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Graphs with multiple sheeted pluripolar hulls

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Abstract. We study the pluripolar hulls of analytic sets. In particular, we show that hulls of graphs of analytic functions can be multiple sheeted and sheets can be separated by a set of zero dimension.

1. Introduction. One of the oldest interesting topics in complex analysis is the problem of analytic extensions: find the maximal analytic object containing a given one. For example, if f is an analytic function we are looking for its analytic continuation and if A is an irreducible analytic set we try to find another one of the same dimension containing A.

The counterpart of an analytic extension in pluripotential theory is the so-called *pluripolar hull.* There are two types of pluripolar hulls of a set A in a domain $D \subset \mathbb{C}^n$. Let $\mathrm{PSH}(D)$ be the set of all plurisubharmonic functions on D and $\mathrm{PSH}_0(D)$ the set of all negative functions from $\mathrm{PSH}(D)$. Define

$$A_D^* = \{z \in D : \forall h \in \mathrm{PSH}(D)\ \ h|_A = -\infty \Rightarrow h(z) = -\infty\},$$
$$A_D^- = \{z \in D : \forall h \in \mathrm{PSH}_0(D)\ \ h|_A = -\infty \Rightarrow h(z) = -\infty\}.$$

For example, if A is an analytic set in a pseudoconvex domain D, then every point of A has a neighborhood V where $A \cap V = \{h_1 = \cdots = h_k = 0\}$ and the functions h_k are holomorphic on this neighborhood. Hence $A \cap V = \{\log \max\{|h_1|, \ldots, |h_k|\} = -\infty\}$. In fact, $A_D^* = A$, because by [2, Cor. 1] there even exists $v \in \mathrm{PSH}(D)$ such that $A = \{v = -\infty\}$.

If such a $v \in \mathrm{PSH}(D)$ exists for A, we call A *pluricomplete* in D. In general, an analytic extension of A is contained in A_D^*.

In the case when $A = \Gamma_f$ is the graph of an analytic function f it was conjectured in [9] that the closure of the analytic extension of A coincides

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with A_D^*. However, A. Edigarian and the second author found in [3] an analytic function f on the unit disk $\mathbb{D}$ that does not extend analytically while the pluripolar hull of its graph is the graph of an analytic function defined on almost the whole plane (cf. [11]). Thus pluripolar hulls and analytic extensions may differ by large sets.

The pluripolar hull of the graph Γ_f of a holomorphic function $f(z)$ on a domain D may well be multi-sheeted over D. The principal value of $\sqrt{z}$ on $\{\Re z > 0\}$ provides the easiest example. Only recently did Zwonek [12], and, independently, Edlund and Jöricke [6], give examples of holomorphic functions f on their *domain of existence* D with the property that the pluripolar hull $(\Gamma_f)^*_{\mathbb{C}^2}$ is multi-sheeted over (parts of) D. In all these examples the topological codimension of ∂D is 1.

As we show in Section 2, this is an intrinsic property of analytic extensions: Let E be a closed subset of $\mathbb{C}^{n+1}$ and A be an analytic set of pure dimension n in $\mathbb{C}^{n+1} \setminus E$. If the set $A^*_{\mathbb{C}^{n+1}}$ is analytic, then $\dim E \geq 2n-1$. However, as we also show, there is a Cantor type set K and a holomorphic function $f(z)$ on $D = \mathbb{C} \setminus K$ such that $(\Gamma_f)^*_{\mathbb{C}^2}$ is 2-sheeted over D. So for pluripolar extensions sheets can be separated by a 0-dimensional cut and this demonstrates another principal difference between analytic extensions and pluripolar hulls.

As a by-product we obtain an example of a uniformly convergent sequence of holomorphic functions such that their pluripolar hulls do *not* converge to the pluripolar hull of the limit.

The set K should be sufficiently fat. Edigarian and the second author showed that if $D = \mathbb{C} \setminus K$, with K a polar compact set in $\mathbb{C}$, and if f is not extendible over K, then $(\Gamma_f)^*_{\mathbb{C}^2} \cap D \times \mathbb{C} = \Gamma_f$; see [4], and [5] for the fact that also over K the hull is at most single sheeted.

