

An extension of Schwick's theorem for normal families

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Abstract. In this paper, the definition of the derivative of meromorphic functions is extended to holomorphic maps from a plane domain into the complex projective space. We then use it to study the normality criteria for families of holomorphic maps. The results obtained generalize and improve Schwick's theorem for normal families.

1. Introduction. The concept of a normal family was introduced by P. Montel [M]. Perhaps the most celebrated criterion for normality in one complex variable is Montel's result which can be stated as follows (see [Y, pp. 53–54]).

THEOREM A. *Let \mathcal{F} be a family of meromorphic functions on a plane domain D which omit three distinct values a, b, c in the extended complex plane. Then \mathcal{F} is normal on D .*

In [D], Theorem A was extended to holomorphic maps into the complex projective space.

THEOREM B ([D]). *Let \mathcal{F} be a family of holomorphic maps of a domain $D \subset \mathbb{C}$ into $\mathbb{P}^N(\mathbb{C})$. Let H_1, \dots, H_{2N+1} be hyperplanes in $\mathbb{P}^N(\mathbb{C})$ in general position. If for each $f \in \mathcal{F}$, f omits H_1, \dots, H_{2N+1} , then \mathcal{F} is normal on D .*

Recall that two nonconstant meromorphic functions f and g share the value a if $f^{-1}(a) = g^{-1}(a)$ as sets (ignoring multiplicities). There are many results concerning this notion in value distribution theory. For example, R. Nevanlinna proved that two meromorphic functions on the complex plane that share five distinct values coincide identically [N, pp. 109–110]. W. Schwick seems to have been the first to draw a connection between normality criteria and shared values.

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THEOREM C ([S]). *Let \mathcal{F} be a family of meromorphic functions on a plane domain D , and a, b, c be three distinct finite complex numbers. If for every $f \in \mathcal{F}$, f and f' share the values a, b, c , then \mathcal{F} is normal on D .*

In [LLP], the authors proved that for two families of functions which share four values, if one is normal, so is the other. Recently, we extended this result to holomorphic maps of a plane domain into the complex projective space [YFP].

In this paper, by applying the generalized Zalcman lemma attributed to Aladro and Krantz [AK], we shall prove some normality criteria for families of holomorphic maps into $\mathbb{P}^N(\mathbb{C})$. This generalizes both Theorem B and Theorem C in some degree. First, we should introduce a sort of derivative for holomorphic maps which possesses similar properties to the derivative of meromorphic functions.

2. Preliminaries and results. We start with relevant notions and definitions. For details see [F1], [R, pp. 99–102], [F2].

2.1. Recall that the N -dimensional complex projective space is $\mathbb{P}^N(\mathbb{C}) = \mathbb{C}^{N+1} - \{0\} / \sim$, where $(a_0, \dots, a_N) \sim (b_0, \dots, b_N)$ if and only if $(a_0, \dots, a_N) = \lambda(b_0, \dots, b_N)$ for some $\lambda \in \mathbb{C}$. We denote by $[a_0 : \dots : a_N]$ the equivalence class of (a_0, \dots, a_N) .

Let H_1, \dots, H_q ($q \geq N+1$) be hyperplanes in $\mathbb{P}^N(\mathbb{C})$ which are given by

$$H_j = \{[x_0 : \dots : x_N]; a_0^j x_0 + \dots + a_N^j x_N = 0\},$$

with nonzero coefficient vectors $\mathbf{a}_j = (a_0^j, \dots, a_N^j) \in \mathbb{C}^{N+1}$ (for $j = 1, \dots, q$). We say that H_1, \dots, H_q are *in general position* if for any injective map $\phi : \{0, 1, \dots, N\} \rightarrow \{1, \dots, q\}$, $\mathbf{a}_{\phi(0)}, \dots, \mathbf{a}_{\phi(N)}$ are linearly independent. When $N = 1$ this just means that H_1, \dots, H_q are mutually distinct points on the Riemann sphere.

