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Existence of solutions of a second order abstract functional Cauchy problem with nonlocal conditions

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Abstract. We establish the existence of mild, strong, classical solutions for a class of second order abstract functional differential equations with nonlocal conditions.

1. Preliminaries. The general purpose of this paper is to establish some results on existence of mild, strong, classical solutions for a class of second order functional differential equations.

Let X be a Banach space endowed with a norm $\|\cdot\|$. Throughout this paper we will write I for the interval $[0,T]$, for some fixed $T>0$. We let $C(I;X)$ denote the space of continuous functions from I into X endowed with the norm of uniform convergence. Let A be the infinitesimal generator of a strongly cosine function of bounded linear operators $C(t)$ on X. We are interested in the initial value problems of the form

$$x''(t) = Ax(t) + f(t, x(t), x(a(t)), x'(t), x'(b(t))), \quad t \in I, \tag{1.1}$$

$$x(0) + p(x, x') = x_0, \tag{1.2}$$

$$x'(0) + q(x, x') = x_1, \tag{1.3}$$

where $a, b : I \to I$, $p, q : C(I;X)^2 \to X$ and $f : I \times X^4 \to X$ are appropriate functions.

The study of initial value problems with nonlocal conditions is motivated by some situations in physics. For the importance of nonlocal conditions in different fields we refer to [1, 3] and the references therein. Specifically, the development of the theory in the abstract framework was initiated by Byszewski in [1–3]. In [1], Byszewski establishes the existence of mild, strong,

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classical solutions of the semilinear nonlocal Cauchy problem

$$x'(t) = Ax(t) + f(t, x(t)), \qquad t \in I,$$
$$x(0) = x_0 + q(t_1, \ldots, t_n, x(\cdot)) \in X,$$

where A is the infinitesimal generator of a strongly continuous semigroup of linear operators on X, $f : [0,T] \times X \to X$ and $q : I^n \times C(I; X) \to X$ are appropriate functions, and $0 < t_1 < \cdots < t_n \leq T$. Here the symbol $q(t_1, \ldots, t_n, u(\cdot))$ indicates that the function $u(\cdot)$ is evaluated only at the points t_i, for example $q(t_1, \ldots, t_n, u(\cdot)) = \sum_{i=1}^{n} \alpha_i u(t_i)$.

On the other hand, some second order nonlocal initial value problems have been studied by Ntouyas & Tsamatos in [11–13]. In these works the authors discuss the existence of solutions of a type of second order delay integrodifferential equations with nonlocal conditions of the form

$$\begin{aligned} x''(t) &= Ax(t) \\ &\quad + f\Big(t, x(\sigma_1(t)), \int_0^t k(t-s)h(s, x(\sigma_2(s)), x'(\sigma_3(s)))\, ds, x(\sigma_4(t))\Big), \\ x(0) &= g(x) + x_0, \qquad x'(0) = \eta, \end{aligned}$$

for $t \in (0, T)$, where A is the generator of a strongly continuous cosine function C of bounded linear operators on X, $g : C(I; X) \to X$, $f : I \times X^3 \to X$ and $h : I \times X^2 \to X$ are appropriate functions, and $x_0, \eta \in X$. The results are obtained using the Leray–Schauder alternative and the strong assumption that the cosine function $C(t)$ is compact on $(0, \infty)$.

The specific object of this paper is to present some results on existence of solutions for problem (1.1)–(1.3), without the hypotheses mentioned previously. Throughout this work $C(t)$ will denote a strongly continuous linear operator cosine function on X with infinitesimal generator A. For the necessary concepts about cosine functions, the reader can consult [6, 15]. Next we introduce some notation, and give a brief review of results needed to establish our results. We let $S(t)$ denote the sine function associated with $C(t)$ which is defined by

$$S(t)x = \int_0^t C(s)x\, ds, \qquad x \in X,\ t \in \mathbb{R}.$$

We let $[D(A)]$ denote the space $D(A)$ endowed with the graph norm

$$\|x\|_A = \|x\| + \|Ax\|, \qquad x \in D(A).$$

Moreover, E stands for the space of vectors $x \in X$ for which the function $C(\cdot)x$ is of class C^1. It was proved by Kisyński [9] that E endowed with the norm

$$\|x\|_1 = \|x\| + \sup_{0 \le t \le 1} \|AS(t)x\|, \qquad x \in E,$$

is a Banach space. The operator-valued function

$$G(t) = \begin{bmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{bmatrix}$$

is a strongly continuous group of linear operators on the space $E \times X$ generated by the operator $\mathcal{A} = \left[\begin{smallmatrix} 0 & I \\ A & 0 \end{smallmatrix}\right]$ defined on $D(A) \times E$. The existence of solutions of the second order abstract Cauchy problem

$$x''(t) = Ax(t) + h(t), \qquad 0 \le t \le T, \tag{1.4}$$

$$x(0) = x_0, \qquad x'(0) = x_1, \tag{1.5}$$

where $h : [0,T] \to X$ is an integrable function, has been discussed in [16]. Similarly, the existence of solutions of semilinear second order abstract Cauchy problems has been treated in [17]. We only mention here that the function $x(\cdot)$ given by

$$x(t) = C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)h(s)\,ds, \qquad 0 \le t \le T, \tag{1.6}$$

is called a *mild solution* of (1.4)–(1.5). If $x_0 \in E$, then $x(\cdot)$ is continuously differentiable and

$$x'(t) = AS(t)x_0 + C(t)x_1 + \int_0^t C(t-s)h(s)\,ds. \tag{1.7}$$

The remaining terminology and notations are those generally used in functional analysis. In particular, $\mathcal{L}(X)$ stands for the Banach space of bounded linear operators from X into X, the prefix $\mathcal{R}$ is used to indicate the range of a map and $B_r(x)$ denotes the closed ball with centre at x and radius r in an appropriate space.

