Graphs with multiple sheeted pluripolar hulls

by Evgeny Poletsky (Syracuse, NY) and Jan Wiegerinck (Amsterdam)

Abstract. We study the pluripolar hulls of analytic sets. In particular, we show that hulls of graphs of analytic functions can be multiple sheeted and sheets can be separated by a set of zero dimension.

1. Introduction. One of the oldest interesting topics in complex analysis is the problem of analytic extensions: find the maximal analytic object containing a given one. For example, if $f$ is an analytic function we are looking for its analytic continuation and if $A$ is an irreducible analytic set we try to find another one of the same dimension containing $A$.

The counterpart of an analytic extension in pluripotential theory is the so-called pluripolar hull. There are two types of pluripolar hulls of a set $A$ in a domain $D \subset \mathbb{C}^n$. Let $\text{PSH}(D)$ be the set of all plurisubharmonic functions on $D$ and $\text{PSH}_0(D)$ the set of all negative functions from $\text{PSH}(D)$. Define

$$A^*_D = \{ z \in D : \forall h \in \text{PSH}(D) \ h|_A = -\infty \Rightarrow h(z) = -\infty \} ,$$

$$A^-_D = \{ z \in D : \forall h \in \text{PSH}_0(D) \ h|_A = -\infty \Rightarrow h(z) = -\infty \} .$$

For example, if $A$ is an analytic set in a pseudoconvex domain $D$, then every point of $A$ has a neighborhood $V$ where $A \cap V = \{ h_1 = \cdots = h_k = 0 \}$ and the functions $h_k$ are holomorphic on this neighborhood. Hence $A \cap V = \{ \log \max \{|h_1|, \ldots , |h_k|\} = -\infty \}$. In fact, $A^*_D = A$, because by [2, Cor. 1] there even exists $v \in \text{PSH}(D)$ such that $A = \{ v = -\infty \}$.

If such a $v \in \text{PSH}(D)$ exists for $A$, we call $A$ pluricomplete in $D$. In general, an analytic extension of $A$ is contained in $A^*_D$.

In the case when $A = \Gamma_f$ is the graph of an analytic function $f$ it was conjectured in [9] that the closure of the analytic extension of $A$ coincides

2000 Mathematics Subject Classification: Primary 32U15; Secondary 32D15.

Key words and phrases: pluripotential theory, pluripolar hulls.

The first author was partially supported by an NSF grant.
with $A^*_D$. However, A. Edigarian and the second author found in [3] an analytic function $f$ on the unit disk $D$ that does not extend analytically while the pluripolar hull of its graph is the graph of an analytic function defined on almost the whole plane (cf. [11]). Thus pluripolar hulls and analytic extensions may differ by large sets.

The pluripolar hull of the graph $\Gamma_f$ of a holomorphic function $f(z)$ on a domain $D$ may well be multi-sheeted over $D$. The principal value of $\sqrt{z}$ on $\{\Re z > 0\}$ provides the easiest example. Only recently did Zwonek [12], and, independently, Edlund and Jöricke [6], give examples of holomorphic functions $f$ on their domain of existence $D$ with the property that the pluripolar hull $\Gamma_f^{\mathbb{C}^2}$ is multi-sheeted over (parts of) $D$. In all these examples the topological codimension of $\partial D$ is 1.

As we show in Section 2, this is an intrinsic property of analytic extensions: Let $E$ be a closed subset of $\mathbb{C}^{n+1}$ and $A$ be an analytic set of pure dimension $n$ in $\mathbb{C}^{n+1}\setminus E$. If the set $A^*_{\mathbb{C}^{n+1}}$ is analytic, then $\dim E \geq 2n-1$. However, as we also show, there is a Cantor type set $K$ and a holomorphic function $f(z)$ on $D = \mathbb{C} \setminus K$ such that $(\Gamma_f)^{\mathbb{C}^2} \cap D \times \mathbb{C} = \Gamma_f$; see [4], and [5] for the fact that also over $K$ the hull is at most single sheeted.

As a by-product we obtain an example of a uniformly convergent sequence of holomorphic functions such that their pluripolar hulls do not converge to the pluripolar hull of the limit.

The set $K$ should be sufficiently fat. Edigarian and the second author showed that if $D = \mathbb{C} \setminus K$, with $K$ a polar compact set in $\mathbb{C}$, and if $f$ is not extendible over $K$, then $(\Gamma_f)^{\mathbb{C}^2} \cap D \times \mathbb{C} = \Gamma_f$; see [4], and [5] for the fact that also over $K$ the hull is at most single sheeted.