2.2. Let D be a domain in \mathbb{C} , $f : D \rightarrow \mathbb{P}^N(\mathbb{C})$ be a holomorphic map and U be an open set in D . Any holomorphic map $\mathbf{f} : U \rightarrow \mathbb{C}^{N+1}$ such that $\mathbb{P}(\mathbf{f}(z)) \equiv f(z)$ in U is called a *representation* of f on U , where $\mathbb{P} : \mathbb{C}^{N+1} - \{0\} \rightarrow \mathbb{P}^N(\mathbb{C})$ is the standard quotient map.

DEFINITION 2.1. For an open subset U of D we call $\mathbf{f} = (f_0, \dots, f_N)$ a *reduced representation* of f on U if f_0, \dots, f_N are holomorphic functions on U without common zeros.

Let $H = \{[x_0 : \dots : x_N]; a_0 x_0 + \dots + a_N x_N = 0\}$ be a hyperplane in $\mathbb{P}^N(\mathbb{C})$. We write

$$\|H\| := \max_{0 \leq i \leq N} |a_i|.$$

Throughout, we only consider normalized hyperplane representations so that $\|H\| = 1$.

Next, for any reduced representation \mathbf{f} of a holomorphic map f , we define the holomorphic function

$$\langle \mathbf{f}(z), H \rangle := \sum_{i=0}^N a_i f_i(z),$$

and put

$$\|\mathbf{f}(z)\| := \left\{ \sum_{i=0}^N |f_i(z)|^2 \right\}^{1/2}.$$

Moreover, we write f instead of \mathbf{f} when the properties are independent of the choice of a reduced representation, for example, we can consider the function $\frac{|(f(z), H)|}{\|f(z)\| \cdot \|H\|}$.

REMARK 2.1. As is easily seen, if both $\mathbf{f}_j : U_j \rightarrow \mathbb{C}^{N+1}$ are reduced representations of f for $j = 1, 2$ with $U_1 \cap U_2 \neq \emptyset$ then there is a holomorphic function $h (\neq 0) : U_1 \cap U_2 \rightarrow \mathbb{C}$ such that $\mathbf{f}_2 = h\mathbf{f}_1$ on $U_1 \cap U_2$.

2.3. Next, we recall the definition of a normal family.

DEFINITION 2.2. A family \mathcal{F} of holomorphic maps of a domain Ω in \mathbb{C}^m into $\mathbb{P}^N(\mathbb{C})$ is said to be *normal* on Ω if any sequence in \mathcal{F} contains a subsequence which converges uniformly on compact subsets of Ω to a holomorphic map of Ω into $\mathbb{P}^N(\mathbb{C})$; and \mathcal{F} is said to be *normal at a point* a in Ω if \mathcal{F} is normal on some neighborhood of a in Ω .

Using the Fubini–Study metric form on $\mathbb{P}^N(\mathbb{C})$, we see that a sequence $\{f_n\}_{n=1}^\infty$ of holomorphic maps of D into $\mathbb{P}^N(\mathbb{C})$ converges uniformly on compact subsets of D to a holomorphic map f if and only if, for any $a \in D$, each f_n has a reduced representation

$$\mathbf{f}_n = (f_{n0}, f_{n1}, \dots, f_{nN})$$

on some fixed neighborhood U of a in D such that $\{f_{ni}\}_{n=1}^\infty$ converges uniformly on compact subsets of U to a holomorphic function f_i (for $i = 0, 1, \dots, N$) on U with the property that

$$\mathbf{f} = (f_0, f_1, \dots, f_N)$$

is a reduced representation of f on U .

2.4. Let $f = [f_0 : \dots : f_N]$ be a holomorphic map of D into $\mathbb{P}^N(\mathbb{C})$, $\mu \in \{0, \dots, N\}$ with $f_\mu \not\equiv 0$, and $d(z)$ be a holomorphic function such that

$$f_\mu^2/d \quad \text{and} \quad W(f_\mu, f_i)/d \quad (i = 0, 1, \dots, N; i \neq \mu)$$

are holomorphic functions without common zeros. Here, as usual,

$$W(f_\mu, f_i) = \begin{vmatrix} f_\mu & f_i \\ f'_\mu & f'_i \end{vmatrix}$$

denotes the Wronskian of f_μ and f_i .

Our definition of an extension of the derivative of meromorphic functions is as follows.