This work contains three sections. In Section 2 we establish the existence of mild, strong, classical solutions of some second order nonlocal initial value Cauchy problems. The results are obtained using the ideas and techniques developed in [8], Sadovskiĭ's fixed point theorem (see [14]) and the contraction mapping principle. In Section 3 we present an application of the results established in Section 2 to the wave equation.

2. Existence of solutions. In this section we study existence of solutions of some nonlocal second order functional abstract Cauchy problem. Henceforth we use $M \ge 1$ to denote a constant such that $\|C(t)\| \le M$ for all $t \in I$.

We first consider the problem

$$x''(t) = Ax(t) + f(t, x(t), x(a(t))), \quad 0 \le t \le T, \tag{2.1}$$

$$x(0) + p(x) = x_0, \tag{2.2}$$

$$x'(0) + q(x) = x_1, \tag{2.3}$$

where $x_0, x_1 \in X$. We assume that the following general conditions hold.

Assumption A.

(i) $a : I \to I$ is continuous.

(ii) $f : I \times X^2 \to X$ satisfies the following Carathéodory conditions:

(a) $f(t, \cdot) : X \times X \to X$ is continuous a.e. $t \in I$;

(b) for each $x, y \in X$, $f(\cdot, x, y) : I \to X$ is strongly measurable.

(iii) $p, q : C(I; X) \to X$ are continuous.

The expression (1.6) motivates the following concept of mild solution.

Definition 2.1. We say that $x : I \to X$ is a *mild solution* of problem (2.1)–(2.3) if x is a continuous function which satisfies the integral equation

$$\begin{aligned} x(t) &= C(t)(x_0 - p(x)) + S(t)(x_1 - q(x)) \\ &\quad + \int_0^t S(t-s) f(s, x(s), x(a(s)))\, ds, \quad t \in I. \end{aligned} \tag{2.4}$$

A cosine function defined on an infinite-dimensional Banach space cannot be compact on an interval of positive length. However, sine functions, especially those that arise in connection with practical situations, are frequently compact ([16]). This fact leads naturally to include that property in the statement of our results. For the theory of condensing maps, we refer to [4]. In what follows we will denote by B_r the closed ball in $C(I; X)$ with centre at 0 and radius r, and for a given $x \in C(I; X)$ we abbreviate $u(s) = (x(s), x(a(s)))$.

Theorem 2.1. *Assume that Assumption* A *and the following conditions hold*:

(H-1) *p and q take closed and bounded sets into bounded sets. For $r > 0$, let*

$$\alpha_r = \sup\{\|p(x)\| : x \in B_r\}, \quad \beta_r = \sup\{\|q(x)\| : x \in B_r\}.$$

(H-2) *The map $C(I; X) \to C(I; X)$, $x \mapsto C(\cdot)p(x)$, is condensing.*

(H-3) *The map $C(I; X) \to C(I; X)$, $x \mapsto S(\cdot)q(x)$, is completely continuous.*

(H-4) *For each $r > 0$, there is a positive function $\gamma_r \in \mathcal{L}^1(I)$ such that $\sup\{\|f(t, x, y)\| : \|x\|, \|y\| \le r\} \le \gamma_r(t)$ a.e. for $t \in I$.*

(H-5) *For each* $0 \le t \le T$ *and* $r > 0$, *the set* $\{S(t)f(s,x,y) : 0 \le s \le T, \|x\|, \|y\| \le r\}$ *is relatively compact.*

(H-6) $$\liminf_{k\to\infty} \frac{M}{k}\Big(\alpha_k + T\beta_k + \int_0^T (T-s)\,\gamma_k(s)\,ds\Big) < 1.$$

Then there is a mild solution of (2.1)–(2.3). *Furthermore*, *if*

(H-6′) $$\limsup_{r\to\infty} \frac{M}{r}\Big(\alpha_r + T\beta_r + \int_0^T (T-s)\,\gamma_r(s)\,ds\Big) < 1,$$

then the set $\mathcal{S}$ *of all mild solutions of* (2.1)–(2.3) *is compact in* $C(I;X)$.

Proof. Let $\mathcal{T} : C(I;X) \to C(I;X)$ be defined by

$$\begin{aligned}\mathcal{T}(x)(t) &= C(t)(x_0 - p(x)) + S(t)(x_1 - q(x)) \\ &\quad + \int_0^t S(t-s)f(s,x(s),x(a(s)))\,ds.\end{aligned} \tag{2.5}$$

Clearly, $\mathcal{T}$ is well defined and Lebesgue's dominated convergence theorem implies that $\mathcal{T}$ is continuous. We claim that there exists $n \in \mathbb{N}$ such that $\mathcal{T} : B_n \to B_n$. In fact, otherwise we can select a sequence $(x_k)_{k\in\mathbb{N}}$ in $C(I;X)$ such that $\|x_k\|_\infty \le k$ and $\|\mathcal{T}(x_k)\|_\infty > k$ for every $k \in \mathbb{N}$. Consequently, by (2.5), for all $k \in \mathbb{N}$, we have

$$k < \|\mathcal{T}(x_k)\|_\infty \le M(\|x_0\| + \alpha_k) + MT(\|x_1\| + \beta_k) + M\int_0^T (T-s)\gamma_k(s)\,ds.$$

It then follows that

$$1 \le \liminf_{k\to\infty} \frac{M}{k}\Big(\alpha_k + T\beta_k + \int_0^T (T-s)\gamma_k(s)\,ds\Big)$$

contrary to (H-6).

Now, we introduce the decomposition $\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2$ where the maps $\mathcal{T}_1, \mathcal{T}_2 : C(I;X) \to C(I;X)$ are defined by

$$\begin{aligned}&\mathcal{T}_1(x)(t) = C(t)(x_0 - p(x)),\\ &\mathcal{T}_2(x)(t) = S(t)(x_1 - q(x)) + \int_0^t S(t-s)f(s,x(s),x(a(s)))\,ds, \qquad t \in I.\end{aligned}$$

It is clear that $\mathcal{T}_1, \mathcal{T}_2$ are continuous maps. Moreover, hypothesis (H-2) implies that $\mathcal{T}_1$ is condensing.

