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On the Cauchy problem for hyperbolic functional-differential equations

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Abstract. We consider the Cauchy problem for a nonlocal wave equation in one dimension. We study the existence of solutions by means of bicharacteristics. The existence and uniqueness is obtained in $W^{1,\infty}_{\mathrm{loc}}$ topology. The existence theorem is proved in a subset generated by certain continuity conditions for the derivatives.

1. Introduction. This paper is devoted to the study of the Cauchy problem for a second-order hyperbolic functional-differential equation. Theorems on existence and uniqueness for this type of problems can be formulated and proved by means of semigroup theory. The differential equation of second order $D_{tt}u - \mathcal{L}u = f(t,x,u)$ (here $\mathcal{L}$ is a second-order differential operator) is transformed to the system of first order differential equations

$$(1.1)\qquad \begin{cases} \dfrac{d}{dt}\mathbf{u}(t) = \begin{bmatrix} 0 & 1 \\ \mathcal{L} & 0 \end{bmatrix}\mathbf{u}(t) + \mathbf{f}(t,x,\mathbf{u}), \\[2ex] \mathbf{u}(0) = \begin{bmatrix} u^0 \\ u^1 \end{bmatrix}, \end{cases}$$

and then semigroup techniques are used. In [E] we can find an example of applications of semigroup theory to homogeneous hyperbolic differential equations of second order.

Another method consists in using sine and cosine families. If $a > 0$ is a constant and $f(t,x,\cdot)$ is a causal operator, then the Cauchy problem for the wave equation $D_{tt}u - a^2 D_{xx}u = f(t,x,u)$ can be transformed to the abstract problem

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(1.2) \qquad \begin{cases} \dfrac{d^2}{dt^2}u(t) = \mathcal{A}u(t) + f(t, u_t), \\ u(0) = u^0, \qquad \dfrac{d}{dt}u(0) = u^1, \end{cases}
$$

where u_t is the Hale operator

$$
u_t(s) = u(t+s)
$$

and the differential operator $\mathcal{A}$ generates a cosine family $C(t)$. The solution of problem (1.2) satisfies the integral equation

$$
u(t) = C(t)u^0 + S(t)u^1 + \int_0^t S(t-s) f(s, u_s)\, ds,
$$

where $S(t) = \int_0^t C(s)\, ds$ is the sine family corresponding to $C(t)$. We find in [TW] an outline of the theory of abstract second-order equations and sine and cosine families. Abstract integrodifferential equations of second order are investigated in [TD1] and [TD2].

We consider a hyperbolic functional-differential equation where the functional model depends on a set generated by bicharacteristics. If we want to transform the PDE problem to the form (1.2), then this functional model becomes difficult and does not seem to be natural.

Among other methods to prove existence theorems for hyperbolic equations, one can mention the fixed point theory, e.g. the method developed by J. Schauder [S], who proved the existence of solutions to the Cauchy problem for quasilinear hyperbolic equations in the normal form. In the case of semigrup theory and fixed point theory, we cannot see that the wave propagates at finite speed and obeys the Huygens principle.

In the second half of the 20th century inner and exterior problems concerning scattering of waves in bounded areas were studied. There are many modern applications of scattering theory [LP]. One can see that inner and exterior problems are mutually conjugated, which leads to nonlocal problems on the whole space $\mathbb{R}^n$, in particular $\mathbb{R}^1$. We regard this as an additional motivation for extensive studies of nonlocal problems. Any extension of boundary value problems to the whole space demands some regularity of solutions (see [LT1, LT2]).

In this paper, we define a functional sequence convergent to the solution of the differential problem which yields existence and uniqueness. This definition is based on the method of bicharacteristics for linear hyperbolic equations with nonconstant coefficient in the normal form. The theory of bicharacteristics for second order partial differential equations is extensively discussed in the monograph [K] which presents both methods: Picard iterations in $W^{1,\infty}$ and Schauder's fixed-point approach with a kind of $W^{2,\infty}$ regularity assumptions. We derive the existence in $W^{1,\infty}$ with more general regularity conditions for first-order derivatives. Namely, our moduli of continuity for partial derivatives are nonlinear.