DEFINITION 2.3 (extension of derivative). We call the holomorphic maps induced by the map

$$(W(f_\mu, f_0), \dots, W(f_\mu, f_{\mu-1}), f_\mu^2, W(f_\mu, f_{\mu+1}), \dots, W(f_\mu, f_N)) : D \rightarrow \mathbb{C}^{N+1}$$

the μ th derived holomorphic map of f and write

$$\nabla_\mu f = [W(f_\mu, f_0)/d : \dots : W(f_\mu, f_{\mu-1})/d : f_\mu^2/d : \\ W(f_\mu, f_{\mu+1})/d : \dots : W(f_\mu, f_N)/d].$$

For simplicity, we also write $\nabla_0 f$ as ∇f .

REMARK 2.2. The definition of $\nabla_\mu f$ does not depend on the choice of a reduced representation of f .

REMARK 2.3. When $N = 1$, ∇f corresponds exactly to the derivative of the meromorphic function f_1/f_0 .

The main result of this paper is the following theorem.

THEOREM 2.1. Let \mathcal{F} be a family of holomorphic maps of a domain D in \mathbb{C} into $\mathbb{P}^N(\mathbb{C})$, H_1, \dots, H_{2N+1} be hyperplanes in $\mathbb{P}^N(\mathbb{C})$ in general position, and δ be a real number with $0 < \delta < 1$. Suppose that for each $f \in \mathcal{F}$ the following conditions are satisfied:

- (i) If $\nabla f(z) \in H_j$, then $f(z) \in H_j$, for $j = 1, \dots, 2N + 1$.
- (ii) If $f(z) \in \bigcup_{j=1}^{2N+1} H_j$, then

$$\frac{|\langle f(z), H_0 \rangle|}{\|f(z)\| \cdot \|H_0\|} \geq \delta,$$

where $H_0 = \{x_0 = 0\}$ is a coordinate hyperplane.

- (iii) If $f(z) \in \bigcup_{j=1}^{2N+1} H_j$, then

$$\frac{|\langle \nabla f(z), H_k \rangle|}{|f_0(z)|^2} \leq \frac{1}{\delta}$$

for $k = 1, \dots, 2N + 1$.

Then \mathcal{F} is normal on D .

REMARK 2.4. Condition (ii) in Theorem 2.1 implies that no hyperplane in $\{H_j\}_{j=1}^{2N+1}$ is H_0 .

We note that all the proofs of this theorem work also when the map is $\nabla_\mu f$ instead of ∇f . In particular when $N = 1$, condition (ii) of Theorem 2.1 holds naturally for some δ . We immediately have the following corollary which contains Theorem C.

COROLLARY 2.2. *Let \mathcal{F} be a family of meromorphic functions on a plane domain D , a_1, a_2, a_3 be three distinct finite complex numbers, and M be a positive real number. Suppose that for each $f \in \mathcal{F}$ the following conditions are satisfied:*

- (i) *If $f'(z) = a_j$, then $f(z) = a_j$, for $j = 1, 2, 3$.*
- (ii) *If $f(z) = a_j$, then $|f'(z)| \leq M$, for $j = 1, 2, 3$.*

Then \mathcal{F} is normal on D .

According to the uniqueness theory of meromorphic maps from \mathbb{C}^m into $\mathbb{P}^N(\mathbb{C})$, here we give the definition of sharing a hyperplane [F3, CRY].

DEFINITION 2.4. *Let f and g be two meromorphic maps from a domain in \mathbb{C}^n into $\mathbb{P}^N(\mathbb{C})$, and H be a hyperplane in $\mathbb{P}^N(\mathbb{C})$. We say f and g share the hyperplane H if $f^{-1}(H) = g^{-1}(H)$ and $f = g$ on $f^{-1}(H)$.*

We have the following improvement of Theorem C.

