^{\infty}$ is, up to a subsequence, weakly convergent in $D(S)$ and strongly convergent in Y. We denote its limit by $\overline{p}$. In the following we denote all the resulting subsequences by the subscript k_i for simplicity. Take a subsequence $\{k_i\}_{i=1}^{\infty}$ such that $\lim_{i\to\infty} \nabla G_{k_i}(\overline{x}) = \nabla G_0(\overline{x})$ weakly.

We will now prove that

$$S\overline{x} = \nabla G_0(\overline{x}).$$

By convexity of G_{k_i} we get, for any $x \in Y$,

$$\langle \nabla G_{k_i}(x_{k_i}) - \nabla G_{k_i}(x), x_{k_i} - x \rangle \geq 0.$$

Hence by Theorem 3.1,

$$\langle Lx_{k_i} + (\nabla G_0(x) - \nabla G_{k_i}(x)) - \nabla G_0(x), x_{k_i} - x \rangle \geq 0.$$

Since $x_{k_i} \to \overline{x}$ strongly in Y and $\nabla G_{k_i}(x) \rightharpoonup \nabla G_0(x)$ weakly in Y we easily obtain

$$\langle(\nabla G_0(x) - \nabla G_{k_i}(x)) - \nabla G_0(x), x_{k_i} - x\rangle \to \langle -\nabla G_0(x), \overline{x} - x\rangle.$$

Moreover $\langle Lx_{k_i}, -x\rangle \to \langle L\overline{x}, -x\rangle$ since L is selfadjoint. It remains to observe that $\langle Lx_{k_i}, x_{k_i}\rangle = \langle Sp_{k_i}, x_{k_i}\rangle \to \langle S\overline{p}, \overline{x}\rangle$. Hence

$$\langle S\overline{p} - \nabla G_0(x), \overline{x} - x\rangle \geq 0 \tag{4.2}$$

for any $x \in D(L)$.

Now we apply the "Minty trick", i.e. we consider the points $\overline{x} + tx$, where $x \in D(L)$ and $t > 0$. By the above inequality we obtain

$$\langle S\overline{p} - \nabla G_0(\overline{x} + tx), x\rangle \leq 0.$$

Since the function $[-1,1] \ni t \mapsto G_0(\overline{x} + tx) \in \mathbb{R}$ is convex, its derivative $[-1,1] \ni t \mapsto \langle \nabla G_0(\overline{x} + tx), x\rangle \in \mathbb{R}$ is continuous. Hence

$$0 \geq \lim_{t\to 0}\langle S\overline{p} - \nabla G_0(\overline{x} + tx), x\rangle = \langle S\overline{p} - \nabla G_0(\overline{x}), x\rangle$$

for any $x \in D(L)$. As $D(L)$ is dense in Y, this means that $S\overline{p} = \nabla G_0(\overline{x})$.

We need to prove that $S\overline{x} = \overline{p}$. We again apply the Minty trick. Obviously $\langle Lx_{k_i} - Lx, x_{k_i} - x\rangle \geq 0$ for any $x \in D(L)$. Moreover $\langle Sp_{k_i} - Lx, x_{k_i} - x\rangle \geq 0$ and taking the limit we get

$$\langle S\overline{p} - Lx, \overline{x} - x\rangle \geq 0.$$

Now by considering points $\overline{x} + tx$ for any $x \in D(L)$ we obtain $-\langle S\overline{p} - L\overline{x}, x\rangle - t\langle Lx, x\rangle \geq 0$. Hence taking the limit as $t \to 0$ we get

$$\langle S\overline{p} - L\overline{x}, x\rangle \leq 0$$

for any $x \in D(L)$. Thus $S\overline{p} = L\overline{x}$ and the proof is finished. ■

5. Applications. In this section we give some applications to concrete problems. We shall check each time that (A1)–(A4) and (A6)–(A7) are satisfied.

5.1. *Existence and stability of solutions for a sixth order Dirichlet problem.* We consider the problem defined in the Introduction, i.e. the family of Dirichlet problems for $k = 0, 1, 2, \ldots$,

$$\begin{gathered} -\frac{d^6}{dt^6}x + \frac{d^4}{dt^4}x - \frac{d^2}{dt^2}x - 2x = \nabla F_k(t,x), \\ x(0) = x(\pi) = \dot{x}(0) = \dot{x}(\pi) = \ddot{x}(0) = \ddot{x}(\pi) = 0, \end{gathered} \tag{5.1}$$

where we define $L = -\frac{d^6}{dt^6}x + \frac{d^4}{dt^4}x - \frac{d^2}{dt^2}x + 2x$ and assume (F1)–(F3).

We observe that each F_k is convex and l.s.c.

THEOREM 5.1. *Under assumptions* (F1)–(F3), *for any* $k = 0, 1, 2, \ldots$ *there exists a solution to the Dirichlet problem* (5.1).

Proof. Fix k. We shall show that all assumptions of Theorem 3.1 are satisfied. First of all we observe that (A1) and (A3) hold by definition of $L^2(0,\pi)$ and by the Poincaré inequality. (A2) follows by the assumptions on F_k. To show (A4) we need to construct a suitable set X_k. We define

$$\overline{X}_k = \Big\{x \in H_0^2(0,\pi) \cap H^3(0,\pi) : x(t) \in [-d_k, d_k] \text{ a.e. on } [0,\pi],$$
$$\Big\|\frac{d^3}{dt^3}x\Big\|_{L^2}^2 \le \sqrt{\frac{12}{\pi}}\, d \text{ and } -\frac{d^6}{dt^6}x + \frac{d^4}{dt^4}x - \frac{d^2}{dt^2}x - 2x \in L^\infty(0,T)\Big\}.$$