2. Pluripolar extensions. Let $E$ be a closed set in a pseudoconvex domain $D \subset \mathbb{C}^n$. If $A \subset D \setminus E$ then, in general, $A^*_D \setminus E$ is a proper subset of $A^*_D \setminus E$. However, as the following statement shows, these sets coincide when $E$ is pluripolar.

**PROPOSITION 2.1.** If $E$ is a closed pluripolar set in a pseudoconvex domain $D \subset \mathbb{C}^n$ and $A \subset D \setminus E$, then $A^*_D \setminus E = A^*_D \setminus E$.

**Proof.** Let $\{D_j\}$ be an increasing sequence of relatively compact subdomains with $\bigcup_j D_j = D$. By [10, Thm. 2.4], $A^*_D = \bigcup_j (A \cap D_j)^{\overline{D_j}}$. If $u \in \text{PSH}_0(D_j \setminus E)$, then $u$ extends as a negative plurisubharmonic function to $D_j$ (see [8, Thm. 2.9.22]). Therefore, $(A \cap D_j)^{\overline{D_j}} \setminus E = (A \cap D_j)^{\overline{D_j \setminus E}}$. Since $(A \cap D_j)^{\overline{D_j \setminus E}} \subset A^*_D \setminus E$, we see that $A^*_D \setminus E \subset A^*_D \setminus E$ and, consequently, $A^*_D \setminus E = A^*_D \setminus E$. □
The proposition below describes the situation when $E$ is a closed set in a pseudoconvex domain $D \subset \mathbb{C}^n$, $A \subset D \setminus E$ is an analytic set and $A_D^*$ is also analytic.

**Proposition 2.2.** Suppose that $E$ is a closed set in a pseudoconvex domain $D \subset \mathbb{C}^n$ and $A \subset D \setminus E$ is an analytic set. If the set $A_D^*$ is analytic then every irreducible component of $A_D^*$ contains a component of $A$ of the same dimension.

**Proof.** Let $X$ be an irreducible component of $A_D^*$. We represent $A_D^*$ as $X \cup Y$, where $Y$ is another analytic set in $D$ and $\dim X \cap Y < \dim X$. As indicated in Section 1, the set $Y$ is pluricomplete and $Y_D^* = Y$. It is easy to check that if $F, G \subset D$, then $(F \cup G)_D^* = F_D^* \cup G_D^*$. So if $B = A \cap (X \setminus Y)$ then $X \setminus Y \subset B_D^*$.

Suppose that $\dim X > \dim B$ and let $R$ be the set of regular points of $X$. The set of singular points of $X$ is analytic and, consequently, pluricomplete. By the argument above $X \setminus Y$ belongs to the pluripolar hull of the set $B' = B \cap R$.

We may assume that $0 \in R$ and let $T$ be the tangent plane to $R$ at 0. If $p$ is a projection of $\mathbb{C}^n$ on $T$, then the set $p(B')$ is pluripolar in $T$ and, consequently, there is a plurisubharmonic function $u$ on $T$ equal to $-\infty$ on $p(B')$. The set $p(R)$ has a non-empty interior in $T$ and, therefore, there is a point $z_0 \in R$ such that $u(p(z_0)) \neq -\infty$. Then the function $v = u \circ p$ is plurisubharmonic on $\mathbb{C}^n$, equal to $-\infty$ on $B'$ and $v(z_0) > -\infty$. Thus $R$ does not belong to the pluripolar hull of the set $B'$. This contradiction proves the proposition. 

Suppose that $A$ is a pluricomplete analytic set of pure dimension $m$ in $D \setminus E$. If $A_D^*$ is an analytic set in $D$ and $A$ is a proper subset of $A_D^* \setminus E$, then the set $E \cap A_D^*$ cuts $A_D^*$ into several pieces and, therefore, its topological dimension must be at least $2m - 1$.

For example, let $D = \{(z, w) \in \mathbb{C}^2\}$ and $E = \{\Im z = 0, \Re z \geq 0\}$. Take a branch $w = f(z)$ of the function $w = \sqrt{z}$ over $\mathbb{C} \setminus E$ and let $A = \{(z, f(z)) : z \in \mathbb{C}\}$. The pluripolar hull of $A$ in $\mathbb{C}^2 \setminus E$ is $A$ because the function $\log |f(z) - z|$ is equal to $-\infty$ exactly on $A$. But $A_D^* = \{(z, w) : z = w^2\}$. In this example $A_D^*$ is an analytic set and the set $A_D^* \cap E$ is the real curve $\{(x^2, x) : x \in \mathbb{R}\}$ which projects 2 to 1 except at 0 and its projection has dimension 1.