Next we prove that $\mathcal{T}_2$ is completely continuous. Define

$$z(t) = \int_0^t S(t-s)f(s,x(s),x(a(s)))\,ds$$

for $x \in C(I;X)$. Applying (H-3), it remains to show that the set $\{z(\cdot) : \|x\|_\infty \le r\}$ is relatively compact in $C(I;X)$ for each $r > 0$. We begin by establishing that this set is equicontinuous. For this, we fix $t \ge 0$ and take $h \ge 0$ such that $t + h \le T$. Since

$$z(t+h) - z(t) = \int_0^t [S(t+h-s) - S(t-s)] f(s, u(s))\, ds + \int_t^{t+h} S(t+h-s) f(s, u(s))\, ds,$$

from (H-4) we obtain the elementary estimate

$$\|z(t+h) - z(t)\| \le Mh \int_0^t \gamma_r(s)\, ds + MT \int_t^{t+h} \gamma_r(s)\, ds,$$

which yields the assertion. Next we show that, for every $0 \le t \le T$, the set $\{z(t) : \|x\|_\infty \le r\}$ is relatively compact in X. Since $S(\cdot)$ is uniformly continuous on I, given $\varepsilon > 0$, let $\delta > 0$ be such that $\|S(t_1) - S(t_2)\| \le \varepsilon$ when $|t_1 - t_2| \le \delta$. We select $0 = s_0 < s_1 < \cdots < s_k = t$ so that $|s_i - s_{i-1}| \le \delta$. We can write

$$\begin{aligned} z(t) &= \sum_{i=1}^{k} \int_{s_{i-1}}^{s_i} S(t-s) f(s, u(s))\, ds \\ &= \sum_{i=1}^{k} \int_{s_{i-1}}^{s_i} [S(t-s) - S(t-s_i)] f(s, u(s))\, ds \\ &\quad + \sum_{i=1}^{k} \int_{s_{i-1}}^{s_i} S(t-s_i) f(s, u(s))\, ds. \end{aligned}$$

Applying (H-5) we deduce easily that the second term on the right hand side of the above expression is included in a compact set. Since we can estimate the first term as

$$\begin{aligned} \Big\| \sum_{i=1}^{k} \int_{s_{i-1}}^{s_i} [S(t-s) - S(t-s_i)] f(s, u(s))\, ds \Big\| &\le \varepsilon \sum_{i=1}^{k} \int_{s_{i-1}}^{s_i} \gamma_r(s)\, ds \\ &\le \varepsilon \int_0^T \gamma_r(s)\, ds \end{aligned}$$

and ε was arbitrarily chosen, the assertion follows.

From these assertions we conclude that $\mathcal{T}$ is a condensing map and combining this result with the first part of the proof and applying Sadovskiĭ's fixed point theorem (see [14]) we infer that $\mathcal{T}$ has a fixed point x in B_n. Clearly x is a mild solution of (2.1)–(2.3).

On the other hand, the continuity of $\mathcal{T}$ implies that $\mathcal{S}$ is closed. Moreover, if we assume that (H-6′) holds, proceeding as at the beginning of the proof, but using (H-6′) instead of (H-6), we can see that $\mathcal{S}$ is bounded. In fact, otherwise there is a sequence $x_k \in \mathcal{S}$ such that $r_k = \|x_k\|_\infty \geq k$. Hence we obtain

$$\begin{aligned}\|x_k(t)\| &= \|\mathcal{T}x_k(t)\| \\ &\leq M(\|x_0\| + \alpha_{r_k}) + MT(\|x_1\| + \beta_{r_k}) + M\int_0^T (T-s)\gamma_{r_k}(s)\,ds,\end{aligned}$$

which yields

$$1 \leq \limsup_{k\to\infty} \frac{M}{r_k}\left(\alpha_{r_k} + T\beta_{r_k} + \int_0^T (T-s)\gamma_{r_k}(s)\,ds\right)$$

contrary to (H-6′). Finally, using the fact that $\mathcal{T}$ is condensing we infer that $\mathcal{S}$ is compact. Thus the proof is complete. ∎

For those functions f which can be considered as perturbations of a linear function, we can establish the following modification of the previous result.

Theorem 2.2. *Assume that conditions* (H-1), (H-3) *and* (H-4) *are fulfilled, that the function a is such that $a(t) \leq t$ for $t \in I$, and that p is completely continuous. Suppose further that the function f can be split as $f = g + \ell$, where $g, \ell : I \times X^2 \to X$ satisfy the following conditions*:

(a) *g, ℓ satisfy the Carathéodory conditions* (ii) *of Assumption* A;
(b) *$\ell(t,\cdot) : X^2 \to X$ is linear and $L(t) = \|\ell(t,\cdot)\|$ is integrable on I*;
(c) *for each $0 \leq t \leq T$ and $r > 0$, the set $\{S(t)g(s,x,y) : 0 \leq s \leq T, \|x\|, \|y\| \leq r\}$ is relatively compact.*

If (H-6) *holds, then there is a mild solution of* (2.1)–(2.3). *Furthermore, if* (H-6′) *holds, then the set $\mathcal{S}$ of mild solutions of* (2.1)–(2.3) *is compact in $C(I;X)$.*

Proof. We define $\widetilde{\mathcal{T}} = \mathcal{T} + \mathcal{T}_0$, where $\mathcal{T}$ is defined by (2.5) with g instead of f and $\mathcal{T}_0$ is given by

$$\mathcal{T}_0 x(t) = \int_0^t S(t-s)\ell(s, x(s), x(a(s)))\,ds.$$

It is clear from the proof of Theorem 2.1 that $\mathcal{T}$ is completely continuous and $\widetilde{\mathcal{T}} : B_n \to B_n$ for some $n \in \mathbb{N}$. Moreover, $\mathcal{T}_0$ is a bounded linear operator and

$$\|\mathcal{T}_0 x(t)\| \leq M\int_0^t (t-s)L(s)(\|x(s)\| + \|x(a(s))\|)\,ds$$
$$\leq 2M\int_0^t (t-s)L(s)\max_{0\leq\xi\leq s}\|x(\xi)\|\,ds.$$