2. Pluripolar extensions. Let E be a closed set in a pseudoconvex domain $D \subset \mathbb{C}^n$. If $A \subset D \setminus E$ then, in general, $A^*_{D\setminus E}$ is a proper subset of $A^*_D \setminus E$. However, as the following statement shows, these sets coincide when E is pluripolar.

PROPOSITION 2.1. *If E is a closed pluripolar set in a pseudoconvex domain $D \subset \mathbb{C}^n$ and $A \subset D \setminus E$, then $A^*_D \setminus E = A^*_{D\setminus E}$.*

Proof. Let $\{D_j\}$ be an increasing sequence of relatively compact subdomains with $\bigcup_j D_j = D$. By [10, Thm. 2.4], $A^*_D = \bigcup_j (A \cap D_j)^-_{D_j}$. If $u \in \mathrm{PSH}_0(D_j \setminus E)$, then u extends as a negative plurisubharmonic function to D_j (see [8, Thm. 2.9.22]). Therefore, $(A \cap D_j)^-_{D_j} \setminus E = (A \cap D_j)^-_{D_j \setminus E}$. Since $(A \cap D_j)^-_{D_j\setminus E} \subset A^*_{D\setminus E}$, we see that $A^*_D \setminus E \subset A^*_{D\setminus E}$ and, consequently, $A^*_D \setminus E = A^*_{D\setminus E}$. ∎

The proposition below describes the situation when E is a closed set in a pseudoconvex domain $D \subset \mathbb{C}^n$, $A \subset D \setminus E$ is an analytic set and A^*_D is also analytic.

PROPOSITION 2.2. *Suppose that E is a closed set in a pseudoconvex domain $D \subset \mathbb{C}^n$ and $A \subset D \setminus E$ is an analytic set. If the set A^*_D is analytic then every irreducible component of A^*_D contains a component of A of the same dimension.*

Proof. Let X be an irreducible component of A^*_D. We represent A^*_D as $X \cup Y$, where Y is another analytic set in D and $\dim X \cap Y < \dim X$. As indicated in Section 1, the set Y is pluricomplete and $Y^*_D = Y$. It is easy to check that if $F, G \subset D$, then $(F \cup G)^*_D = F^*_D \cup G^*_D$. So if $B = A \cap (X \setminus Y)$ then $X \setminus Y \subset B^*_D$.

Suppose that $\dim X > \dim B$ and let R be the set of regular points of X. The set of singular points of X is analytic and, consequently, pluricomplete. By the argument above $X \setminus Y$ belongs to the pluripolar hull of the set $B' = B \cap R$.

We may assume that $0 \in R$ and let T be the tangent plane to R at 0. If p is a projection of $\mathbb{C}^n$ on T, then the set $p(B')$ is pluripolar in T and, consequently, there is a plurisubharmonic function u on T equal to $-\infty$ on $p(B')$. The set $p(R)$ has a non-empty interior in T and, therefore, there is a point $z_0 \in R$ such that $u(p(z_0)) \neq -\infty$. Then the function $v = u \circ p$ is plurisubharmonic on $\mathbb{C}^n$, equal to $-\infty$ on B' and $v(z_0) > -\infty$. Thus R does not belong to the pluripolar hull of the set B'. This contradiction proves the proposition. ∎

Suppose that A is a pluricomplete analytic set of pure dimension m in $D \setminus E$. If A^*_D is an analytic set in D and A is a proper subset of $A^*_D \setminus E$, then the set $E \cap A^*_D$ cuts A^*_D into several pieces and, therefore, its topological dimension must be at least $2m - 1$.

For example, let $D = \{(z, w) \in \mathbb{C}^2\}$ and $E = \{\Im z = 0,\ \Re z \geq 0\}$. Take a branch $w = f(z)$ of the function $w = \sqrt{z}$ over $\mathbb{C} \setminus E$ and let $A = \{(z, f(z)) : z \in \mathbb{C}\}$. The pluripolar hull of A in $\mathbb{C}^2 \setminus E$ is A because the function $\log |f(z) - z|$ is equal to $-\infty$ exactly on A. But $A^*_D = \{(z, w) : z = w^2\}$. In this example A^*_D is an analytic set and the set $A^*_D \cap E$ is the real curve $\{(x^2, x) : x \in \mathbb{R}\}$ which projects 2 to 1 except at 0 and its projection has dimension 1.

