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Generalized m-quasi-Einstein metric within the framework of Sasakian and K-contact manifolds

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Abstract. We consider generalized m-quasi-Einstein metric within the framework of Sasakian and K-contact manifolds. First, we prove that a complete Sasakian manifold M admitting a generalized m-quasi-Einstein metric is compact and isometric to the unit sphere S^{2n+1}. Next, we generalize this to complete K-contact manifolds with $m \neq 1$.

1. Introduction. Einstein metrics [4] and their generalizations are very much interesting both in mathematics and physics. One such important generalization is the so called *Ricci solitons* which are fixed points of Hamilton's Ricci flow: $\frac{\partial}{\partial t} g_{ij} = -2S_{ij}$ up to diffeomorphisms and scalings, and they correspond to self-similar solutions for this flow. In the recent literature, we find the study of Riemannian manifolds admitting Einstein-like metrics, which are natural generalizations of Einstein metrics and Ricci solitons [7]. These include *Ricci almost solitons* (introduced and studied by Pigola el al. [18]), *m-quasi-Einstein metrics* [8], and *(m, ρ)-quasi Einstein metrics* [14]. In [9] Catino introduced and studied Riemannian manifolds satisfying a more general structural equation that covers all the aforementioned examples as particular cases. These are known as generalized quasi-Einstein manifolds. A smooth connected Riemannian manifold (M^n, g) of dimension $n > 2$ is said to be a *generalized quasi-Einstein manifold* if there exist three smooth functions f, α and β such that

$$S + \nabla^2 f - \alpha df \otimes df = \beta g, \tag{1.1}$$

where S denotes the Ricci tensor of M and $\nabla^2 f$ is the Hessian of f. Catino [9] proved a local classification of such metrics with harmonic Weyl conformal curvature tensor.

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Generalized quasi-Einstein metrics generalize:

- Gradient Ricci solitons: $\alpha = 0$ and $\beta = \lambda \in \mathbb{R}$.
- Gradient Ricci almost solitons: $\alpha = 0$ and $\beta = \lambda \in C^\infty(M)$.
- m-quasi-Einstein metrics: $\alpha = 1/m$, where m is a positive integer, and $\beta = \lambda \in \mathbb{R}$, introduced and studied by Case et al. [8]; they have natural geometric interpretations with warped product Einstein metrics. For $m = 1$ these include the static metrics, and have been studied in depth for their connections to the positive mass theorem and general relativity (see [1]).
- Generalized quasi-Einstein metrics: $\alpha = 1/m$, where m is a positive integer, and $\beta = \lambda \in C^\infty(M)$. Existence of such metrics has been confirmed by Barros and Ribeiro, Jr. [3]. For instance, the standard unit sphere (S^n, g_0) and Euclidean space $(\mathbb{R}^n, g_0)$ belong to this class.
- (m, ρ)-quasi-Einstein metrics: $\alpha = 1/m$, where m is a positive integer, and $\beta = \rho r + \lambda$ with r the scalar curvature of g [14]. If r is constant, then (m, ρ)-quasi-Einstein metrics reduce to the usual m-quasi-Einstein metrics.

In this paper, we consider a particular case of a generalized quasi-Einstein metric which is defined as follows:

Definition 1.1. A Riemannian manifold (M^{2n+1}, g) is said to be a *generalized m-quasi-Einstein-metric* if there exist smooth functions f, λ and a real constant m $(0 < m \leq \infty)$ such that

$$S + \nabla^2 f - \frac{1}{m} df \otimes df = \lambda g, \tag{1.2}$$

where $S + \nabla^2 f - \frac{1}{m} df \otimes df$ is known as the *m-Bakry–Emery* Ricci tensor.

A generalized m-quasi-Einstein manifold $M^n(g, f, m)$ is said to be *trivial* if the potential function f is constant, and the triviality condition implies that M^n is an Einstein manifold.