As the following statement shows, this is the minimal possible dimension.

**Proposition 2.3.** Let $E$ be a closed set in a pseudoconvex domain $D \subset \mathbb{C}^n$ and let $A$ be an irreducible analytic set of dimension $m$ in $D \setminus E$ such that $A_D^*$ is also analytic. If $p$ is a projection of $\mathbb{C}^n$ onto $\mathbb{C}^m \subset \mathbb{C}^n$ such that
measure of the set $A$ also has dimension at most 2

is a projection such that the restriction $p$ to $Y$ has rank less than $m$. Since $p(A)$ has non-empty interior in $\mathbb{C}^m$, the set $Y' = Y \setminus X'$ is non-empty and relatively open in $Y$. The set $X'$ is analytic and, therefore, its dimension is at most $m - 1$. Hence the set $Y'$ is an irreducible complex submanifold of $D \setminus (X \cup X')$ and the set $A' = A \cap Y'$ is a non-empty submanifold of $Y' \setminus E$ of dimension $m$.

Suppose that the topological dimension of $p(E)$ is at most $2m - 2$. The projection $p$ is locally homeomorphically on $Y'$ and, therefore, the set $E \cap Y'$ also has dimension at most $2m - 2$. Since $E$ is closed the set $A'$ is relatively open in $Y'$. The set $Y'' = Y' \setminus (A' \cup E)$ is also relatively open in $Y'$. Thus $Y'$ is a topological manifold of dimension $2m$ and equals the union of two open sets $A'$ and $Y''$ and a closed set $E$ of dimension at most $2m - 2$. By the Urysohn–Menger Theorem a set of dimension $2m - 2$ cannot disconnect a $2m$-dimensional manifold (see [7, Ch. IV.5, Cor. 1]). Hence either $A'$ or $Y''$ is empty. Since $A'$ is not empty, $Y' \setminus E = A'$.

Since $A$ is closed in $A_D^* \setminus E$, it contains $X \setminus E$ and $X' \setminus E$. So $A_D^* \setminus E = A$. ■

When $A$ is an analytic set in $D \setminus E$ we denote by $A_E$ the intersection of $E$ and the closure of $A$ in $D$. If $A_D^* \setminus E \neq A$, we will say that $A$ has a non-trivial pluripolar extension through $E$ in $D$.

The following theorem lists some limitations on the set $A_E$ when a non-trivial pluripolar extension takes place. Following [1] we call a set $G$ in a domain $Y \subset \mathbb{C}^p$ locally removable if $G$ is closed and for every open set $V$ in $Y$ every bounded holomorphic function $f$ on $V \setminus G$ extends holomorphically to $V$.

**Theorem 2.4.** Suppose that $D$ is a pseudoconvex domain in $\mathbb{C}^n$, $E$ is a closed set in $D$ and $A$ is an analytic set of pure dimension $m$ in $D \setminus E$ with a non-trivial pluripolar extension through $E$ in $D$. Then the $(2m - 1)$-Hausdorff measure of the set $A_E$ is not equal to zero and if, additionally, $p: \mathbb{C}^n \to \mathbb{C}^m$ is a projection such that the restriction $p|_{A_E}$ is proper and $A \cap p^{-1}(z)$ is empty for all $z \in p(A_E)$, then $p(A_E)$ is not locally removable in $\mathbb{C}^m$.

**Proof.** The set $A$ is analytic in $D \setminus A_E$. If the $(2m - 1)$-Hausdorff measure of $E$ is zero, then by Shiffman’s theorem (see [1, 4.4]) the closure $\overline{A}$ of $A$ in $D$ is an analytic set in $D$. Since the domain $D$ is pseudoconvex there is
a holomorphic function \( f \) on \( D \) such that \( \overline{A} = \{ f = 0 \} \). Thus \( A^*_D = \overline{A} \) and 
this contradicts the assumption that the extension is non-trivial.

In the second case, if \( p(A_E) \) is locally removable in \( \mathbb{C}^m \), then by the \nProposition in [1, 18.1] the closure \( \overline{A} \) of \( A \) is an analytic set in \( D \) as before 
and the same argument leads to a contradiction. 