This estimate implies that, for m sufficiently large, $\mathcal{T}_0^m$ is a contraction, which in turn implies that the spectrum of $\mathcal{T}_0$ is included in the open unit ball. Using Lemma 4.4.4 of [7], we infer that the space $C(I;X)$ can be renormed by $|||\cdot|||$ so that $|||\mathcal{T}_0||| < 1$. Consequently, $\widetilde{\mathcal{T}}$ is a condensing map, and since B_n is a bounded, closed, convex set, Sadovskiĭ's theorem shows that $\widetilde{\mathcal{T}}$ has a fixed point in B_n. The proof is completed by arguing as in the proof of Theorem 2.1 ∎

When p or q satisfies a Lipschitz condition, we can establish a slightly different result. To state it, we introduce the conditions:

(H-7) $\|p(x) - p(y)\| \leq N_p\|x-y\|_\infty$ for $x, y \in C(I;X)$.

(H-8) $\|q(x) - q(y)\| \leq N_q\|x-y\|_\infty$ for $x, y \in C(I;X)$.

Here N_p, N_q are positive constants.

THEOREM 2.3. *Assume that Assumption* A *and conditions* (H-4) *and* (H-5) *are fulfilled. Suppose also that at least one of the following statements holds*:

(i) *Conditions* (H-1), (H-3) *and* (H-7) *are satisfied*, *and*

$$MN_p + \liminf_{k\to\infty}\frac{M}{k}\Big(T\beta_k + \int_0^T (T-s)\,\gamma_k(s)\,ds\Big) < 1. \tag{2.6}$$

(ii) *Conditions* (H-7) *and* (H-8) *are satisfied*, *and*

$$MN_p + MTN_q + \liminf_{k\to\infty}\frac{M}{k}\int_0^T (T-s)\,\gamma_k(s)\,ds < 1. \tag{2.7}$$

Then there is a mild solution of (2.1)–(2.3).

Proof. We prove the statement in case (i). Let $\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2$ be as in the proof of Theorem 2.1. Using condition (2.6), we can select $n \in \mathbb{N}$ sufficiently large such that

$$MN_p n + M\|x_0 - p(0)\| + MT(\|x_1\| + \beta_n) + M\int_0^T (T-s)\,\gamma_n(s)\,ds \leq n.$$

We claim that $\mathcal{T} : B_n \to B_n$. In fact, for $x(\cdot) \in B_r$ and $t \in I$, we have

$$
\begin{aligned}
\|\mathcal{T}(x)(t)\| &\le \|C(t)(p(0)-p(x)) + C(t)(x_0-p(0))\| + \|S(t)(x_1-q(x))\| \\
&\quad + \Big\| \int_0^t S(t-s) f(s, x(s), x(a(s)))\, ds \Big\| \\
&\le MN_p n + M\|x_0 - p(0)\| + MT(\|x_1\| + \beta_n) \\
&\quad + M \int_0^T (T-s)\,\gamma_n(s)\, ds \\
&\le n,
\end{aligned}
$$

which yields the assertion. Moreover, it is immediate that $\mathcal{T}_1$ is a contraction and, proceeding as in the proof of Theorem 2.1, it follows that $\mathcal{T}_2$ is completely continuous. Thus $\mathcal{T}$ is a condensing map and Sadovskiĭ's theorem implies that $\mathcal{T}$ has a fixed point in B_n.

The proof when condition (ii) holds is similar, except that in this case we modify the definition of $\mathcal{T}_1$ and $\mathcal{T}_2$ as

$$
\begin{aligned}
\mathcal{T}_1(x)(t) &= C(t)(x_0 - p(x)) + S(t)(x_1 - q(x)), \\
\mathcal{T}_2(x)(t) &= \int_0^t S(t-s) f(s, x(s), x(a(s)))\, ds, \qquad t \in I.
\end{aligned}
$$

We omit the details of this case. ■

When we can ensure that the set $\mathcal{S}$ of mild solutions is bounded, we infer its compactness. Thus the following result can be proved with the argument employed in the proof of Theorem 2.1. We omit the details.

Corollary 2.1. *Assume that Assumption* A *and conditions* (H-4) *and* (H-5) *are fulfilled. Suppose further that at least one of the following two statements holds*:

(i) *Conditions* (H-1), (H-3) *and* (H-7) *are satisfied*, *and*

$$
MN_p + \limsup_{r\to\infty} \frac{M}{r}\Big(T\beta_r + \int_0^T (T-s)\gamma_r(s)\, ds\Big) < 1.
$$

(ii) *Conditions* (H-7) *and* (H-8) *are satisfied*, *and*

$$
MN_p + MTN_q + \limsup_{r\to\infty} \frac{M}{r} \int_0^T (T-s)\gamma_r(s)\, ds < 1.
$$

Then the set $\mathcal{S}$ *is compact.*

As already mentioned, in most situations of practical interest, the sine function $S(t)$ is compact ([16]). For this reason we state separately the following result.

COROLLARY 2.2. *Assume that Assumption* A *holds*, *the operator* $S(t)$ *is compact for every* $t \in \mathbb{R}$, *and one of the following statements is satisfied*:

(i) *The map* p *is completely continuous and* (H-1), (H-4) *and* (H-6) *hold.*
(ii) *Conditions* (H-1), (H-4), (H-7) *and* (2.6) *hold.*
(iii) *Conditions* (H-4), (H-7), (H-8) *and* (2.7) *hold.*

Then there is a mild solution of (2.1)–(2.3).