In this paper we expand our earlier work where we investigate rigidity phenomena coming from generalizing the odd-dimensional Goldberg conjecture [13], which states that any compact K-contact Einstein manifold is Sasakian, and was proven by Boyer and Galicki [6]. Now, one may generalize the aforementioned result in two ways. The first one is to keep the Einstein condition and relax the hypothesis of K-contactness to contact metric structure; or to keep K-contactness and relax the hypothesis on the Einstein condition to either the gradient Ricci soliton [19], or the gradient Ricci almost soliton [11], or the m-quasi-Einstein metric [12]. It is interesting to note that the former is not possible, as was pointed out by Apostolov–Drăghici–Moroianu [2], and in all the latter cases we found that the manifold becomes Einstein and Sasakian provided it is complete.

So, it would be relevant if we consider a generalized m-quasi-Einstein condition in the framework of K-contact and Sasakian manifold in order to generalize the results of Boyer–Galicki [6], Sharma [19] and the author [11], [12].

Recently, the author proved that if a complete (m, ρ)-quasi-Einstein metric represents a K-contact metric and $m \neq 1$, then it is trivial (Einstein) and Sasakian. It is worth pointing out that any unit sphere S^{2n+1} with standard contact metric admits a generalized m-quasi-Einstein metric and a gradient Ricci almost soliton while it admits no (m, ρ)-quasi-Einstein metrics or m-quasi-Einstein metrics (see [10]). Moreover in [11], the author proved that if a compact K-contact metric represents a gradient Ricci almost soliton, then it is isometric to a unit sphere S^{2n+1}. Thus, the question may arise: Is the aforementioned result true if one replaces gradient Ricci almost soliton by generalized m-quasi-Einstein metric? Here, we prove that indeed the answer to this question is affirmative, even if we relax the compactness hypothesis to completeness.

The organization of this paper is as follows. In Section 2, we collect some basic definitions and fundamental formulas for contact metric structures, in particular for K-contact and Sasakian manifolds. Thereafter, we study Sasakian manifolds M admitting a generalized m-quasi-Einstein metric, and prove that M is locally isometric to S^{2n+1} provided M is complete. Next, we study complete K-contact manifolds M admitting a generalized quasi-Einstein metric, and prove that M is compact, Einstein, and Sasakian.

2. Preliminaries. We start by recalling basic definitions and collecting some formulas for contact metric manifolds. Let M be a smooth Riemannian manifold of dimension $2n+1$. Then M is said to be a *contact manifold* if it has a global 1-form η such that $\eta \wedge (d\eta)^n$ is non-vanishing everywhere on M. We call this 1-form a *contact* 1*-form.* For such a 1-form η there exists a unit vector field ξ, called the *Reeb vector field*, such that $\eta(\xi) = 1$ and $d\eta(\xi, \cdot) = 0$. Moreover, it is always possible to define a Riemannian metric g and a $(1,1)$ tensor field φ such that

$$d\eta(X,Y) = g(X, \varphi Y), \qquad \eta(X) = g(X, \xi), \qquad \varphi^2 X = -X + \eta(X)\xi.$$

Clearly these equations also imply that

$$\varphi\xi = 0, \qquad \eta \circ \varphi = 0, \qquad g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y).$$

A Riemannian manifold M^{2n+1} together with the structure (φ, ξ, η, g) is said to be a *contact metric manifold*. The operator $h = \frac{1}{2}\pounds_\xi \varphi$ is known to be symmetric and trace free. If the vector field ξ is Killing (equivalently $h = 0$), then the contact metric manifold M is said to be *K-contact*. On a

K-contact (Sasakian) manifold the following formulas are known [5]:

$$\nabla_X \xi = -\varphi X, \tag{2.1}$$

$$Q\xi = 2n\xi, \tag{2.2}$$

$$(\nabla_X \varphi)Y = R(\xi, X)Y. \tag{2.3}$$

A contact metric manifold is said to be *Sasakian* if

$$(\nabla_X \varphi)Y = g(X,Y)\xi - \eta(Y)X. \tag{2.4}$$

On a Sasakian manifold the curvature tensor satisfies

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y. \tag{2.5}$$

Also, the contact metric structure on M is said to be *Sasakian* if the almost Kähler structure on the metric cone $(M \times \mathbb{R}^+, r^2 g + dr^2)$ over M is Kähler. Sasakian manifolds are K-contact, and K-contact 3-manifolds are Sasakian. For details about contact metric manifolds (including K-contact and Sasakian ones), we refer to [5].