We take any $u \in \overline{X}_k$. A solution to

$$\begin{aligned} -\frac{d^6}{dt^6}x + \frac{d^4}{dt^4}x - \frac{d^2}{dt^2}x - 2x &= \nabla F_k(t,u), \\ x(0) = x(\pi) = \dot{x}(0) = \dot{x}(\pi) = \ddot{x}(0) &= \ddot{x}(\pi) = 0, \end{aligned} \tag{5.2}$$

obviously exists. Moreover by a direct calculation using the Poincaré inequality and the fact that the derivative of a convex function is nondecreasing we get

$$\begin{aligned} \int_0^\pi \Big|\frac{d^3}{dt^3}\,x\Big|^2 dt &\le \int_0^\pi \Big(-\frac{d^6}{dt^6}x(t) + \frac{d^4}{dt^4}x(t) - \frac{d^2}{dt^2}x(t) - 2x(t)\Big)x(t)\,dt \\ &= \int_0^\pi \nabla F_k(t,u(t))x(t)\,dt \le \operatorname*{ess\,sup}_{t\in[0,T]} |\nabla F_k(t,d)| \int_0^\pi \Big|\frac{d^3}{dt^3}x\Big|^2 dt. \end{aligned}$$

Hence

$$\Big\|\frac{d^3}{dt^3}x\Big\|_{L^2}^2 \le \sqrt{\frac{12}{\pi}}\, d.$$

So by Sobolev's inequality we get

$$|x(t)| \le \max_{s\in[0,\pi]} |x(s)| \le \sqrt{\frac{\pi}{12}}\,\Big\|\frac{d^3}{dt^3}x\Big\|_{L^2}^2 \le d.$$

Thus $x \in \overline{X}_k$ and we may put $X_k = \overline{X}_k$. Of course X_k and $F_k(X_k)$ are relatively weakly compact in $H^3(0,\pi)$ and $L^2(0,\pi)$. Hence all the assumptions of Theorem 3.1 are satisfied and we infer the existence of solutions to the Dirichlet problem (5.1). ∎

The set B (see (A6)) may due to (F1) be given by $B = X_0$. Therefore (A6), (A7) are satisfied. Now by Theorem 4.1 we easily obtain Theorem 1.1.

5.2. *Dependence on parameters.* We now consider a similar problem but concentrate on the continuous dependence on parameters for the Dirichlet problem

$$(5.3)\qquad -\frac{d^6}{dt^6}x + \frac{d^4}{dt^4}x - \frac{d^2}{dt^2}x - 2x = \nabla F(t, x(t), u(t)),$$
$$x(0) = x(\pi) = \dot{x}(0) = \dot{x}(\pi) = \ddot{x}(0) = \ddot{x}(\pi) = 0,$$

where $u : [0,\pi] \to \mathbb{R}^m$ is a functional parameter from the set

$$L_M = \{u : [0,\pi] \to \mathbb{R}^m : u \text{ is measurable, } u(t) \in M \text{ a.e.}\}$$

and $M \subset \mathbb{R}^m$ is a given compact set. We also assume:

(Fp1) There exist numbers $d \le d_0$ such that

$$\nabla F(t,d,u), \nabla F(t,-d,u), \nabla F(t,d_0,u), \nabla F(t,-d_0,u) \in L^\infty(0,T)$$

for all $u \in L_M$.

(Fp2) $F, \nabla F : [0,T] \times [-d,d] \times M$ are Carathéodory functions, F is continuously differentiable and convex with respect to the second variable in $[-d_0,d_0]$ for a.e. $t \in [0,T]$ and equals $+\infty$ outside $[0,T] \times [-d_0,d_0]$; for all $u \in L_M$,

$$\operatorname*{ess\,sup}_{t\in[0,T]} |\nabla F(t,d,u)| \le \sqrt{\frac{12}{\pi}}\, d, \quad \operatorname*{ess\,sup}_{t\in[0,T]} |\nabla F(t,-d,u)| \le \sqrt{\frac{12}{\pi}}\, d.$$

(Fp3) $\nabla F(t,0,u) \ne 0$ for a.e. $t \in [0,T]$ and all $u \in L_M$; $t \mapsto F(t,0,u)$ and $t \mapsto F^*(t,0,u)$ are integrable for all $u \in L_M$.

We have the following theorem which is a direct consequence of Theorem 4.1.

THEOREM 5.2. *Assume that* (Fp1)–(Fp3) *hold and that* $\{u_k\}_{k=1}^\infty \subset L_M$ *with* $u_k \to \overline{u}$ *in* $L^2(0,\pi)$. *For each* $k = 1,2,\ldots$ *there exists a solution* x_k *to problem* (5.3) *and there exists a subsequence* $\{x_{k_i}\}_{i=1}^\infty$ *of* $\{x_k\}_{k=1}^\infty$ *and* $\overline{x} \in X_0$ *such that* $\lim_{i\to\infty} x_{k_i} = \overline{x}$ *strongly in* $L^2(0,\pi)$ *and*

$$-\frac{d^6}{dt^6}\overline{x}(t) + \frac{d^4}{dt^4}\overline{x}(t) - \frac{d^2}{dt^2}\overline{x}(t) - 2\overline{x}(t) = \nabla F(t, \overline{x}(t), \overline{u}(t)).$$

Proof. By the Krasnosel'skiĭ theorem we get

$$\nabla F(\cdot, x(\cdot), u_k(\cdot)) \underset{k\to\infty}{\to} \nabla F(\cdot, x(\cdot), \overline{u}(\cdot))$$

strongly in $L^2(0,\pi)$. Hence Theorem 4.1 applies with

$$F_k(\cdot, x(\cdot)) = F(\cdot, x(\cdot), u_k(\cdot)). \ \blacksquare$$

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