Hyperbolic nonlinear PDE's in two independent variables are used in hydro- and gas dynamics, superconductivity, chemical technology and many other application areas. For example, the waves in two-conductor transmission lines having small transverse dimensions are modeled by the telegraph equation

$$D_{tt}u = D_x(F(u)D_xu) + G'(u)D_xu,$$

where u, $F(u)$ and $G(u)$ are respectively: the voltage between the conductors, the leakage current per unit length and the differential capacitance. Nonlinear telegraph equations are investigated in many papers, in particular in [G], [B] and [HZ]. Another example of applications of our new results is the one-dimensional heat propagation in a rigid body. It is described by the equation

$$D_{xx}\theta = D_t\left(\frac{\tau_0}{\chi}C(\theta)D_t\theta\right) + \frac{1}{\chi}D_\theta\Big[\int C(\theta)\,d\theta\Big]D_t\theta,$$

where $C(\theta)$ is the special heat, τ_0 is the thermal relaxation time and χ is the thermal conductivity. Concerning the heat propagation in a rigid body we refer to [HZ].

Notation.

- $C(X)$ is the space of continuous functions from a metric space X to $\mathbb{R}$.
- $C^1(X)$ denotes the space of continuous functions from X to $\mathbb{R}$ that have continuous first-order derivatives; here X is a subset of a normed space.
- $W^{1,\infty}(X)$ means the Sobolev space of L^∞-functions from $X \subset \mathbb{R}^n$ to $\mathbb{R}$ having first-order derivatives of the same class $L^\infty(X)$.
- If $u = u(t,x)$, then u_1 and u_2 denote partial first-order derivatives of u in t and x, respectively.
- If $\eta = \eta(s,t,x)$, then η_1, η_2 and η_3 denote partial first-order derivatives of η in s, t and x, respectively.

2. Integral equations. Let $E = [0,T] \times \mathbb{R}$. In this section, we consider the classical wave equation, where $a \in C^1(E)$, $a > 0$ on E and $g \in C(E)$. It is natural to provide a decomposition of the operator $D_{tt} - a^2D_{xx}$, which leads to integral equations. We first consider the Cauchy problem for the classical wave equation with homogeneous initial conditions

$$\begin{cases} D_{tt}u - a^2D_{xx}u = g & \text{in } E, \\ u(0,x) = 0, \quad D_tu(0,x) = 0 & \text{for } x \in \mathbb{R}. \end{cases} \tag{2.1}$$

More general initial-value problems with the initial conditions can be easily reduced to the Cauchy problem (2.1). Indeed, if u satisfies the IVP

$$D_{tt}u - a^2 D_{xx}u = g, \quad u(0,x) = \phi(x), \quad D_t u(0,x) = \psi(x),$$

then $\tilde{u} = u - w$ satisfies (2.1) where w solves the Cauchy problem for the homogeneous PDE

$$D_{tt}w - a^2 D_{xx}w = 0, \quad w(0,x) = \phi(x), \quad D_t w(0,x) = \psi(x).$$

Denote $A = a_1 + aa_2$ where $D_t a = a_1$ and $D_x a = a_2$. It is natural to consider bicharacteristics η, θ of the hyperbolic equation (2.1) passing through $(t,x) \in E$:

$$\begin{aligned} \eta'(s) &= a(s, \eta(s)), & \eta(t) &= x, \\ \theta'(s) &= -a(s, \theta(s)), & \theta(t) &= x. \end{aligned}$$