THEOREM 2.3. *Let \mathcal{F} be a family of holomorphic maps of a domain D in \mathbb{C} into $\mathbb{P}^N(\mathbb{C})$, H_1, \dots, H_{2N+1} be hyperplanes in $\mathbb{P}^N(\mathbb{C})$ in general position, and δ be a real number with $0 < \delta < 1$. Suppose that for each $f \in \mathcal{F}$ the following conditions are satisfied:*

- (i) *The maps f and ∇f share H_j on D for $j = 1, \dots, 2N + 1$.*
- (ii) *If $f(z) \in \bigcup_{j=1}^{2N+1} H_j$, then $\frac{|\langle f(z), H_0 \rangle|}{\|f(z)\| \cdot \|H_0\|} \geq \delta$, where $H_0 = \{x_0 = 0\}$.*

Then \mathcal{F} is normal on D .

3. Proofs. The following is the general version of Zalcman's lemma.

LEMMA 3.1 ([AK]). *Let \mathcal{F} be a family of holomorphic maps of a domain Ω in \mathbb{C}^m into $\mathbb{P}^N(\mathbb{C})$. The family \mathcal{F} is not normal on Ω if and only if there exist sequences $\{f_n\} \subset \mathcal{F}$, $\{z_n\} \subset \Omega$ with $z_n \rightarrow z_0 \in \Omega$, $\{\varrho_n\}$ with $\varrho_n > 0$ and $\varrho_n \rightarrow 0$ and $\{e_n\} \subset \mathbb{C}^m$ Euclidean unit vectors, such that*

$$g_n(\xi) := f_n(z_n + \varrho_n e_n \xi)$$

converges uniformly on compact subsets of \mathbb{C} to a nonconstant holomorphic map g of \mathbb{C} into $\mathbb{P}^N(\mathbb{C})$.

The degenerate second main theorem in Nevanlinna theory shows the following fact.

LEMMA 3.2 (see [R, p. 141]). *Let $f : \mathbb{C} \rightarrow \mathbb{P}^N(\mathbb{C})$ be a holomorphic map, and H_1, \dots, H_q ($q \geq 2N + 1$) be hyperplanes in $\mathbb{P}^N(\mathbb{C})$ in general position. If for each $j = 1, \dots, q$, either $f(\mathbb{C})$ is contained in H_j , or $f(\mathbb{C})$ omits H_j , then f is constant.*

Proof of Theorem 2.1. Suppose \mathcal{F} is not normal on D . Then by Lemma 3.1, there exist points $z_n \rightarrow z_0 \in D$, positive numbers $\varrho_n \rightarrow 0$ and holomorphic maps $f_n \in \mathcal{F}$ such that

$$g_n(\xi) := f_n(z_n + \varrho_n \xi)$$

converges uniformly on compact subsets of \mathbb{C} to a nonconstant holomorphic map $g : \mathbb{C} \rightarrow \mathbb{P}^N(\mathbb{C})$. Set

$$H_j = \{[x_0 : \cdots : x_N]; a_0^j x_0 + \cdots + a_N^j x_N = 0\} \quad \text{for } j = 1, \dots, 2N + 1.$$

Since H_1, \dots, H_{2N+1} are in general position, by Lemma 3.2 we may assume that the holomorphic function $\langle g, H_1 \rangle$ does not vanish identically and has at least one zero in \mathbb{C} . Take any fixed zero of $\langle g, H_1 \rangle$, say ξ_0 , that is,

$$(3.1) \quad \langle g(\xi_0), H_1 \rangle = 0.$$

There exists a small neighborhood $U := U(\xi_0)$ of ξ_0 such that $\langle g, H_1 \rangle$ has no other zeros in U . Moreover, each g_n has a reduced representation

$$\mathbf{g}_n(\xi) = (g_{n0}(\xi), \dots, g_{nN}(\xi)) = (f_{n0}(z_n + \varrho_n \xi), \dots, f_{nN}(z_n + \varrho_n \xi))$$

on U such that $\{g_{ni}\}_{n=1}^\infty$ converges uniformly on U to a holomorphic function g_i (for $i = 0, 1, \dots, N$) on U with the property that

$$\mathbf{g} = (g_0, \dots, g_N)$$

is a reduced representation of g on U . Thus, $\sum_{i=0}^N a_i^1 g_{ni}(\xi)$ converges uniformly on U to $\sum_{i=0}^N a_i^1 g_i(\xi)$.