As the following statement shows, this is the minimal possible dimension.

PROPOSITION 2.3. *Let E be a closed set in a pseudoconvex domain $D \subset \mathbb{C}^n$ and let A be an irreducible analytic set of dimension m in $D \setminus E$ such that A^*_D is also analytic. If p is a projection of $\mathbb{C}^n$ onto $\mathbb{C}^m \subset \mathbb{C}^n$ such that*

$p(A)$ has a non-empty interior in $\mathbb{C}^m$ and the topological dimension of $p(E)$ is less than $2m-1$, then $A^*_D \setminus E = A$.

Proof. By Proposition 2.2 every irreducible component of A^*_D contains a component of A of the same dimension. Thus A^*_D is also irreducible and has dimension m. We denote by X the set of singular points of A^*_D. This set is analytic and its dimension is at most $m-1$. Hence the set $Y = A^*_D \setminus X$ is an irreducible complex submanifold of $D \setminus X$. Let $X' \subset Y$ be the set where the restriction of the projection p to Y has rank less than m. Since $p(A)$ has non-empty interior in $\mathbb{C}^m$, the set $Y' = Y \setminus X'$ is non-empty and relatively open in Y. The set X' is analytic and, therefore, its dimension is at most $m-1$. Hence the set Y' is an irreducible complex submanifold of $D \setminus (X \cup X')$ and the set $A' = A \cap Y'$ is a non-empty submanifold of $Y' \setminus E$ of dimension m.

Suppose that the topological dimension of $p(E)$ is at most $2m-2$. The projection p is locally homeomorphic on Y' and, therefore, the set $E \cap Y'$ also has dimension at most $2m-2$. Since E is closed the set A' is relatively open in Y'. The set $Y'' = Y' \setminus (A' \cup E)$ is also relatively open in Y'. Thus Y' is a topological manifold of dimension $2m$ and equals the union of two open sets A' and Y'' and a closed set E of dimension at most $2m-2$. By the Urysohn–Menger Theorem a set of dimension $2m-2$ cannot disconnect a $2m$-dimensional manifold (see [7, Ch. IV.5, Cor. 1]). Hence either A' or Y'' is empty. Since A' is not empty, $Y' \setminus E = A'$.

Since A is closed in $A^*_D \setminus E$, it contains $X \setminus E$ and $X' \setminus E$. So $A^*_D \setminus E = A$. ∎

When A is an analytic set in $D \setminus E$ we denote by A_E the intersection of E and the closure of A in D. If $A^*_D \setminus E \neq A$, we will say that A has a *non-trivial pluripolar extension* through E in D.

The following theorem lists some limitations on the set A_E when a non-trivial pluripolar extension takes place. Following [1] we call a set G in a domain $Y \subset \mathbb{C}^p$ *locally removable* if G is closed and for every open set V in Y every bounded holomorphic function f on $V \setminus G$ extends holomorphically to V.

THEOREM 2.4. *Suppose that D is a pseudoconvex domain in $\mathbb{C}^n$, E is a closed set in D and A is an analytic set of pure dimension m in $D \setminus E$ with a non-trivial pluripolar extension through E in D. Then the $(2m-1)$-Hausdorff measure of the set A_E is not equal to zero and if, additionally, $p : \mathbb{C}^n \to \mathbb{C}^m$ is a projection such that the restriction $p|_{A_E}$ is proper and $A \cap p^{-1}(z)$ is empty for all $z \in p(A_E)$, then $p(A_E)$ is not locally removable in $\mathbb{C}^m$.*

Proof. The set A is analytic in $D \setminus A_E$. If the $(2m-1)$-Hausdorff measure of E is zero, then by Shiffman's theorem (see [1, 4.4]) the closure $\overline{A}$ of A in D is an analytic set in D. Since the domain D is pseudoconvex there is a holomorphic function f on D such that $\overline{A} = \{f = 0\}$. Thus $A_D^* = \overline{A}$ and this contradicts the assumption that the extension is non-trivial.