These bicharacteristics will be denoted by $\eta = \eta(s) = \eta^{t,x}(s)$ and $\theta = \theta(s) = \theta^{t,x}(s)$, respectively. It is also convenient to write them in some formulas as follows: $\eta(s) = \eta(s;t,x)$ and $\theta(s) = \theta(s;t,x)$. In that case we denote partial first-order derivatives of η in s, t, x by η_1, η_2, η_3, respectively. A similar notation θ_1, θ_2, θ_3 applies to θ. If u is a solution of (2.1) then

$$(D_t + aD_x)(D_t - aD_x)u = g - AD_x u.$$

From this representation it is seen that the Cauchy problem (2.1) is equivalent to the system of first-order equations

$$\begin{cases} D_t u - aD_x u = v & \text{in } E, \\ D_t v + aD_x v = g - AD_x u & \text{in } E, \\ u(0,x) = 0, \quad v(0,x) = 0 & \text{for } x \in \mathbb{R}. \end{cases} \tag{2.2}$$

System (2.2) can be integrated along the curves $\theta^{t,x}$ and $\eta^{t,x}$, which leads to the integral equations

$$\begin{aligned} u(t,x) &= \int_0^t v(s, \theta^{t,x}(s))\, ds, \\ v(t,x) &= \int_0^t [g(s, \eta^{t,x}(s)) - A(s, \eta^{t,x}(s)) D_x u(s, \eta^{t,x}(s))]\, ds. \end{aligned}$$

If we substitute v from the second equation to the first one, we get an integral equation for u:

$$\begin{aligned} u(t,x) &= \int_0^t \int_0^\tau [g(s, \eta^{\tau,\theta^{t,x}(\tau)}(s)) - A(s, \eta^{\tau,\theta^{t,x}(\tau)}(s)) D_x u(s, \eta^{\tau,\theta^{t,x}(\tau)}(s))]\, ds\, d\tau \\ &= \int_0^t \int_s^t [g(s, \eta^{\tau,\theta^{t,x}(\tau)}(s)) - A(s, \eta^{\tau,\theta^{t,x}(\tau)}(s)) D_x u(s, \eta^{\tau,\theta^{t,x}(\tau)}(s))]\, d\tau\, ds. \end{aligned}$$

We change variables

$$[s,t] \ni \tau \mapsto y = \eta^{\tau,\theta^{t,x}(\tau)}(s). \tag{2.3}$$

If we have the inverse mapping $y \mapsto \tau = T(y;s,t,x)$, we denote

$$B(s,y,t,x) = a(\tau,\theta^{t,x}(\tau)) \exp\Big(\int_\tau^s a_2(z,\eta^{\tau,\theta^{t,x}(\tau)}(z))\,dz\Big).$$

With this notation the above integral equation takes the form

$$u(t,x) = \frac{1}{2}\int_0^t \int_{\eta^{t,x}(s)}^{\theta^{t,x}(s)} \frac{g(s,y) - A(s,y)D_x u(s,y)}{B(s,y,t,x)}\,dy\,ds.$$

Recall the notation $\eta_1 = D_s\eta$, $\eta_2 = D_t\eta$, $\eta_3 = D_x\eta$, $\theta_1 = D_s\theta$, $\theta_2 = D_t\theta$ and $\theta_3 = D_x\theta$. We see that

$$\begin{aligned} D_x u(t,x) = \frac{1}{2}\int_0^t \bigg\{ & \frac{g(s,y) - A(s,y)D_x u(s,y)}{B(s,y,t,x)}\theta_3(s;t,x)\bigg|_{y=\theta^{t,x}(s)} \\ & - \frac{g(s,y) - A(s,y)D_x u(s,y)}{B(s,y,t,x)}\eta_3(s;t,x)\bigg|_{y=\eta^{t,x}(s)}\bigg\}\,ds \\ & - \frac{1}{2}\int_0^t \int_{\eta^{t,x}(s)}^{\theta^{t,x}(s)} \frac{g(s,y) - A(s,y)D_x u(s,y)}{B^2(s,y,t,x)}\frac{\partial B}{\partial x}(s,y,t,x)\,dy\,ds, \end{aligned}$$