In our main example, \( n = 2, m = 1 \). In this case Theorem 2.4 can be 
reformulated as follows:

**Corollary 2.5.** If in the assumptions of Theorem 2.4, \( n = 2 \) and \( m = 1 \), then the 
first Hausdorff measure of \( A_E \) is not zero and, under additional 
assumptions, the first Hausdorff measure of \( p(A_E) \) is not zero.

### 3. A holomorphic function on the complement of a Cantor type set 
with 2-sheeted hull

**Definition 3.1.** A Cantor type set \( K \) is a compact perfect subset of \( \mathbb{R} \) 
with empty interior.

It is a well known fact from elementary point set topology that such a 
\( K \) is homeomorphic to Cantor’s middle thirds set. It is of the form 
\( [a_0, b_0] \setminus \bigcup_{j=1}^{\infty} I_j \) where \( I_j \) are open intervals in 
\( [a_0, b_0] \), \( I_j \cap I_k = \emptyset \) if \( j \neq k \) and \( \bigcup_{j=1}^{\infty} I_j \) is dense in 
\( [a, b] \). We can assume that the length of \( I_j = (a_j, b_j) \) 
decreases with \( j \).

It is useful to enumerate the set \( \{a_j, b_j : j = 0, \ldots, n \} \) as \( \{\alpha_{jn}, \beta_{jn}\} \) 
so that \( \alpha_{0n} = a_0, \alpha_{jn} < \beta_{jn} < \alpha_{j+1,n} \) and \( \beta_{nn} = b_0 \). Note that \( [a_0, b_0] \setminus \bigcup_{j=1}^{n} I_j = \bigcup_{j=0}^{n} [\alpha_{jn}, \beta_{jn}] \) and that \( I_m \cap [\alpha_{jn}, \beta_{jn}] \neq \emptyset \) implies \( I_m \subset [\alpha_{jn}, \beta_{jn}] \). 
Let
\[
g_n(z) = \frac{z - a_0}{z - b_0} \cdot \frac{z - b_1}{z - a_1} \cdots \frac{z - b_n}{z - a_n}.
\]
Then
\[
g_n(z) = \frac{z - \alpha_{0n}}{z - \beta_{0n}} \cdot \frac{z - \alpha_{1n}}{z - \beta_{1n}} \cdots \frac{z - \alpha_{nn}}{z - \beta_{nn}}.
\]
Each fraction \( \frac{z - \alpha_{jn}}{z - \beta_{jn}}, j = 0, 1, \ldots, n \), has a holomorphic branch \( \sqrt{\frac{z - \alpha_{jn}}{z - \beta_{jn}}} \) 
of its square root outside \( [\alpha_{jn}, \beta_{jn}] \) that equals 1 at infinity. Let
\[
(3.1) \quad f_n(z) = \sqrt{\frac{z - \alpha_{0n}}{z - \beta_{0n}}} \cdot \sqrt{\frac{z - \alpha_{1n}}{z - \beta_{1n}}} \cdots \sqrt{\frac{z - \alpha_{nn}}{z - \beta_{nn}}} = g_n(z).
\]
Then \( f_n(\infty) = 1 \), \( f_n^2 = g_n \) and \( f_n \) is holomorphic on \( G_n = (\mathbb{C} \setminus [a_0, b_0]) \cup \bigcup_{j=1}^{n} I_j \).

The maximal analytic extension of \( f_n \) is a branched 2-sheeted cover \( X_n = \{(z, w) : w^2 = g_n(z)\} \) of \( \mathbb{C} \) that branches over \( \{a_j, b_j : j = 0, 1, \ldots, n\} \). The 
pluripolar hull \( (f_n)_C^* \) equals \( X_n \).
LEMMA 3.2. Keeping the above notation, the sequence \( \{g_n\} \) converges normally to an analytic function \( g \) on \( \mathbb{C} \setminus K \). Moreover, the function \( g \) extends analytically over a point \( x \in K \) if and only if for some \( \alpha < x < \beta \) the length of the set \( K \cap (\alpha, \beta) \) is zero.