3. Main results. Before proving our main results we prove the following

Lemma 3.1. *Let* $(M^{2n+1}, g, m, \lambda)$ *be a generalized quasi-Einstein manifold. If* g *represents a* K*-contact metric, then*

$$\begin{aligned} g(R(X,Y)Df, \xi) &= g(Q\varphi Y, X) - g(Q\varphi X, Y) + 4ng(\varphi X, Y) \\ &\quad + \frac{\lambda - 2n}{m}[(Yf)\eta(X) - (Xf)\eta(Y)] \\ &\quad + [(X\lambda)\eta(Y) - (Y\lambda)\eta(X)], \end{aligned} \tag{3.1}$$

$$\frac{m-1}{m} S(Y, Df) = \frac{1}{2}(Yr) + \frac{2n\lambda - r}{m}(Yf) - 2n(Y\lambda). \tag{3.2}$$

Proof. We write equation (1.2) as

$$\nabla_Y Df + QY = \frac{1}{m} g(Y, Df)Df + \lambda Y, \tag{3.3}$$

where D is the gradient operator of g. Straightforward computations using the well-known expression of the curvature tensor,

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]},$$

and the repeated use of equation (3.3), give

$$\begin{aligned} R(X,Y)Df &= (\nabla_Y Q)X - (\nabla_X Q)Y + \frac{\lambda}{m}[(Yf)X - (Xf)Y] \\ &\quad + \frac{1}{m}[(Xf)QY - (Yf)QX] + [(X\lambda)Y - (Y\lambda)X]. \end{aligned} \tag{3.4}$$

Next, differentiating (2.2) and using (2.1) gives

$$(\nabla_X Q)\xi = Q\varphi X - 2n\varphi X. \tag{3.5}$$

Taking the scalar product of (3.4) with ξ and recalling (3.5) and (2.2) gives (3.1). To prove (3.2), we only need to take the trace of (3.4) over X. This completes the proof.

THEOREM 3.2. *Let $(M^{2n+1}, g, m, \lambda)$ be a generalized quasi-Einstein manifold. If g represents a Sasakian metric, then it is Einstein. If in addition M is complete, then it is compact and isometric to the unit sphere S^{2n+1}.*

Proof. We know that on a Sasakian manifold, Q and φ commute, i.e. $Q\varphi = \varphi Q$ (see [5]). Therefore, equation (3.1) reduces to

$$\begin{aligned} g(R(X,Y)Df,\xi) = &-2g(Q\varphi X,Y) + 4ng(\varphi X,Y) \\ &+ \frac{\lambda-2n}{m}[(Yf)\eta(X) - (Xf)\eta(Y)] \\ &+ [(X\lambda)\eta(Y) - (Y\lambda)\eta(X)]. \end{aligned}$$

Replacing X by φX, Y by φY, and noting that $R(\varphi X,\varphi Y)\xi = 0$ (follows from (2.5)), we find that $\varphi QX = 2n\varphi X$. Next, applying φ to the last equation and then recalling (2.2) yields $QX = 2nX$. Hence M is Einstein. Consequently, (3.3) reduces to

$$\nabla_Y Df = \frac{1}{m}g(Y,Df)Df + (\lambda - 2n)Y. \tag{3.6}$$

We now introduce a new function $u = e^{-f/m}$ on M. Then it follows that $Du = -\frac{u}{m}Df$. From this we also have (see e.g. [8])

$$\nabla_Y Df - \frac{1}{m}g(Y,Df)Df = -\frac{m}{u}\nabla_Y Du. \tag{3.7}$$

Combining (3.6) with (3.7) yields

$$\nabla_Y Du = \frac{(2n-\lambda)u}{m}Y. \tag{3.8}$$

Since M is Einstein, its scalar curvature is $r = 2n(2n+1)$. Consequently, equation (3.2) transforms to $(\lambda - 2n - m)Df = mD\lambda$. By virtue of $Du = -\frac{u}{m}Df$ the foregoing equation reduces to

$$(\lambda - 2n - m)Du = -uD\lambda.$$

From this we can deduce that $\lambda Du + uD\lambda = (2n+m)Du$, which can be written as $D(\lambda u) = (2n+m)Du$. In other words, $\lambda u = (2n+m)u + k$, where k is a constant. Making use of this in (3.8) we immediately infer that

$$\nabla_Y Du = \left(-u - \frac{k}{m}\right)Y.$$

We now apply Tashiro's theorem [20]: "If a complete Riemannian manifold M^n of dimension ≥ 2 admits a special concircular field ρ satisfying $\nabla\nabla\rho = (-c^2\rho + b)g$, then it is isometric to a sphere $S^n(c^2)$" to conclude that M is isometric to the unit sphere S^{2n+1}. ∎