$$\begin{aligned} D_t u(t,x) = \frac{1}{2}\int_0^t \bigg\{ & \frac{g(s,y) - A(s,y)D_x u(s,y)}{B(s,y,t,x)}\theta_2(s;t,x)\bigg|_{y=\theta^{t,x}(s)} \\ & - \frac{g(s,y) - A(s,y)D_x u(s,y)}{B(s,y,t,x)}\eta_2(s;t,x)\bigg|_{y=\eta^{t,x}(s)}\bigg\}\,ds \\ & - \frac{1}{2}\int_0^t \int_{\eta^{t,x}(s)}^{\theta^{t,x}(s)} \frac{g(s,y) - A(s,y)D_x u(s,y)}{B^2(s,y,t,x)}\frac{\partial B}{\partial t}(s,y,t,x)\,dy\,ds. \end{aligned}$$

In order to get $\frac{\partial B}{\partial x}$ and $\frac{\partial B}{\partial t}$ we calculate partial derivatives. From the implicit function theorem we get

$$\frac{\partial \tau}{\partial x} = -\frac{\eta_3(s;\tau,\theta^{t,x}(\tau))\theta_3(\tau;t,x)}{\eta_2(s;\tau,\theta^{t,x}(\tau)) + \eta_3(s;\tau,\theta^{t,x}(\tau))\theta_1(\tau;t,x)},$$

$$\frac{\partial \tau}{\partial t} = -\frac{\eta_3(s;\tau,\theta^{t,x}(\tau))\theta_2(\tau;t,x)}{\eta_2(s;\tau,\theta^{t,x}(\tau)) + \eta_3(s;\tau,\theta^{t,x}(\tau))\theta_1(\tau;t,x)}.$$

Note that $\theta_1(\tau;t,x) = -a(\tau,\theta^{t,x}(\tau))$. From the theorem on differentiability of solutions with respect to initial conditions (see [W]) we have

$$\eta_2(s;\tau,\theta^{t,x}(\tau)) = -a(\tau,\theta^{t,x}(\tau))\exp\Big(\int_\tau^s a_2(z,\eta^{\tau,\theta^{t,x}(\tau)}(z))\,dz\Big),$$
$$\eta_3(s;\tau,\theta^{t,x}(\tau)) = \exp\Big(\int_\tau^s a_2(z,\eta^{\tau,\theta^{t,x}(\tau)}(z))\,dz\Big),$$
$$\theta_2(\tau;t,x) = a(t,x)\exp\Big(-\int_t^\tau a_2(z,\theta^{t,x}(z))\,dz\Big),$$
$$\theta_3(\tau;t,x) = \exp\Big(-\int_t^\tau a_2(z,\theta^{t,x}(z))\,dz\Big).$$

Thus

$$\frac{\partial\tau}{\partial x} = \frac{\exp\big(\int_\tau^t a_2(z,\theta^{t,x}(z))\,dz\big)}{2a(\tau,\theta^{t,x}(\tau))},$$
$$\frac{\partial\tau}{\partial t} = \frac{a(t,x)\exp\big(\int_\tau^t a_2(z,\theta^{t,x}(z))\,dz\big)}{2a(\tau,\theta^{t,x}(\tau))}.$$

We calculate the derivatives of B:

$$\frac{\partial B}{\partial x}(s,y,t,x) = \frac{a_1(\tau,\theta^{t,x}(\tau))}{2a(\tau,\theta^{t,x}(\tau))}\exp\Big(\int_\tau^t a_2(t',\theta^{t,x}(t'))\,dt' + \int_\tau^s a_2(t',\eta^{\tau,\theta^{t,x}(\tau)}(t'))\,dt'\Big),$$
$$\frac{\partial B}{\partial t}(s,y,t,x) = \frac{a(t,x)a_1(\tau,\theta^{t,x}(\tau))}{2a(\tau,\theta^{t,x}(\tau))}\exp\Big(\int_\tau^t a_2(t',\theta^{t,x}(t'))\,dt' + \int_\tau^s a_2(t',\eta^{\tau,\theta^{t,x}(\tau)}(t'))\,dt'\Big),$$

where $\tau = T(y;s,t,x)$.