Proof. Let \( L \) be a compact set in \( \mathbb{C} \setminus K \). Let us show that

\[
(3.2) \quad \frac{z - a_0}{z - b_0} \prod_{j=1}^{\infty} \frac{z - b_j}{z - a_j} = \lim_{n \to \infty} g_n(z)
\]

is uniformly convergent on \( L \). There exists \( n_0 \) such that \( L \subset G_n \) for \( n > n_0 \) and moreover, for some \( \delta > 0 \),

\[
L \subset \{ z : |z - a_j| > \delta, j = 0, 1, \ldots \).
\]

Hence, for \( z \in L \),

\[
\left| \frac{z - b_j}{z - a_j} - 1 \right| = \left| \frac{b_j - a_j}{z - a_j} \right| \leq \frac{b_j - a_j}{\delta}.
\]

Since \( \sum (b_j - a_j) \) is finite, the product in (3.2) converges uniformly on \( L \) to a function \( g \) that is holomorphic on \( \mathbb{C} \setminus K \).

Suppose that \( g \) extends analytically over a point \( x \in K \) so that \( g \) is analytic on \( (\mathbb{C} \setminus K) \cup (\alpha, \beta) \). We may assume that \( \alpha \in I_k, b_k \leq x, \) and \( \beta \in I_m, a_m \geq x, \) and

\[
g_{1n}(z) = \prod_{j} \frac{z - \alpha_{jn}}{z - \beta_{jn}},
\]

where the product runs over all \( j \) such that either \( \beta_{jn} < \alpha \) or \( \alpha_{jn} > \beta \). Let

\[
g_{2n}(z) = \prod_{j} \frac{z - \alpha_{jn}}{z - \beta_{jn}},
\]

where the product runs over all \( j \) such that \( \alpha < \alpha_{jn} \) and \( \beta_{jn} < \beta \). Then \( g_n = g_{1n}g_{2n} \) and by the argument above the sequences \( \{g_{1n}\} \) and \( \{g_{2n}\} \) converge uniformly on compacta in \( \mathbb{C} \setminus (K \setminus (\alpha, \beta)) \) and \( \mathbb{C} \setminus (K \cap (\alpha, \beta)) \) respectively. We denote their respective limits by \( g_1 \) and \( g_2 \).

For the derivative of \( g_{2n} \) we find \( g'_{2n}(\infty) = \sum (\beta_{jn} - \alpha_{jn}) = l_n \), where the sum runs over all \( j \) such that \( \alpha < \alpha_{jn} \) and \( \beta_{jn} < \beta \). Thus \( g_{2n}(\infty) \) is equal to the length \( l_n \) of the intervals \( (\alpha_{jn}, \beta_{jn}) \) lying in \( (\alpha, \beta) \) and \( g'_2(\infty) \) is the length of the set \( K \cap (\alpha, \beta) \). If this length is positive, then the function \( g_2 \) is not constant and, therefore, does not extend to \( K \cap (\alpha, \beta) \).

If this length is 0 then for \( z \in \mathbb{C} \) such that \( |z - y| \geq 1 \) for all \( y \in (\alpha, \beta) \) we have

\[
|1 - g_{2n}(z)| = \left| 1 - \prod_{j} \left( 1 + \frac{\beta_{jn} - \alpha_{jn}}{z - \beta_{jn}} \right) \right| \leq e^{l_n} - 1.
\]

Hence the sequence \( \{g_{2n}\} \) converges to 1 near \( \infty \), \( g_2 \equiv 1 \) and \( g \) extends analytically over \( (\alpha, \beta) \).
**Lemma 3.3.** If $f = f_K$ and the length of $K$ is positive, then the union $\Gamma_f \cup \Gamma_{-f}$ of the closures of the graphs of $f$ and $-f$ is not an analytic set.

**Proof.** If $A = \Gamma_f \cup \Gamma_{-f}$ is an analytic set, then there is a holomorphic function $h = h(z, w)$ on $\mathbb{C}^2$ such that $h \equiv 0$ on $A$. We have $A = \Gamma_f \cup \Gamma_{-f} \cup E$, where $E \subset K \times \mathbb{C}$.

Let us show that for every $z_0 \in K$ the analytic set $E_{z_0} = \{ w : (z_0, w) \in E \}$ $= A \cap \{ z_0 \} \times \mathbb{C}$ consists of at most two points. If it contains three points, then at least two of them belong to, say, $\Gamma_f$. Thus there are sequences $\{ z_j \}$ and $\{ z_j' \}$ converging to $z_0$ such that the sequences $\{ f(z_j) \}$ and $\{ f(z_j') \}$ have distinct limits. Connecting each $z_j$ and $z_j'$ by small curves in $\mathbb{C} \setminus K$ and looking at their limits we see that the cluster set of $f$ at $z_0$ contains a continuum. Hence, $E_{z_0} = \mathbb{C}$ and $h(z_0, w) \equiv 0$.