By the argument principle we may find a sequence $\{\xi_n\}$ tending to ξ_0 such that, for large n , $\sum_{i=0}^N a_i^1 g_{ni}(\xi_n) = 0$, that is,

$$(3.2) \quad \sum_{i=0}^N a_i^1 f_{ni}(z_n + \varrho_n \xi_n) = 0.$$

We have

$$|g_{n0}(\xi_n)| \geq \delta \|\mathbf{g}_n(\xi_n)\|,$$

by (3.2) and condition (ii). Setting $n \rightarrow \infty$ yields $|g_0(\xi_0)| \geq \delta \|\mathbf{g}(\xi_0)\| > 0$. Thus, $g_0(\xi_0) \neq 0$.

Shrinking the neighborhood U if necessary, we may assume that $g_0(\xi) \neq 0$ and thus $g_{n0}(\xi) \neq 0$ on U when n is sufficiently large. Then for each such n ,

$$(g_{n0}^2(\xi), W(g_{n0}, g_{n1})(\xi), \dots, W(g_{n0}, g_{nN})(\xi))$$

is a reduced representation of $\nabla g_n(\xi)$ on U . Note that

$$(3.3) \quad g'_{ni}(\xi) = \varrho_n f'_{ni}(z_n + \varrho_n \xi)$$

for $i = 0, 1, \dots, N$.

Now, by (3.2), (3.3) and condition (iii) we have

$$\left| a_0^k + \sum_{i=1}^N \frac{W(g_{n0}, g_{ni})(\xi_n)}{\varrho_n g_{n0}^2(\xi_n)} \right| \leq \frac{1}{\delta},$$

and hence

$$(3.4) \quad \left| \varrho_n a_0^k + \sum_{i=1}^N \frac{W(g_{n0}, g_{ni})(\xi_n)}{g_{n0}^2(\xi_n)} \right| \leq \frac{\varrho_n}{\delta}$$

for $k = 2, \dots, 2N + 1$. Put

$$\varphi_k(\xi) = \sum_{i=1}^N a_i^k \frac{W(g_0, g_i)(\xi)}{g_0^2(\xi)}, \quad \xi \in U.$$

Letting $n \rightarrow \infty$ in (3.4), we obtain $\varphi_k(\xi_0) = 0$ for $k = 2, \dots, 2N + 1$.

CLAIM. *There are at most N hyperplanes in $\{H_k\}_{k=2}^{2N+1}$ such that $\varphi_k(\xi) \equiv 0$.*

Suppose, to the contrary, that $\varphi_2(\xi) \equiv \dots \equiv \varphi_{N+2}(\xi) \equiv 0$. Since

$$\varphi_j(\xi) = \left(\sum_{i=1}^N a_i^j \frac{g_i}{g_0} \right)' \equiv 0,$$

there exist complex numbers c_j such that

$$(3.5) \quad \sum_{i=1}^N a_i^j \frac{g_i}{g_0} \equiv c_j \quad \text{for } j = 2, \dots, N + 2.$$

Since H_1, \dots, H_{2N+1} are in general position, linear algebra shows that either the system of equations

$$\sum_{i=1}^N a_i^j x_i \equiv c_j \quad \text{for } j = 2, \dots, N + 2$$

has no solution, or the solution is unique. This means g is constant from (3.5), a contradiction. The claim is true.

We now return to the proof of the theorem. By the claim, we can suppose $\varphi_k(\xi) \not\equiv 0$ for $k = 2, \dots, N + 1$. Then for each $k \in \{2, \dots, N + 1\}$,

$$\varrho_n a_0^k + \varrho_n \sum_{i=1}^N a_i^k \frac{W(f_{n0}, f_{ni})}{f_{n0}^2}(z_n + \varrho_n \xi)$$

converges uniformly to $\varphi_k(\xi)$ on U . Again, the argument principle implies that there exist $\xi_n^* \rightarrow \xi_0$ such that

$$\langle \nabla f_n(z_n + \varrho_n \xi_n^*), H_k \rangle = 0.$$

It follows from condition (i) that

$$\langle f_n(z_n + \varrho_n \xi_n^*), H_k \rangle = 0.$$

Setting $n \rightarrow \infty$ yields

$$\langle g(\xi_0), H_k \rangle = 0 \quad \text{for } k = 2, \dots, N + 1.$$

Noting (3.1), we have

$$\langle g(\xi_0), H_k \rangle = 0 \quad \text{for } k = 1, \dots, N + 1,$$

which contradicts the hyperplanes $\{H_k\}_{k=1}^{N+1}$ being in general position. Hence, \mathcal{F} is normal on D . ■