REMARK 2.1. We apply the method of bicharacteristics and the above decomposition to the Cauchy problem

$$\begin{cases} D_{tt}u - a^2 D_{xx}u = g(t,x) & \text{in } E,\\ u(0,x) = \phi(x), \quad D_t u(0,x) = \psi(x) & \text{for } x\in\mathbb{R}.\end{cases} \tag{2.4}$$

Then we obtain the fixed-point equation

$$u(t,x) = \frac12\int_0^t\int_{\eta^{t,x}(s)}^{\theta^{t,x}(s)} \frac{g(s,y) - A(s,y)D_x u(s,y)}{B(s,y,t,x)}\,dy\,ds + \frac12\int_{\eta^{t,x}(0)}^{\theta^{t,x}(0)} \frac{\psi(y) - a(0,y)\phi'(y)}{C(y,t,x)}\,dy + \phi(\theta^{t,x}(0)), \tag{2.5}$$

where

$$C(y,t,x) = a(s,\theta^{t,x}(s)) \exp\Big(-\int_0^s a_2(z,\eta^{s,\theta^{t,x}(s)}(z))\,dz\Big).$$

Here $y = \eta^{s,\theta^{t,x}(s)}(0)$, and we have the inverse mapping $y \mapsto s = S(y;t,x)$.

Remark 2.2. If $a = \mathrm{const} > 0$ then $A = 0$, and we get the well known d'Alembert formula

$$u(t,x) = \frac{1}{2a}\int_0^t \int_{x+as-at}^{x-as+at} g(s,y)\,dy\,ds$$

for problem (2.1), and the formula

$$u(t,x) = \frac{1}{2a}\int_0^t \int_{x+as-at}^{x-as+at} g(s,y)\,dy\,ds + \frac{1}{2a}\int_{x-at}^{x+at} \psi(y)\,dy$$
$$+ \tfrac{1}{2}[\phi(x+at)+\phi(x-at)]$$

for problem (2.4).

3. Main results. In this section, we present an existence and uniqueness theorem, another existence theorem and some applications.

3.1. Functional-differential equation. We use the integral representation (2.4) to the differential-functional problem to be formulated in this subsection. Our functional model is associated with the area of the wave dependence. Let $(t,x) \in E$ and $u \in C(E,\mathbb{R})$. Then $u_{|E_{t,x}} : E_{t,x} \to \mathbb{R}$ is the restriction of u to the set $E_{t,x}$, where

$$E_{t,x} = \{(s,y) \in [0,t]\times\mathbb{R} : \eta^{t,x}(s) \le y \le \theta^{t,x}(s)\}.$$

By formula (2.5) for solutions of the Cauchy problem (2.4), we see that the value of u at any point $(t,x) \in E$ depends only on the values given in the bounded region $E_{t,x}$ (compare with the Huygens principle). It is seen that the initial condition at the point $(0,x)$ affects only that part of the solution in the bounded region

$$\{(s,y) \in [0,t]\times\mathbb{R} : \theta^{t,x}(s) + x - \theta^{t,x}(0) \le y \le \eta^{t,x}(s) + x - \eta^{t,x}(0)\}.$$

This illustrates the finite speed of wave propagation. We deal with the following Cauchy problem for a second-order partial functional-differential equation:

$$(3.1)\qquad \begin{cases} D_{tt}u(t,x) - a^2(t,x)D_{xx}u(t,x) = f(t,x,u_{|E_{t,x}}) & \text{for } (t,x)\in E,\\ u(0,x) = 0, \quad D_t u(0,x) = 0 & \text{for } x \in \mathbb{R}, \end{cases}$$

where $a : E \to \mathbb{R}$ and $f(t,x,\cdot) : W^{1,\infty}(E_{t,x}) \to \mathbb{R}$ for all $(t,x) \in E$.