In the second case, if $p(A_E)$ is locally removable in $\mathbb{C}^m$, then by the Proposition in [1, 18.1] the closure $\overline{A}$ of A is an analytic set in D as before and the same argument leads to a contradiction. ■

In our main example, $n = 2$, $m = 1$. In this case Theorem 2.4 can be reformulated as follows:

COROLLARY 2.5. *If in the assumptions of Theorem* 2.4, $n = 2$ *and* $m = 1$, *then the first Hausdorff measure of* A_E *is not zero and*, *under additional assumptions*, *the first Hausdorff measure of* $p(A_E)$ *is not zero.*

3. A holomorphic function on the complement of a Cantor type set with 2-sheeted hull

DEFINITION 3.1. A *Cantor type set* K is a compact perfect subset of $\mathbb{R}$ with empty interior.

It is a well known fact from elementary point set topology that such a K is homeomorphic to Cantor's middle thirds set. It is of the form $[a_0, b_0] \setminus \bigcup_{j=1}^{\infty} I_j$ where I_j are open intervals in $[a_0, b_0]$, $\overline{I}_j \cap \overline{I}_k = \emptyset$ if $j \neq k$ and $\bigcup_{j=1}^{\infty} I_j$ is dense in $[a, b]$. We can assume that the length of $I_j = (a_j, b_j)$ decreases with j.

It is useful to enumerate the set $\{a_j, b_j : j = 0, \ldots, n\}$ as $\{\alpha_{jn}, \beta_{jn}\}$ so that $\alpha_{0n} = a_0$, $\alpha_{jn} < \beta_{jn} < \alpha_{j+1,n}$ and $\beta_{nn} = b_0$. Note that $[a_0, b_0] \setminus \bigcup_{j=1}^{n} I_j = \bigcup_{j=0}^{n} [\alpha_{jn}, \beta_{jn}]$ and that $I_m \cap [\alpha_{jn}, \beta_{jn}] \neq \emptyset$ implies $I_m \subset [\alpha_{jn}, \beta_{jn}] \neq \emptyset$.

Let

$$g_n(z) = \frac{z - a_0}{z - b_0} \frac{z - b_1}{z - a_1} \cdots \frac{z - b_n}{z - a_n}.$$

Then

$$g_n(z) = \frac{z - \alpha_{0n}}{z - \beta_{0n}} \frac{z - \alpha_{1n}}{z - \beta_{1n}} \cdots \frac{z - \alpha_{nn}}{z - \beta_{nn}}.$$

Each fraction $\frac{z-\alpha_{jn}}{z-\beta_{jn}}$, $j = 0, 1, \ldots, n$, has a holomorphic branch $\sqrt{\frac{z-\alpha_{jn}}{z-\beta_{jn}}}$ of its square root outside $[\alpha_{jn}, \beta_{jn}]$ that equals 1 at infinity. Let

$$f_n(z) = \sqrt{\frac{z - \alpha_{0n}}{z - \beta_{0n}}} \sqrt{\frac{z - \alpha_{1n}}{z - \beta_{1n}}} \cdots \sqrt{\frac{z - \alpha_{nn}}{z - \beta_{nn}}} = \sqrt{g_n(z)}. \tag{3.1}$$

Then $f_n(\infty) = 1$, $f_n^2 = g_n$ and f_n is holomorphic on $G_n = (\mathbb{C} \setminus [a_0, b_0]) \cup \bigcup_{j=1}^{n} I_j$.

The maximal analytic extension of f_n is a branched 2-sheeted cover $X_n = \{(z, w) : w^2 = g_n(z)\}$ of $\mathbb{C}$ that branches over $\{a_j, b_j : j = 0, 1, \ldots, n\}$. The pluripolar hull $(\Gamma_{f_n})_{\mathbb{C}^2}^*$ equals X_n.