Proof of Theorem 2.3. We will show that the assumptions imply the conditions in Theorem 2.1. It suffices to prove that condition (iii) of Theorem 2.1 holds. For this, fix a map $f \in \mathcal{F}$ and a point $z_0 \in \bigcup_{j=1}^{2N+1} f^{-1}(H_j)$. Then there exists a reduced representation of f , say, $\mathbf{f}(z) = (f_0(z), \dots, f_N(z))$ on some fixed neighborhood U of z_0 in D . By condition (ii) of Theorem 2.3, $f_0(z_0) \neq 0$. Then

$$(f_0^2(z), W(f_0, f_1)(z), \dots, W(f_0, f_N)(z))$$

is a reduced representation of $\nabla f(z)$ on U .

Using condition (i) and Definition 2.4, we obtain

$$(3.6) \quad W(f_0, f_i)(z_0) = f_0(z_0) \cdot f_i(z_0) \quad \text{for } i = 1, \dots, N.$$

If we set

$$H_k = \{a_0^k x_0 + \dots + a_N^k x_N = 0\} \quad \text{for } k = 1, \dots, 2N + 1$$

then for each k ,

$$\frac{|\langle \nabla f(z_0), H_k \rangle|}{|f_0^2(z_0)|} = \frac{|\langle f(z_0), H_k \rangle|}{|f_0(z_0)|} \leq \frac{\|f(z_0)\| \cdot \|H_k\|}{|f_0(z_0)|} \leq \frac{1}{\delta}$$

by condition (ii) and (3.6). Hence, condition (iii) of Theorem 2.1 holds. This proves Theorem 2.3. ■

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References

- [AK] G. Aladro and S. G. Krantz, *A criterion for normality in \mathbb{C}^n* , J. Math. Anal. Appl. 161 (1991), 1–8.
- [CRY] Z. H. Chen, M. Ru and Q. M. Yan, *The truncated second main theorem and uniqueness theorems*, Sci. China Math. 53 (2010), 605–616.
- [D] J. Dufresnoy, *Théorie nouvelle des familles complexes normales. Applications à l'étude des fonctions algébroides*, Ann. Sci. École Norm. Sup. 61 (1944), 1–44.
- [F1] H. Fujimoto, *On families of meromorphic maps into the complex projective space*, Nagoya Math. J. 54 (1974), 21–51.
- [F2] H. Fujimoto, *Value Distribution Theory of the Gauss Map of Minimal Surfaces in \mathbb{R}^m* , Vieweg, 1993.

- [F3] H. Fujimoto, *Uniqueness problem with truncated multiplicities in value distribution theory*, Nagoya Math. J. 152 (1998), 131–152.
- [LLP] X. J. Liu, S. H. Li and X. C. Pang, *A normal criterion about two families of meromorphic functions concerning shared values*, Acta Math. Sinica (English Ser.) 29 (2013), 151–158.
- [M] P. Montel, *Sur les suites infinies des fonctions*, Ann. École Norm. Sup. 24 (1907), 233–334.
- [N] R. Nevanlinna, *Le théorème de Picard–Borel et la théorie des fonctions méromorphes*, Gauthier-Villars, Paris, 1929.
- [R] M. Ru, *Nevanlinna Theory and its Relation to Diophantine Approximation*, World Sci., 2001.
- [S] W. Schwick, *Sharing values and normality*, Arch. Math. (Basel), 59 (1992), 50–54.
- [Y] L. Yang, *Value Distribution Theory*, Springer, 1993.
- [YFP] L. Yang, C. Y. Fang and X. C. Pang, *Normal families of holomorphic mappings into complex projective space concerning shared hyperplanes*, Pacific J. Math. 272 (2014), 245–256.

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