The Cauchy problem (3.1) is equivalent to the integral fixed-point equation

$$u(t,x) = \frac{1}{2}\int_0^t \int_{\eta^{t,x}(s)}^{\theta^{t,x}(s)} \frac{f(s,y,u_{|E_{s,y}}) - A(s,y)D_x u(s,y)}{B(s,y,t,x)}\,dy\,ds. \tag{3.2}$$

We will give sufficient conditions for $W^{1,\infty}_{\mathrm{loc}}$ convergence of Picard iterations corresponding to (3.2). Next we provide a Schauder-type existence theorem in a similar metric space.

Lemma 3.1. *If $a \in C^1(E)$ and $a(t,x) > 0$ for $(t,x) \in E$, then the bicharacteristics $\theta^{t,x}$ and $\eta^{t,x}$ are unique, $\eta^{t,x}(s) < \theta^{t,x}(s)$ for $0 \le s < t \le T$, and $E_{s,y} \subset E_{t,x}$ for $(s,y) \in E_{t,x}$.*

Proof. Since the solutions η and θ are unique, we see that if $(s,y) \in E_{t,x}$ then $s \le t$ and

$$\eta^{s,y}(t') \ge \eta^{t,x}(t') \quad \text{and} \quad \theta^{s,y}(t') \le \theta^{t,x}(t') \quad \text{for } 0 \le t' \le t.$$

Thus $E_{s,y} \subset E_{t,x}$. ∎

Theorem 3.2. *Suppose that:*

(a) *$a \in C^1(E)$ and $a(t,x) > 0$ for $(t,x) \in E$.*
(b) *$f(\cdot,\cdot,0) : E \to \mathbb{R}$ is continuous and $f(t,x,\cdot) : W^{1,\infty}(E_{t,x}) \to \mathbb{R}$.*
(c) *There is a nonnegative function $L \in C(E)$ such that $L(s,y) \le L(t,x)$ for $E_{s,y} \subset E_{t,x}$ and*

$$|f(t,x,w) - f(t,x,v)| \le L(t,x)\|w - v\|_{W^{1,\infty}(E_{t,x})} \quad \text{for } (t,x) \in E.$$

Then there exists exactly one solution of problem (3.1) in the class $W^{1,\infty}_{\mathrm{loc}}(E)$.

Assumption (c) of Theorem 3.2 can be formulated with any nonnegative $\tilde{L} \in C(E)$, and then this function can be modified as follows: $L(t,x) = \sup_{E_{t,x}} \tilde{L}(s,y)$. Its monotonicity with respect to the sets $E_{t,x}$ is justified by Lemma 3.1.

Remark 3.3. Theorem 3.2 can be used to investigate the existence and uniqueness of solution of variable-coefficient delayed nonlinear telegraph equations of the form

$$D_{tt}u = D_x(a(t,x)D_x u) + g(t,x,u(t/2,x),D_x u(t/2,x)),$$

where $a \in C^1(E)$, $a > 0$ and g satisfies the Lipschitz condition.

3.2. Existence of solutions. We deal with the existence of solutions to the Cauchy problem (3.1), not necessarily unique. The main assumptions are: the sublinearity of f and the existence of moduli of continuity for the unknown functions and their derivatives. It turns out that these moduli are more general than the Lipschitz condition: they are generated by Perron comparison conditions. A function $\sigma : [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a

Perron function if σ is continuous, $\sigma(t,0)=0$ for $t\in[0,T]$, and the unique solution of the Cauchy problem

$$y'(t)=\sigma(t,y(t)),\qquad y(0)=0,$$

is $y(t)=0$ for $t\in[0,T]$.