LEMMA 3.2. *Keeping the above notation, the sequence $\{g_n\}$ converges normally to an analytic function g on $\mathbb{C} \setminus K$. Moreover, the function g extends analytically over a point $x \in K$ if and only if for some $\alpha < x < \beta$ the length of the set $K \cap (\alpha, \beta)$ is zero.*

Proof. Let L be a compact set in $\mathbb{C} \setminus K$. Let us show that

$$\frac{z - a_0}{z - b_0} \prod_{j=1}^{\infty} \frac{z - b_j}{z - a_j} = \lim_{n\to\infty} g_n(z) \tag{3.2}$$

is uniformly convergent on L. There exists n_0 such that $L \subset G_n$ for $n > n_0$ and moreover, for some $\delta > 0$,

$$L \subset \{z : |z - a_j| > \delta,\ j = 0, 1, \dots\}.$$

Hence, for $z \in L$,

$$\left| \frac{z - b_j}{z - a_j} - 1 \right| = \left| \frac{b_j - a_j}{z - a_j} \right| \le \frac{b_j - a_j}{\delta}.$$

Since $\sum (b_j - a_j)$ is finite, the product in (3.2) converges uniformly on L to a function g that is holomorphic on $\mathbb{C} \setminus K$.

Suppose that g extends analytically over a point $x \in K$ so that g is analytic on $(\mathbb{C} \setminus K) \cup (\alpha, \beta)$. We may assume that $\alpha \in I_k$, $b_k \le x$, and $\beta \in I_m$, $a_m \ge x$, and

$$g_{1n}(z) = \prod \frac{z - \alpha_{jn}}{z - \beta_{jn}},$$

where the product runs over all j such that either $\beta_{jn} < \alpha$ or $\alpha_{jn} > \beta$. Let

$$g_{2n}(z) = \prod \frac{z - \alpha_{jn}}{z - \beta_{jn}},$$

where the product runs over all j such that $\alpha < \alpha_{jn}$ and $\beta_{jn} < \beta$. Then $g_n = g_{1n} g_{2n}$ and by the argument above the sequences $\{g_{1n}\}$ and $\{g_{2n}\}$ converge uniformly on compacta in $\mathbb{C} \setminus (K \setminus (\alpha, \beta))$ and $\mathbb{C} \setminus (K \cap (\alpha, \beta))$ respectively. We denote their respective limits by g_1 and g_2.

For the derivative of g_{2n} we find $g'_{2n}(\infty) = \sum (\beta_{jn} - \alpha_{jn}) = l_n$, where the sum runs over all j such that $\alpha < \alpha_{jn}$ and $\beta_{jn} < \beta$. Thus $g_{2n}(\infty)$ is equal to the length l_n of the intervals $(\alpha_{jn}, \beta_{jn})$ lying in (α, β) and $g'_2(\infty)$ is the length of the set $K \cap (\alpha, \beta)$. If this length is positive, then the function g_2 is not constant and, therefore, does not extend to $K \cap (\alpha, \beta)$.

If this length is 0 then for $z \in \mathbb{C}$ such that $|z - y| \ge 1$ for all $y \in (\alpha, \beta)$ we have

$$|1 - g_{2n}(z)| = \left| 1 - \prod \left(1 + \frac{\beta_{jn} - \alpha_{jn}}{z - \beta_{jn}} \right) \right| \le e^{l_n} - 1.$$

Hence the sequence $\{g_{2n}\}$ converges to 1 near ∞, $g_2 \equiv 1$ and g extends analytically over (α, β). ∎

LEMMA 3.3. *If $f = f_K$ and the length of K is positive, then the union $\overline{\Gamma}_f \cup \overline{\Gamma}_{-f}$ of the closures of the graphs of f and $-f$ is not an analytic set.*

Proof. If $A = \overline{\Gamma}_f \cup \overline{\Gamma}_{-f}$ is an analytic set, then there is a holomorphic function $h = h(z,w)$ on $\mathbb{C}^2$ such that $h \equiv 0$ on A. We have $A = \Gamma_f \cup \Gamma_{-f} \cup E$, where $E \subset K \times \mathbb{C}$.

Let us show that for every $z_0 \in K$ the analytic set $E_{z_0} = \{w : (z_0, w) \in E\} = A \cap \{z_0\} \times \mathbb{C}$ consists of at most two points. If it contains three points, then at least two of them belong to, say, $\overline{\Gamma}_f$. Thus there are sequences $\{z_j\}$ and $\{z'_j\}$ converging to z_0 such that the sequences $\{f(z_j)\}$ and $\{f(z'_j)\}$ have distinct limits. Connecting each z_j and z'_j by small curves in $\mathbb{C} \setminus K$ and looking at their limits we see that the cluster set of f at z_0 contains a continuum. Hence, $E_{z_0} = \mathbb{C}$ and $h(z_0, w) \equiv 0$.