From the Taylor expansion of $h$ we immediately derive that $h(z, w) = (z - z_0)^n h_1(z_0, w)$, where $h_1$ is holomorphic on $\mathbb{C}^2$ and $h_1(z_0, w) \neq 0$. But for every point $w \in \mathbb{C}$ there is a sequence of $z_j$ converging to $z_0$ such that, say, $f(z_j)$ converges to $w$. Since $h(z_j, f(z_j)) = 0$ we see that $h_1(z_0, w) = 0$. This contradiction shows that $E_{z_0}$ has at most two points and the intersection of $\Gamma_f$ or $\Gamma_{-f}$ with $E$ consists of at most one point.

It follows that $f$ extends continuously to $K$. Since $K$ lies on the real line, $f$ extends holomorphically to $\mathbb{C}$; but this impossible by Lemma 3.2. $\blacksquare$

**Example 3.4.** If the set $K$ has Lebesgue measure 0, then

$$\lim_{n \to \infty} g_n(z) = 1$$

uniformly on any compact set $L$ not meeting $K$. It follows that $f \equiv 1$ and $(\Gamma_f)_{\mathbb{C}^2} = \{(z, 1)\}$. But the Hausdorff limit of the sets $X_n$ over $D$ equals $\{(z, w) : w = 1 \text{ or } -1\}$.

We will need the next lemma, whose proof is similar to the proof of Theorem 2.1 in [4].

**Lemma 3.5.** Let $f$ be a holomorphic function on a domain $V \subset \mathbb{C}^N$ containing a closed ball $B$ and let $\{ r_n \}$ be a sequence of rational functions of degree $n$ with poles outside $V$ and such that the sup-norm $\| f - r_n \|_B^{1/n}$ tends to 0 as $n \to \infty$. Then there is a plurisubharmonic function $v$ on $\mathbb{C}^{N+1}$ such that $\{ v = -\infty \} \cap (V \times \mathbb{C}) = \Gamma_f$. Thus, $(\Gamma_f)^*_n \cap (V \times \mathbb{C}) = \Gamma_f$.

**Proof.** The functions $r_n$ are ratios of polynomials $p_n$ and $q_n$ of degree $n$. We may assume that $B$ is the closed unit ball centered at the origin and $\| q_n \|_B = 1$. Then $|q_n(z)| \leq \max \{ 1, |z|^n \}$, $\| p_n \|_B$ does not exceed some constant $C$ and $|p_n(z)| \leq C \max \{ 1, |z|^n \}$.

Consider the plurisubharmonic functions

$$u_n(z, w) = \frac{1}{n} \log |q_n(z)w - p_n(z)|$$
on \( \mathbb{C}^{N+1} \). From the estimates on \( p_n \) and \( q_n \) there is a constant \( C_1 \) such that
\[
\log |z| + \log |w| + C_1 \quad \text{when} \quad |z| \geq 1 \quad \text{and} \quad u_n(z, w) \leq \log |w| + C_1 \quad \text{when} \quad |z| \leq 1.
\]
We take \( z_n \in B \) such that \( q_n(z_n) = a_n, |a_n| = 1 \). Let \( w_n = (p_n(z_n) + 1)/a_n \).
Then \( |w_n| \leq C + 1 \) and \( u_n(z_n, w_n) = 0 \). If \( B_n \) is a ball in \( \mathbb{C}^{N+1} \) centered at \((z_n, w_n)\) and of radius \( r_n = C + 5 \), then \( B_n \) contains the unit ball \( B' \) centered at the origin and
\[
\int_{B_n} u_n \, dV \geq cu_n(z_n, w_n) = 0.
\]
It is immediate from the upper estimates on \( u_n \) that there is a constant \( C_2 \) such that
\[
\int_{B'} u_n \, dV \geq C_2.
\]
By our assumption there is a sequence \( \{d_n\} \) converging to \( \infty \) such that
\[
\frac{1}{n} \log |q_n(z)| + \frac{1}{n} \log |f(z) - \frac{p_n(z)}{q_n(z)}| \leq -d_n
\]
when \( |z| \leq 1 \).
Let us take a sequence \( \{c_n\} \) of positive reals such that \( \sum c_n = 1 \) while \( \sum c_n d_n = \infty \). Let
\[
v(z, w) = \sum_{n=1}^{\infty} c_n \max\{u_n(z, w), d_n\}.
\]
Since
\[
\int_{B} v \, dV \geq C_2,
\]
we see that \( v \neq -\infty \) and, therefore, it is a plurisubharmonic function on \( \mathbb{C}^{N+1} \). Clearly, \( v(z, f(z)) = -\infty \) when \( |z| \leq 1 \). Therefore, \( v = -\infty \) on \( \Gamma_f \).