Proof. The compactness of $S(t)$ implies easily that condition (H-3) is satisfied. Furthermore, the operator $\mathcal{T}_0 : C(I;X) \to C(I;X)$ defined by

$$\mathcal{T}_0(x)(t) = \int_0^t S(t-s) f(s, x(s), x(a(s)))\, ds$$

is completely continuous. We repeat the argument used in the proof of Theorem 2.1 to establish that $\mathcal{T}_2$ is completely continuous. Therefore, it suffices to show that the set $\{\mathcal{T}_0 x(t) : \|x\|_\infty \le r\}$, for $r > 0$, is relatively compact in X, for every $0 \le t \le T$. To prove this, we point out first that if we take $\xi > 0$, then we can write $S(n\xi) = S(\xi)P_n$ for $n \in \mathbb{N}$, where P_n is a certain bounded linear operator on X.

Since $S(\cdot)$ is uniformly continuous on I, given $\varepsilon > 0$, there is $\delta > 0$ such that $\|S(t_1) - S(t_2)\| \le \varepsilon$ when $|t_1 - t_2| \le \delta$. We select $n \in \mathbb{N}$ such that $\xi = t/n < \delta$ and define $s_i = i\xi$ for $i = 0, 1, \ldots, n$. Hence we can write

$$\begin{aligned}
\mathcal{T}_0 x(t) &= \sum_{i=1}^{n} \int_{s_{i-1}}^{s_i} S(t-s) f(s, u(s))\, ds \\
&= \sum_{i=1}^{n} \int_{s_{i-1}}^{s_i} [S(t-s) - S(t-s_i)] f(s, u(s))\, ds \\
&\quad - S(\xi) \sum_{i=1}^{n-1} P_i \int_{s_i}^{s_{i+1}} f(s, u(s))\, ds.
\end{aligned}$$

Applying (H-4), we can establish the estimates

$$\Big\| \sum_{i=1}^{k} \int_{s_{i-1}}^{s_i} [S(t-s) - S(t-s_i)] f(s, u(s))\, ds \Big\| \le \varepsilon \int_0^T \gamma_r(s)\, ds,$$

$$\Big\| \sum_{i=1}^{n-1} P_i \int_{s_i}^{s_{i+1}} f(s, u(s))\, ds \Big\| \le \max_{i=1,\ldots,n-1} \|P_i\| \int_0^T \gamma_r(s)\, ds,$$

which are valid independent of $x(\cdot)$. Since $S(\xi)$ is a compact operator, and since ε was arbitrarily chosen, the assertion follows.

We complete the proof by observing that if condition (i) is satisfied, then the statement is a consequence of Theorem 2.1, while if (ii) or (iii) holds, then the assertion follows from Theorem 2.3. ∎

From the properties of the abstract Cauchy problem mentioned in the preliminaries we know that if $x_0 - p(x) \in E$, then the mild solution x is continuously differentiable on I and

$$x'(t) = AS(t)(x_0 - p(x)) + C(t)(x_1 - q(x)) + \int_0^t C(t-s)f(s, x(s), x(a(s)))\,ds. \tag{2.8}$$

Next we study the differentiability of the function $x'(t)$. We first consider the following concept of strong solution.

DEFINITION 2.2. A function $x : [0, T] \to X$ is a *strong solution* of (2.1)–(2.3) if $x \in W^{2,1}([0, T]; X)$, the equation (2.1) is satisfied a.e., and the conditions (2.2) and (2.3) are satisfied.

In our next results, we consider Banach spaces that have the Radon–Nikodym property (abbreviated, RNP). We refer to [5] for a discussion about this matter.

THEOREM 2.4. *Assume that X has the RNP and that the hypotheses of Theorem* 2.1 *or* 2.3 *are fulfilled. Suppose*, *in addition*, *that the following assertions hold*:

(H-9) $\mathcal{R}(p) \subseteq x_0 + D(A)$ *and* $\mathcal{R}(q) \subseteq x_1 + E$.
(H-10) $\gamma_k \in \mathcal{L}^\infty(I)$.
(H-11) *For each bounded set $D \subseteq X$*, *the functions $C(\cdot)f(t, x, y)$ for $t \in I$ and $x, y \in D$ are uniformly Lipschitz continuous on I.*

Then each mild solution $x(\cdot)$ of (2.1)–(2.3) *is a strong solution.*

Proof. Let $x(\cdot)$ be a mild solution of (2.1)–(2.3). It follows from (2.8) and (H-9) that

$$\begin{aligned} x'(t+s) - x'(t) &= (AS(t+s) - AS(t))(x_0 - p(x)) \\ &\quad + (C(t+s) - C(t))(x_1 - q(x)) \\ &\quad + \int_0^t (C(t+s-\xi) - C(t-\xi))f(\xi, x(\xi), x(a(\xi)))\,d\xi \\ &\quad + \int_t^{t+s} C(t+s-\xi)f(\xi, x(\xi), x(a(\xi)))\,d\xi, \end{aligned}$$

which, by applying jointly (H-10) and (H-11), implies that $x'(\cdot)$ is Lipschitz continuous. Combining this with the fact that X has the RNP, it follows that $x \in W^{2,1}(I; X)$. The assertion is now a consequence of Proposition 3.3 in [8]. ∎

Relating to this result, it is worthwhile to point out that if X has the RNP and the function $C(\cdot)z$ is locally Lipschitz continuous, then $z \in E$ ([8]). In addition, from the results of [9], it follows that (H-11) holds if $\mathcal{R}(f) \subseteq E$ and $f : I \times D \times D \to E$ is bounded.

Next we discuss existence of classical solutions. We begin with the following definition.

DEFINITION 2.3. A function $x : I \to X$ is said to be a *classical solution* of problem (2.1)–(2.3) if x is of class C^2 and satisfies the equation (2.1) and the initial conditions (2.2) and (2.3).