THEOREM 3.3. *Let $(M^{2n+1}, g, m, \lambda)$ be a complete generalized quasi-Einstein manifold. If g represents a K-contact metric and $m \neq 1$, then it is compact, Einstein, Sasakian, and isometric to the unit sphere S^{2n+1}.*

Proof. Note that for a K-contact manifold, ξ is Killing, and hence $\pounds_\xi Q = 0$. In view of (2.1), this is equivalent to $\nabla_\xi Q = Q\varphi - \varphi Q$. Next, replacing X by ξ in (3.4) and taking into account the preceding equation and (2.3), we get

$$\begin{aligned} -g((\nabla_Y\varphi)X, Df) &= g(\varphi QY, X) - 2ng(\varphi Y, X) \\ &\quad + \frac{\xi f}{m} g(QY, X) + \left[\xi\lambda - \frac{\lambda}{m}(\xi f)\right] g(Y, X) \\ &\quad + \left[\frac{\lambda - 2n}{m}\right](Yf)\eta(X) - (Y\lambda)\eta(X). \end{aligned} \tag{3.9}$$

Replacing X by φX and Y by φY in (3.9), adding the resulting equation to (3.9) and then recalling the well-known formula (see [5, p. 95] or [17])

$$(\nabla_Y\varphi)X + (\nabla_{\varphi Y}\varphi)\varphi X = 2g(Y, X)\xi - \eta(X)(Y + \eta(Y)\xi),$$

we obtain

$$\begin{aligned} -2(\xi f)g(Y, X) + (Yf)\eta(X) + (\xi f)\eta(Y)\eta(X) &= g((Q\varphi + \varphi Q)Y, X) \\ - 4ng(\varphi Y, X) + \frac{\xi f}{m} g(QY - \varphi Q\varphi Y, X) &+ 2\left[\xi\lambda - \frac{\lambda}{m}(\xi f)\right] g(Y, X) \\ - \left[\xi\lambda - \frac{\lambda}{m}(\xi f)\right]\eta(Y)\eta(X) &+ \frac{\lambda - 2n}{m}(Yf)\eta(X) - (Y\lambda)\eta(X). \end{aligned}$$

Anti-symmetrizing the foregoing equation yields

$$\begin{aligned} (Yf)\eta(X) - (Xf)\eta(Y) &= 2g((Q\varphi + \varphi Q)Y, X) - 8ng(\varphi Y, X) \\ &\quad + \frac{\lambda - 2n}{m}[(Yf)\eta(X) - (Xf)\eta(Y)] \\ &\quad + (X\lambda)\eta(Y) - (Y\lambda)\eta(X). \end{aligned} \tag{3.10}$$

Choosing $X, Y \perp \xi$ in (3.10) provides

$$g((Q\varphi + \varphi Q)Y, X) = 4ng(\varphi Y, X).$$

Clearly for any vector field X, $X - \eta(X)\xi$ is orthogonal to ξ. Therefore, replacing X by $X - \eta(X)\xi$ and Y by $Y - \eta(Y)\xi$ in the preceding equation, we ultimately obtain

$$(Q\varphi + \varphi Q)Y = 4n\varphi Y \tag{3.11}$$

for any vector field Y. In view of this, equation (3.10) transforms to

$$\frac{\lambda - 2n - m}{m}\{(Yf)\eta(X) - (Xf)\eta(Y)\} = (Y\lambda)\eta(X) - (X\lambda)\eta(Y). \tag{3.12}$$

Let $\sigma = \lambda - 2n - m$. Then $D\sigma = D\lambda$. Thus (3.12) takes the form

$$\sigma Df - mD\sigma = \{\sigma(\xi f) - m(\xi\sigma)\}\xi. \tag{3.13}$$

Taking covariant differentiation of (3.13) along X and using (2.1) gives

$$(X\sigma)Df + \sigma\nabla_X Df - m\nabla_X D\sigma = X\{\sigma(\xi f) - m(\xi\sigma)\}\xi - \{\sigma(\xi f) - m(\xi\sigma)\}\varphi X.$$