The zeros of the polynomials \( q_n \) are in \( \mathbb{C}^N \setminus V \). Hence the functions \( h_n(z) = n^{-1} \log |q_n(z)| \) are harmonic on \( V \) and uniformly bounded above on compacta. So if \( z \in V \) and \( \liminf h_n(z) = -\infty \), then there is a subsequence \( \{n_k\} \) such that \( \lim h_{n_k}(z) = -\infty \). Therefore, the functions \( h_{n_k} \) converge to \( -\infty \) uniformly on compacta in \( V \). But \( h_{n_k}(z_{n_k}) = 0 \), and this contradiction tells us that \( \liminf h_n(z) > -\infty \). So if \( w \neq f(z) \), then
\[
v(z, w) \geq \sum_{n=1}^{\infty} c_n h_n(z) + \sum_{n=1}^{\infty} \frac{c_n}{n} \log |w - \frac{p_n(z)}{q_n(z)}| > -\infty.
\]
Hence \( \{v = -\infty\} \cap (V \times \mathbb{C}) = \Gamma_f \) and \( (\Gamma_f)_{\mathbb{C}^{N+1}}^* \cap (V \times \mathbb{C}) = \Gamma_f \). \( \blacksquare \)

Now we can present our main example.

**Theorem 3.6.** There exists a Cantor type set \( K \), obtained by deleting intervals \( I_i = (a_i, b_i) \) from \([-1, 1] \), such that the function \( f = f_K \) given by
Lemma 3.3 has the following property:

$$(\Gamma_f)^* \cap (\mathbb{C} \setminus K) \times \mathbb{C} = \Gamma_f \cup \Gamma_{-f}.$$ 

**Proof.** We will construct $K$ by deleting a sequence of open intervals $(a_i, b_i)$ from the interval $[-1, 1]$. For convenience, set $a_0 = 1$, $b_0 = -1$. In order to choose the intervals appropriately, we have to construct certain subdomains $D_n$ in the open unit disk $\mathbb{D}$ in the process. The domains $D_n$ will contain the set $\{ |\Im z| > 1/2 \}$. Thus the closed discs $S = \{ |z + 3i/4| \leq 1/8 \}$ and $X = \{ |z - 3i/4| \leq 1/8 \}$ will be contained in $D_n$.

For a compact set $F$ in a domain $D \subset \mathbb{C}$ let

$$\omega(z, F, D) = -\sup\{ h(z) : h \in \text{PSH}(D), \limsup_{w \to K} h(w) \leq -1 \}$$

be the harmonic measure of $F$ in $D$. We set $D_0 = \mathbb{D}$ and observe that $\omega(z, S, D_0) > c_0$ for some positive $c_0$.

Let $\{c_n\}$ be a sequence of positive real numbers converging to $\infty$. Suppose that the intervals $I_1, \ldots, I_n$ have been chosen. We take as $a_{n+1}$ the midpoint of the largest interval in their complement. Next take $b_{n+1} > a_{n+1}$ so small that the interval $[a_{n+1}, b_{n+1}]$ does not intersect the intervals $I_1, \ldots, I_n$, $d_{n+1} = b_{n+1} - a_{n+1} < 4^{-(n+1)c_{n+1}}$ and, moreover,

$$\omega(z, S, D_{n+1}) > c_0, \quad z \in X.$$ 

Here we define $D_{n+1} = D_n \setminus \mathbb{D}(a_{n+1}, (b_{n+1} - a_{n+1})2^{n+1})$, where $\mathbb{D}(a, r)$ is the open disk centered at $a$ and of radius $r$.

Observe that for $j \leq n$,

$$\left| \frac{z - b_j}{z - a_j} - 1 \right| = \left| \frac{d_j}{z - a_j} \right| < \frac{1}{2^j}$$

on $D_n$. It follows that

$$\prod_{j=1}^{n} \frac{z - b_j}{z - a_j}$$

is bounded on $D_n$ independently of $n$.