From the Taylor expansion of h we immediately derive that $h(z,w) = (z-z_0)^n h_1(z_0, w)$, where h_1 is holomorphic on $\mathbb{C}^2$ and $h_1(z_0,w) \not\equiv 0$. But for every point $w \in \mathbb{C}$ there is a sequence of z_j converging to z_0 such that, say, $f(z_j)$ converges to w. Since $h(z_j, f(z_j)) = 0$ we see that $h_1(z_0, w) = 0$. This contradiction shows that E_{z_0} has at most two points and the intersection of $\overline{\Gamma}_f$ or $\overline{\Gamma}_{-f}$ with E consists of at most one point.

It follows that f extends continuously to K. Since K lies on the real line, f extends holomorphically to $\mathbb{C}$; but this impossible by Lemma 3.2. ∎

EXAMPLE 3.4. If the set K has Lebesgue measure 0, then

$$\lim_{n\to\infty} g_n(z) = 1$$

uniformly on any compact set L not meeting K. It follows that $f \equiv 1$ and $(\Gamma_f)^*_{\mathbb{C}^2} = \{(z,1)\}$. But the Hausdorff limit of the sets X_n over D equals $\{(z,w) : w = 1 \text{ or } -1\}$.

We will need the next lemma, whose proof is similar to the proof of Theorem 2.1 in [4].

LEMMA 3.5. *Let f be a holomorphic function on a domain $V \subset \mathbb{C}^N$ containing a closed ball B and let $\{r_n\}_n$ be a sequence of rational functions of degree n with poles outside V and such that the sup-norm $\|f - r_n\|_B^{1/n}$ tends to 0 as $n \to \infty$. Then there is a plurisubharmonic function v on $\mathbb{C}^{N+1}$ such that $\{v = -\infty\} \cap (V \times \mathbb{C}) = \Gamma_f$. Thus, $(\Gamma_f)^*_{\mathbb{C}^{N+1}} \cap (V \times \mathbb{C}) = \Gamma_f$.*

Proof. The functions r_n are ratios of polynomials p_n and q_n of degree n. We may assume that B is the closed unit ball centered at the origin and $\|q_n\|_B = 1$. Then $|q_n(z)| \le \max\{1, |z|^n\}$, $\|p_n\|_B$ does not exceed some constant C and $|p_n(z)| \le C \max\{1, |z|^n\}$.

Consider the plurisubharmonic functions

$$u_n(z,w) = \frac{1}{n} \log |q_n(z) w - p_n(z)|$$

on $\mathbb{C}^{N+1}$. From the estimates on p_n and q_n there is a constant C_1 such that $u_n(z,w) \le 2\log|z| + \log|w| + C_1$ when $|z| \ge 1$ and $u_n(z,w) \le \log|w| + C_1$ when $|z| \le 1$.

We take $z_n \in B$ such that $q_n(z_n) = a_n$, $|a_n| = 1$. Let $w_n = (p_n(z_n)+1)/a_n$. Then $|w_n| \le C+1$ and $u_n(z_n,w_n) = 0$. If B_n is a ball in $\mathbb{C}^{N+1}$ centered at (z_n,w_n) and of radius $r_n = C+5$, then B_n contains the unit ball B' centered at the origin and

$$\int_{B_n} u_n\,dV \ge cu_n(z_n,w_n) = 0.$$

It is immediate from the upper estimates on u_n that there is a constant C_2 such that

$$\int_{B'} u_n\,dV \ge C_2.$$

By our assumption there is a sequence $\{d_n\}$ converging to ∞ such that

$$u_n(z,f_n(z)) = \frac{1}{n}\log|q_n(z)| + \frac{1}{n}\log\left|f(z) - \frac{p_n(z)}{q_n(z)}\right| \le -d_n$$

when $|z| \le 1$.