THEOREM 2.5. *Assume that X has the RNP and that condition* (H-9) *as well as the hypotheses of Theorem* 2.1 *or* 2.3 *are fulfilled. Suppose also that the following two conditions hold*:

(H-12) *a is Lipschitz continuous.*

(H-13) *For each bounded set $D \subseteq X$, $f : I \times D \times D \to X$ is Lipschitz continuous.*

Then each mild solution $x(\cdot)$ of (2.1)–(2.3) *is a classical solution.*

Proof. It follows from (H-9) that every mild solution $x(\cdot)$ is of class C^1. Applying (H-12) and (H-13), it is easy to see that $t \mapsto f(t, x(t), x(a(t)))$ is Lipschitz continuous. The assertion is now a consequence of Theorem 3.1 of [8]. ∎

We can omit the RNP condition for X whenever $f(\cdot)$ and $a(\cdot)$ are differentiable. We state the following result without proof (see [16, Corollary 3.5]).

PROPOSITION 2.1. *Assume that the hypotheses in Theorem* 2.1 *or* 2.3 *are satisfied. Suppose further that* (H-9) *holds and that $f(\cdot)$, $a(\cdot)$ are continuously differentiable. Then each mild solution $x(\cdot)$ of* (2.1)–(2.3) *is a classical solution.*

Next we establish an existence result for the linear nonhomogeneous case. We henceforth assume that the space $X \times X$ is endowed with the norm

$$\|(x, y)\| = \max\{\|x\|, \|y\|\}.$$

In the next result we assume that $f(t, x, y) = \ell(x, y) + h(t)$ where $\ell : X \times X \to X$ is a bounded linear map and $h \in \mathcal{L}^1(I; X)$. Furthermore, $p, q : C(I; X) \to X$ are bounded linear maps which can be represented as the Riemann–Stieltjes integrals

$$p(x) = \int_0^T d_s P(s) x(s), \qquad q(x) = \int_0^T d_s Q(s) x(s)$$

where the operator-valued maps P, $Q : I \to \mathcal{L}(X)$ have bounded variation. We let $V(P)$ (resp. $V(Q)$) denote the variation of P (resp. Q).

THEOREM 2.6. *Assume that*

$$M\left(V(P) + TV(Q) + \frac{1}{2}T^2\|\ell\|\right) < 1. \tag{2.9}$$

Then problem (2.1)–(2.3) *has a unique mild solution* $x(\cdot)$. *Furthermore*, *if* X *has the RNP*, $h \in \mathcal{L}^\infty(I;X)$, *and* (H-9) *holds*, *then* x *is a strong solution. If*, *in addition*, a *and* h *are Lipschitz continuous*, *then* x *is a classical solution.*

Proof. The inequality (2.9) implies that the operator $\mathcal{T} : C(I;X) \to C(I;X)$ defined by (2.5) is a contraction. Consequently, the existence of a unique mild solution $x(\cdot)$ follows from the contraction principle. Using the corresponding hypotheses, according to Theorem 2.4 (respectively, Theorem 2.5), x is a strong solution (respectively, a classical solution). ∎

This result shows the importance of distinguishing between strong and classical solutions.

In what follows we consider the second order initial value problem (1.1)–(1.3). We assume that the following three general conditions are satisfied:

ASSUMPTION B.

(i) $a, b : I \to I$ are continuous.
(ii) $f : I \times X^4 \to X$ satisfies the following Carathéodory conditions:
  (a) $f(t,\cdot) : X^4 \to X$ is continuous a.e. for $t \in I$;
  (b) for each $u \in X^4$, $f(\cdot,u) : I \to X$ is strongly measurable.
(iii) $p, q : C(I;X)^2 \to X$ are continuous.

We begin by introducing the concept of mild solution.

DEFINITION 2.4. A function $x : I \to X$ is said to be a *mild solution* of problem (1.1)–(1.3) if x is a continuously differentiable function which satisfies the integral equation

$$\begin{aligned} x(t) &= C(t)(x_0 - p(x,x')) + S(t)(x_1 - q(x,x')) \\ &\quad + \int_0^t S(t-s) f(s, x(s), x(a(s)), x'(s), x'(b(s)))\, ds, \qquad t \in I. \end{aligned} \tag{2.10}$$

It is clear that if x is a function of class C^1 that satisfies (2.10), then $x_0 - p(x,x') \in E$ and x' satisfies the equation

$$\begin{aligned} x'(t) &= AS(t)(x_0 - p(x,x')) + C(t)(x_1 - q(x,x')) \\ &\quad + \int_0^t C(t-s) f(s, x(s), x(a(s)), x'(s), x'(b(s)))\, ds. \end{aligned} \tag{2.11}$$

Since the operator map $AS(\cdot)$ is strongly continuous with values in $\mathcal{L}(E;X)$, there is a constant N such that $\|AS(t)\| \le N$ for $0 \le t \le T$. Next we establish our first existence result.

THEOREM 2.7. *Assume that Assumption* B *and the following conditions hold*:

(H-14) *p and q are completely continuous. For each $r > 0$, set*

$$\alpha_r = \sup\{\|p(x,y)\| : x,\, y \in B_r\},$$
$$\beta_r = \sup\{\|q(x,y)\| : x,\, y \in B_r\}.$$

(H-15) *One of the following two conditions is fulfilled*:

(a) *$x_0 - p$ is completely continuous with values in E.*
(b) *$x_0 - p : X \times X \to [D(A)]$ maps closed bounded sets into bounded sets and $S(\cdot)$ is compact.*

Set $e_r = \sup\{\|x_0 - p(x,y)\|_1 : x,\, y \in B_r\}$.