Taking the inner product of this equation with Y and anti-symmetrizing provides

$$(X\sigma)(Yf) - (Y\sigma)(Xf) = X\{\sigma(\xi f) - m(\xi\sigma)\}\eta(Y) - Y\{\sigma(\xi f) - m(\xi\sigma)\}\eta(X) + 2\{\sigma(\xi f) - m(\xi\sigma)\}d\eta(X,Y), \tag{3.14}$$

where we have used $g(\nabla_X Df, Y) = g(\nabla_Y Df, X)$, which holds for any smooth function f on M. Now (3.13) can be written as

$$X\sigma = \frac{\sigma}{m}(Xf) - \frac{1}{m}\{\sigma(\xi f) - m(\xi\sigma)\}\eta(X).$$

Making use of this in (3.14), choosing $X, Y \perp \xi$ and noting that $d\eta \neq 0$ (by definition of contact metric structure), we get $\sigma(\xi f) - m(\xi\sigma) = 0$. Hence from (3.13) we have

$$(\lambda - 2n - m)(Xf) = m(X\lambda). \tag{3.15}$$

Let $\{e_a, \varphi e_a, \xi\}$, $a = 1, \ldots, n$, be a φ-basis of M such that $Qe_a = \sigma_a e_a$. Then $\varphi Q e_a = \sigma_a \varphi e_a$. Taking e_a instead of Y in (3.11) and using the foregoing equations we obtain $Q\varphi e_a = (4n - \sigma_a)\varphi e_a$. Computing the scalar curvature $r = g(Q\xi, \xi) + \sum_{a=1}^{n}[g(Qe_a, e_a) + g(Q\varphi e_a, \varphi e_a)]$ with the use of (2.2) we get $r = 2n(2n+1)$. By virtue of this and (3.15) one can deduce from (3.2) that $QDf = 2nDf$, as $m \neq 1$. Differentiating this along an arbitrary vector field X, recalling (3.3) and $QDf = 2nDf$, we obtain

$$(\nabla_X Q)Df - Q^2X + (\lambda + 2n)QX - 2n\lambda X = 0.$$

Contracting the foregoing equation over X, noting that the scalar curvature is constant, we deduce $|Q|^2 = 2nr$. By virtue of this and $r = 2n(2n+1)$, we compute

$$\begin{aligned}\left|Q - \frac{r}{2n+1}I\right|^2 &= |Q|^2 - \frac{2r^2}{2n+1} + \frac{r^2}{2n+1} = 2nr - \frac{r^2}{2n+1}\\ &= 4n^2(2n+1) - 4n^2(2n+1) = 0.\end{aligned}$$

Since the length of the symmetric tensor $Q - \frac{r}{2n+1}I$ vanishes, we must have $Q = \frac{r}{2n+1}I = 2nI$. This shows that M is Einstein with Einstein constant $2n$. Since M is complete and Einstein, it is compact by Myers' theorem [15]. Thus applying the result of Boyer–Galicki [6], we conclude that M is Sasakian. Finally, using the last theorem we complete the proof. ■

In [11] the author proved that if a compact K-contact metric represents a Ricci almost soliton, then it is locally isometric to S^{2n+1}. But following the proof of the above theorem, it is easy to observe that the above result is true if one relaxes the compactness hypothesis to completeness. In fact, we have the following

COROLLARY 3.4. *Let $M^{2n+1}(\varphi, \xi, \xi, g)$ be a complete K-contact manifold. If g represents a gradient Ricci almost soliton, then it is isometric to a unit sphere.*

Further, we remark that Theorem 3.3 also generalizes the following result on K-contact manifolds admitting an m-quasi-Einstein metric.

COROLLARY 3.5. *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a complete K-contact manifold. If g represents an m-quasi-Einstein metric and $m \neq 1$, then f is constant and M is compact, Einstein, and Sasakian.*

REMARK 3.6. Examples of generalized quasi-Einstein structures on S^n are given in [3]; the construction given there is valid for any $m > 0$.

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