Let us take a sequence $\{c_n\}$ of positive reals such that $\sum c_n = 1$ while $\sum c_n d_n = \infty$. Let

$$v(z,w) = \sum_{n=1}^{\infty} c_n \max\{u_n(z,w), d_n\}.$$

Since

$$\int_{B} v\,dV \ge C_2,$$

we see that $v \not\equiv -\infty$ and, therefore, it is a plurisubharmonic function on $\mathbb{C}^{N+1}$. Clearly, $v(z,f(z)) = -\infty$ when $|z| \le 1$. Therefore, $v = -\infty$ on Γ_f.

The zeros of the polynomials q_n are in $\mathbb{C}^N \setminus V$. Hence the functions $h_n(z) = n^{-1}\log|q_n(z)|$ are harmonic on V and uniformly bounded above on compacta. So if $z \in V$ and $\liminf h_n(z) = -\infty$, then there is a subsequence $\{n_k\}$ such that $\lim h_{n_k}(z) = -\infty$. Therefore, the functions h_{n_k} converge to $-\infty$ uniformly on compacta in V. But $h_{n_k}(z_{n_k}) = 0$, and this contradiction tells us that $\liminf h_n(z) > -\infty$. So if $w \ne f(z)$, then

$$v(z,w) \ge \sum_{n=1}^{\infty} c_n h_n(z) + \sum_{n=1}^{\infty} \frac{c_n}{n}\log\left|w - \frac{p_n(z)}{q_n(z)}\right| > -\infty.$$

Hence $\{v = -\infty\} \cap (V \times \mathbb{C}) = \Gamma_f$ and $(\Gamma_f)^*_{\mathbb{C}^{N+1}} \cap (V \times \mathbb{C}) = \Gamma_f$. ∎

Now we can present our main example.

THEOREM 3.6. *There exists a Cantor type set K, obtained by deleting intervals $I_i = (a_i, b_i)$ from $[-1,1]$, such that the function $f = f_K$ given by*

Lemma 3.3 *has the following property*:

$$(\varGamma_f)^*_{\mathbb{C}^2} \cap (\mathbb{C} \setminus K) \times \mathbb{C} = \varGamma_f \cup \varGamma_{-f}.$$

Proof. We will construct K by deleting a sequence of open intervals (a_i, b_i) from the interval $[-1, 1]$. For convenience, set $a_0 = 1$, $b_0 = -1$. In order to choose the intervals appropriately, we have to construct certain subdomains D_n in the open unit disk $\mathbb{D}$ in the process. The domains D_n will contain the set $\mathbb{D} \cap \{|\Im z| > 1/2\}$. Thus the closed discs $S = \{|z+3i/4| \le 1/8\}$ and $X = \{|z - 3i/4| \le 1/8\}$ will be contained in D_n.

For a compact set F in a domain $D \subset \mathbb{C}$ let

$$\omega(z, F, D) = -\sup\{h(z) : h \in \mathrm{PSH}_0(D), \limsup_{w \to K} h(w) \le -1\}$$

be the harmonic measure of F in D. We set $D_0 = \mathbb{D}$ and observe that $\omega(z, S, D_0) > c_0$ for some positive c_0.

Let $\{c_n\}$ be a sequence of positive real numbers converging to ∞. Suppose that the intervals $I_1, \ldots, I_n$ have been chosen. We take as a_{n+1} the midpoint of the largest interval in their complement. Next take $b_{n+1} > a_{n+1}$ so small that the interval $[a_{n+1}, b_{n+1}]$ does not intersect the intervals $I_1, \ldots, I_n$, $d_{n+1} = b_{n+1} - a_{n+1} < 4^{-(n+1)c_{n+1}}$ and, moreover,

$$\omega(z, S, D_{n+1}) > c_0, \qquad z \in X.$$

Here we define $D_{n+1} = D_n \setminus \mathbb{D}(a_{n+1}, (b_{n+1} - a_{n+1})2^{n+1})$, where $\mathbb{D}(a, r)$ is the open disk centered at a and of radius r.

Observe that for $j \le n$,

$$\left| \frac{z - b_j}{z - a_j} - 1 \right| = \left| \frac{d_j}{z - a_j} \right| < \frac{1}{2^j}$$

on D_n. It follows that

$$\prod_{j=1}^{n} \frac{z - b_j}{z - a_j}$$

is bounded on D_n independently of n.