(H-16) *For each $r > 0$, the set $f(I \times B_r^4)$ is relatively compact. Define $\gamma_r = \sup\{\|z\| : z \in f(I \times B_r^4)\}$.*

(H-17)
$$\liminf_{k\to\infty} \frac{M}{k}\left(\alpha_k + T\beta_k + \frac{1}{2}T^2\gamma_k\right) < 1,$$
$$\liminf_{k\to\infty} \frac{1}{k}(Ne_k + M\beta_k + MT\gamma_k) < 1.$$

Then there is a mild solution of (1.1)–(1.3). *If*, *in addition*,

(H-18)
$$\limsup_{r\to\infty} \frac{M}{r}\left(\alpha_r + T\beta_r + \frac{1}{2}T^2\gamma_r\right) < 1,$$
$$\limsup_{r\to\infty} \frac{1}{r}\left(Ne_r + M\beta_r + MT\gamma_r\right) < 1,$$

then the set $\mathcal{S}$ of mild solutions of (1.1)–(1.3) *is compact in $C^1(I;X)$.*

Proof. We define the map $\mathcal{T} : C(I;X)^2 \to C(I;X)^2$ by

$$\mathcal{T}(x,y) = (\mathcal{T}^1(x,y), \mathcal{T}^2(x,y))$$

where $\mathcal{T}^i : C(I;X)^2 \to C(I;X)$ are given by

$$\mathcal{T}^1(x,y)(t) = C(t)(x_0 - p(x,y)) + S(t)(x_1 - q(x,y)) + \int_0^t S(t-s)f(s,u(s),v(s))\,ds, \tag{2.12}$$

$$\mathcal{T}^2(x,y)(t) = AS(t)(x_0 - p(x,y)) + C(t)(x_1 - q(x,y)) + \int_0^t C(t-s)f(s,u(s),v(s))\,ds, \tag{2.13}$$

where, for brevity, we have set $u(t) = (x(t), x(a(t)))$ and $v(t) = (y(t), y(b(t)))$.

Now we can repeat the argument used in the proof of Theorem 2.1 to conclude that $\mathcal{T}$ is a completely continuous map and that there exists $n \in \mathbb{N}$ such that $\mathcal{T} : B_n^2 \to B_n^2$. Hence, by Schauder's theorem, $\mathcal{T}$ has a fixed point $(x, y) \in B_n^2$. It is clear from (2.12)–(2.13) that $y = x'$, which by (2.10) implies that x is a mild solution of (1.1)–(1.3).

Finally, in a similar way we prove that the set $\widetilde{\mathcal{S}} = \{(x, x') : x \in \mathcal{S}\}$ is compact in $C(I; X)^2$, which implies that $\mathcal{S}$ is compact in $C^1(I; X)$. ∎

Remark. Arguing as in the proof of Theorem 2.2, we can generalize this result to cover some functions f which are compact perturbations of linear operators.

3. Application to the wave equation. The one-dimensional wave equation modelled as an abstract Cauchy problem has been studied extensively (see [18] and the references therein). In this section, we apply some results of the preceding section to the wave equation with nonlocal conditions. Specifically, we consider the boundary value problem

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial \xi^2} + F(\xi, t, w(\xi, t), w(\xi, a(t))), \qquad 0 \le \xi \le \pi, \tag{3.1}$$

$$w(0, t) = w(\pi, t) = 0, \tag{3.2}$$

for $0 \le t \le T$, where $F : [0, \pi] \times [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies appropriate conditions. It is well known that this equation can be modelled as an abstract Cauchy problem on the space $X = L^2(0, \pi)$ defining $x(t) = w(\cdot, t)$. The operator A is given by $A\varphi = \varphi''$ on the domain

$$D(A) = \{x \in H^2(0, \pi) : x(0) = x(\pi) = 0\}.$$

This operator generates a cosine function C on X. Furthermore, A has discrete spectrum, the eigenvalues of A are $-n^2$ for $n \in \mathbb{N}$ with corresponding normalized eigenvectors $z_n(\xi) = (2/\pi)^{1/2} \sin(n\xi)$, and the following properties hold:

(a) $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of X.

(b) If $\varphi \in D(A)$, then $A\varphi = -\sum_{n=1}^{\infty} n^2 \langle \varphi, z_n \rangle z_n$.

(c) For each $\varphi \in X, C(t)\varphi = \sum_{n=1}^{\infty} \cos nt \langle \varphi, z_n \rangle z_n$. Therefore, $\|C(t)\| = 1$.

(d) The corresponding sine function is given by

$$S(t)\varphi = \sum_{n=1}^{\infty} \frac{\sin nt}{n} \langle \varphi, z_n \rangle z_n.$$

It follows that $\|S(t)\| = 1$ and that $S(t)$ is a compact operator.

(e) If G denotes the group of translations on X defined by $G(t)x(\xi) = \tilde{x}(\xi + t)$, where $\tilde{x}$ is the odd extension of x with period 2π, then

$C(t) = \frac{1}{2}(G(t) + G(-t))$. Hence $A = B^2$, where B is the infinitesimal generator of G and $E = \{x \in H^1(0,\pi) : x(0) = x(\pi) = 0\}$ (see [6]).

We assume that F satisfies the following Carathéodory conditions :

(a) $F(\xi, t, \cdot) : \mathbb{R}^2 \to \mathbb{R}$ is continuous a.e. for $\xi \in [0,\pi], t \in I$.
(b) For each $w_1, w_2 \in \mathbb{R}$, $F(\cdot, w_1, w_2) : [0,\pi] \times [0,T] \to \mathbb{R}$ is measurable.
(c) There are a positive constant N_1 and a measurable positive function η defined on $[0,\pi] \times [0,T]$, with $\int_0^T (\int_0^\pi \eta^2(\xi,t)\,d\xi)^{1/2}\,dt < \infty$, such that

$$|F(\xi, t, w_1, w_2)| \le \eta(\xi, t) + N_1(|w_1| + |w_2|).$$

With these conditions (we refer to [10] for a similar result) the substitution operator $f : [0,T] \times X^2 \to X$ defined by

$$f(t,x,y)(\xi) = F(\xi, t, x(\xi), y(\xi))$$

satisfies Assumption A.