Let $z_0 \in X$. We will show that $(z_0, -f(z_0)) \in (\Gamma_f)^* \cap (\mathbb{C} \setminus K) \times \mathbb{C}$. Then $\Gamma_{-f}$ is also in the hull and we are done. Consider the function $\tilde{f}_n$ defined on $\mathbb{D} \setminus \bigcup_{j=1}^{n} I_j$ by

$$\tilde{f}_n(z) = \begin{cases} 
  f_n(z) & \text{if } \Im z < 0, \\
  -f_n(z) & \text{if } \Im z > 0, \\
  \lim_{y \to 0} f(x + iy) & \text{if } x \in [-1, 1] \setminus \bigcup I_j.
\end{cases}$$

The function $\tilde{f}_n$ is holomorphic. Let $c_n = \tilde{f}_n(z_0) + f_K(z_0).$ Then $c_n \to 0$ as $n \to \infty$. The functions $h_n = \tilde{f}_n - c_n$ tend to $f_K$ uniformly on compact sets in $\mathbb{D} \cap \{ \Im z < 0 \}$ and $h_n(z_0) = -f_K(z_0).$
Now let $u$ be a plurisubharmonic function on $\mathbb{C}^2$ that equals $-\infty$ on $\Gamma_f$. The function $u(z, h_n(z))$ is subharmonic on the domain $D_n$, and because $h_n(z)$ is bounded on $D_n$ independently of $n$, $u(z, h_n(z))$ is bounded by a constant $M$ independently of $n$.

Next we apply the two constants theorem and find

$$u(z_0, -f_K(z_0)) \leq M(1 - \omega(z_0, S, D_n)) + \max_{z \in S} u(z, h_n(z)) \omega(z_0, S, D_n) \to -\infty \quad \text{as } n \to \infty.$$  

Hence, $(\Gamma_f)^* \supset \Gamma_f \cup \Gamma_{-f}$.

To get the equality we will show that the sup-norm satisfies

$$\|g - g_n\|_{L^1}^{1/n} \to 0, \quad n \to \infty,$$

on compacta $L$ outside $K$. For this we write

$$|g - g_n| = |g_n| \prod_{k=n+1}^{\infty} \frac{z - a_k}{z - b_k} - 1.$$

The first factor is bounded by a constant $C$ depending on $L$. To estimate the second factor we let $\delta$ be the distance from $L$ to $K$ and write the factor as

$$\prod_{k=n+1}^{\infty} \left(1 + \frac{d_k}{z - b_k}\right) - 1 \leq \prod_{k=n+1}^{\infty} \left(1 + \frac{d_k}{\delta}\right) - 1 \leq \exp \left(\sum_{k=n+1}^{\infty} \frac{d_k}{\delta}\right) - 1.$$

Since $d_k < 4^{-kc_k}$ we see that

$$\prod_{k=n+1}^{\infty} \frac{z - a_k}{z - b_k} - 1 \leq \exp \left(\frac{4^{-nc_n}}{(1 - 4^{-c_n})\delta}\right) - 1.$$

Hence

$$\|g - g_n\|_{L^1}^{1/n} \leq 2 \left(\frac{C_L}{\delta}\right)^{1/n} 4^{-c_n}$$

when $n$ is sufficiently large and

$$\|g - g_n\|_{L^1}^{1/n} \to 0, \quad n \to \infty.$$

By Lemma 3.5, $\Gamma_g^* = \Gamma_g$ in $\mathbb{C}^2 \setminus (K \times \mathbb{C})$. Thus for any $(z_0, w_0)$ with $z_0 \in \mathbb{C} \setminus K$ and $w_0 \neq g(z_0)$, there is a function $u \in \text{PSH}(\mathbb{C}^2)$ such that $u|_{\Gamma_g} = -\infty$ and $u(z_0, w_0) \neq -\infty$. Then the function $v(z, w) = u(z, w^2)$ is equal to $-\infty$ on $\Gamma_f \cup \Gamma_{-f}$ and $v(z_0, \pm \sqrt{w_0}) \neq -\infty$. Hence $(\Gamma_f)^* \cap (\mathbb{C} \setminus K) \times \mathbb{C} = \Gamma_f \cup \Gamma_{-f}$. 

References


Department of Mathematics
215 Carnegie Hall
Syracuse University
Syracuse, NY 13244, U.S.A.
E-mail: eapolets@syr.edu

Korteweg–de Vries Institute for Mathematics
Plantage Muidergracht 24
1018 TV Amsterdam, The Netherlands
E-mail: janwieg@science.uva.nl

Received 27.10.2005