Let $z_0 \in X$. We will show that $(z_0, -f(z_0)) \in (\varGamma_f)^*_{\mathbb{C}^2}$. Then $\varGamma_{-f}$ is also in the hull and we are done. Consider the function $\widetilde{f}_n$ defined on $\mathbb{D} \setminus \bigcup_{j=1}^{n} I_j$ by

$$\widetilde{f}_n(z) = \begin{cases} f_n(z) & \text{if } \Im z < 0, \\ -f_n(z) & \text{if } \Im z > 0, \\ \lim_{y \uparrow 0} f(x + iy) & \text{if } x \in [-1, 1] \setminus \bigcup I_j. \end{cases}$$

The function $\widetilde{f}_n$ is holomorphic. Let $c_n = \widetilde{f}_n(z_0) + f_K(z_0)$. Then $c_n \to 0$ as $n \to \infty$. The functions $h_n = \widetilde{f}_n - c_n$ tend to f_K uniformly on compact sets in $\mathbb{D} \cap \{\Im z < 0\}$ and $h_n(z_0) = -f_K(z_0)$.

Now let u be a plurisubharmonic function on $\mathbb{C}^2$ that equals $-\infty$ on Γ_f. The function $u(z,h_n(z))$ is subharmonic on the domain D_n, and because $h_n(z)$ is bounded on D_n independently of n, $u(z,h_n(z))$ is bounded by a constant M independently of n.

Next we apply the two constants theorem and find

$$\begin{aligned} u(z_0,-f_K(z_0)) &\le M(1-\omega(z_0,S,D_n) \\ &\quad + \max_{z\in S} u(z,h_n(z))\omega(z_0,S,D_n) \to -\infty \quad \text{as } n\to\infty. \end{aligned} \tag{3.3}$$

Hence, $(\Gamma_f)^*_{\mathbb{C}^2} \supset \Gamma_f \cup \Gamma_{-f}$.

To get the equality we will show that the sup-norm satisfies

$$\|g-g_n\|_L^{1/n} \to 0, \qquad n\to\infty,$$

on compacta L outside K. For this we write

$$|g-g_n| = |g_n| \left| \prod_{k=n+1}^{\infty} \frac{z-a_k}{z-b_k} - 1 \right|.$$

The first factor is bounded by a constant C depending on L. To estimate the second factor we let δ be the distance from L to K and write the factor as

$$\left| \prod_{k=n+1}^{\infty} \left(1+\frac{d_k}{z-b_k}\right) - 1 \right| \le \prod_{k=n+1}^{\infty} \left(1+\frac{d_k}{\delta}\right) - 1 \le \exp\left(\sum_{k=n+1}^{\infty} \frac{d_k}{\delta}\right) - 1.$$

Since $d_k < 4^{-kc_k}$ we see that

$$\left| \prod_{k=n+1}^{\infty} \frac{z-a_k}{z-b_k} - 1 \right| \le \exp\left(\frac{4^{-nc_n}}{(1-4^{-c_n})\delta}\right) - 1.$$

Hence

$$\|g-g_n\|_L^{1/n} \le 2\left(\frac{C_L}{\delta}\right)^{1/n} 4^{-c_n}$$

when n is sufficiently large and

$$\|g-g_n\|_L^{1/n} \to 0, \quad n\to\infty.$$

By Lemma 3.5, $\Gamma_g^* = \Gamma_g$ in $\mathbb{C}^2 \setminus (K\times\mathbb{C})$. Thus for any (z_0,w_0) with $z_0 \in \mathbb{C}\setminus K$ and $w_0 \ne g(z_0)$, there is a function $u\in \mathrm{PSH}(\mathbb{C}^2)$ such that $u|_{\Gamma_g} = -\infty$ and $u(z_0,w_0) \ne -\infty$. Then the function $v(z,w) = u(z,w^2)$ is equal to $-\infty$ on $\Gamma_f \cup \Gamma_{-f}$ and $v(z_0, \pm\sqrt{w_0}) \ne -\infty$. Hence $(\Gamma_f)^*_{\mathbb{C}^2} \cap (\mathbb{C}\setminus K)\times\mathbb{C} = \Gamma_f\cup\Gamma_{-f}$. ∎

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