We consider problem (3.1)–(3.2) subject to the following initial conditions:

$$w(\xi,0) + \int_0^T P(w(\cdot,s))(\xi)\,d\mu(s) = \varphi(\xi), \qquad 0 \le \xi \le \pi, \tag{3.3}$$

$$\frac{\partial w(\xi,0)}{\partial t} + \int_0^T Q(w(\cdot,s))(\xi)\,d\nu(s) = \psi(\xi), \qquad 0 \le \xi \le \pi, \tag{3.4}$$

where $\mu, \nu : [0,T] \to \mathbb{R}$ are functions of bounded variation. We also assume that $P, Q : X \to X$ are continuous, P is completely continuous and Q takes closed bounded sets to bounded sets. We introduce the positive constants $\alpha_{1,r}$ and $\beta_{1,r}$ with $\|P(x)\|_2 \le \alpha_{1,r}$ and $\|Q(x)\|_2 \le \beta_{1,r}$ when $\|x\|_2 \le r$. We refer to [10] for examples of operators with these properties. We define $p, q : C(I;X) \to X$ by

$$p(x) = \int_0^T P(x(s))\,d\mu(s), \tag{3.5}$$

$$q(x) = \int_0^T Q(x(s))\,d\nu(s). \tag{3.6}$$

It is not difficult to see that p, q are continuous and p is completely continuous. Furthermore, for $x \in C(I;X)$ with $\|x\|_\infty \le r$, it follows from (3.5) that

$$\|p(x)\|_2 \le \max_{0 \le s \le T} \|P(x(s))\|_2 V(\mu) \le \alpha_{1,r} V(\mu).$$

Similarly, it follows from (3.6) that

$$\|q(x)\|_2 \le \beta_{1,r} V(\nu).$$

On the other hand, if $x, y \in X$ and $\|x\|_2, \|y\|_2 \le r$, then

$$\begin{aligned} \|f(t,x,y)\|_2 = \|F(\xi,t,x(\xi),y(\xi))\|_2 &= \Big(\int_0^\pi |F(\xi,t,x(\xi),y(\xi))|^2 \, d\xi\Big)^{1/2} \\ &\le \Big(\int_0^\pi |\eta(\xi,t) + N_1(|x(\xi)| + |y(\xi)|)|^2 \, d\xi\Big)^{1/2} \\ &\le \Big(\int_0^\pi \eta^2(\xi,t)\, d\xi\Big)^{1/2} + N_1(\|x\|_2 + \|y\|_2) \\ &\le \Big(\int_0^\pi \eta^2(\xi,t)\, d\xi\Big)^{1/2} + 2N_1 r. \end{aligned}$$

We denote by $\gamma_r(t)$ the last right hand side. Applying our previous estimate, it is clear that

$$\begin{aligned} \alpha_k + T\beta_k + \int_0^T (T-s)\,\gamma_k(s)\, ds \le \alpha_{1,k} V(\mu) + T\beta_{1,k} V(\nu) + N_1 T^2 k \\ + \int_0^T (T-s)\Big(\int_0^\pi \eta^2(\xi,t)\, d\xi\Big)^{1/2} ds, \end{aligned}$$

which yields

$$\begin{aligned} \liminf_{k\to\infty} \frac{M}{k}\Big(\alpha_k + T\beta_k + \int_0^T (T-s)\,\gamma_k(s)\, ds\Big) \\ \le N_1 T^2 + \liminf_{k\to\infty} \frac{1}{k}(\alpha_{1,k} V(\mu) + T\beta_{1,k} V(\nu)). \end{aligned}$$

Similarly we obtain

$$\begin{aligned} \limsup_{r\to\infty} \frac{M}{r}\Big(\alpha_r + T\beta_r + \int_0^T (T-s)\,\gamma_r(s)\, ds\Big) \\ \le N_1 T^2 + \limsup_{r\to\infty} \frac{1}{r}(\alpha_{1,r} V(\mu) + T\beta_{1,r} V(\nu)). \end{aligned}$$

Employing these properties and Corollary 2.2, Theorem 2.4 and Theorem 2.5 we can establish the following consequences:

(i) If

$$N_1 T^2 + \liminf_{k\to\infty} \frac{1}{k}\,(\alpha_{1,k} V(\mu) + T\beta_{1,k} V(\nu)) < 1, \tag{3.7}$$

then problem (3.1)–(3.4) has a mild solution.

(ii) Suppose that (3.7) hold and

(a) $\mathcal{R}(p) \subseteq \varphi + D(A)$ and $\mathcal{R}(q) \subseteq \psi + E$;

(b) $\sup_{0 \le t \le T} \int_0^\pi \eta^2(\xi, t)\, d\xi < \infty$;

(c) for each bounded set $D \subseteq X$, there is a sequence $(a_n)_n$ of positive numbers with $\sum_{n=1}^\infty n a_n < \infty$ such that

$$\left| \int_0^\pi F(\xi, t, x(\xi), y(\xi)) z_n(\xi)\, d\xi \right| \le a_n$$

for all $t \in I$ and every $x, y \in D$.

Then the mild solutions of (3.1)–(3.4) are strong solutions.

(iii) Suppose that (3.7) and

(a) $\mathcal{R}(p) \subseteq \varphi + D(A)$ and $\mathcal{R}(q) \subseteq \psi + E$;

(b) a is Lipschitz continuous;

(c) there is a positive function $N_2 \in L^2(0, \pi)$ and a constant $N_3 > 0$ such that

$$|F(\xi, t_2, x_2, y_2 - F(\xi, t_1, x_1, y_1)| \le N_2(\xi)|t_2 - t_1| \\ + N_3(|x_2 - x_1| + |y_2 - y_1|)$$

for a.e. $\xi \in [0, \pi]$, all $t_1, t_2 \in I$ and all $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

Then each mild solution of (3.1)–(3.4) is a classical solution.

(iv) If

$$N_1 T^2 + \limsup_{r \to \infty} \frac{1}{r} \left(\alpha_{1,r} V(\mu) + T \beta_{1,r} V(\nu) \right) < 1,$$

then the set of mild solutions of problem (3.1)–(3.4) is compact.

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