Denote $Dw(t,x)=(D_tw(t,x),D_xw(t,x))$ and

$$|Dw(t,x)|=\max\{|D_tw(t,x)|,\,|D_xw(t,x)|\}.$$

Let $w:E_{t,x}\to\mathbb{R}$, $\bar w:E_{\bar t,\bar x}\to\mathbb{R}$ and $\epsilon>0$. Then

$$\operatorname{dist}_\epsilon(w,\bar w)=\sup_{(s,y)\in E_{t,x},\,(\bar s,\bar y)\in E_{\bar t,\bar x},\,|s-\bar s|\le\epsilon,\,|y-\bar y|\le\epsilon}|w(s,y)-w(\bar s,\bar y)|.$$

THEOREM 3.4. *Suppose that:*

(a) $a\in C^1(E)$ *and* $a(t,x)>0$ *for* $(t,x)\in E$.

(b) *For every* $x_0\in\mathbb{R}$ *there are* $K>0$ *and continuous functions* $M_f,N_f:[0,T]\times\mathbb{R}_+\to\mathbb{R}_+$ *such that* $M_f(t,0)=N_f(t,0)=0$ *for* $t\in[0,T]$, M_f,N_f *are nondecreasing with respect to the second variable, for every constant* $C>0$, *the map*

$$[0,T]\times\mathbb{R}_+\ni(t,r)\mapsto M_f(t,r)+Cr$$

is a Perron function, and

$$\begin{aligned}|f(t,x,w)-f(t,\bar x,\bar w)|&\\ &\le N_f(t,\operatorname{dist}_{K|x-\bar x|}(w,\bar w))+M_f(t,\operatorname{dist}_{K|x-\bar x|}(Dw,D\bar w))\end{aligned}$$

for all $(t,x),(t,\bar x)\in E_{T,x_0}$, $w\in W^{1,\infty}(E_{t,x})$, $\bar w\in W^{1,\infty}(E_{t,\bar x})$.

(c) *There is a nonnegative function* $L\in C(E)$ *such that*

$$|f(t,x,w)|\le L(t,x)(1+\|w\|_{W^{1,\infty}(E_{t,x})})\qquad \text{for } (t,x)\in E,\ w\in W^{1,\infty}(E).$$

Then there exists a solution of problem (3.1) *in the class* $W^{1,\infty}_{\mathrm{loc}}(E)$.

3.3. Applications. Based on Theorem 3.2, we investigate the local existence of solutions of the Cauchy problem

$$\text{(3.3)}\quad\begin{cases}D_{tt}u(t,x)-(u(t,x)+1)^2D_{xx}u(t,x)=f(t,x,u_{|E_{t,x}}) & \text{for } (t,x)\in E,\\ u(0,x)=0,\quad D_tu(0,x)=0 & \text{for } x\in\mathbb{R}.\end{cases}$$

The nonlinear telegraph equation is our motivation for studying the above differential-functional problem. The coefficient $(u+1)^2$ can be replaced by any regular function $g:E\times\mathbb{R}\to\mathbb{R}$ such that $g(t,x,0)>0$ for $(t,x)\in E$. The Cauchy problem (3.3) is equivalent to the integral fixed-point equation

$$u(t,x) = \frac{1}{2}\int_0^t \int_{\eta^{t,x}(s)}^{\theta^{t,x}(s)} \frac{f(s,y,u_{|E_{s,y}})}{B(s,y,t,x)}\,dy\,ds$$
$$- \frac{1}{2}\int_0^t \int_{\eta^{t,x}(s)}^{\theta^{t,x}(s)} \frac{[D_t u(s,y) + (u(s,y)+1)D_x u(s,y)]D_x u(s,y)}{B(s,y,t,x)}\,dy\,ds,$$

where η, θ depend on u and satisfy the ODE's

$$\begin{aligned} \eta'(s) &= u(s,\eta(s)) + 1, &\quad \eta(t) &= x,\\ \theta'(s) &= -u(s,\theta(s)) - 1, &\quad \theta(t) &= x, \end{aligned}$$

and

$$B(s,y,t,x) = (u(\tau,\theta^{t,x}(\tau)) + 1)\exp\Big(\int_\tau^s u_2(z,\eta^{\tau,\theta^{t,x}(\tau)}(z))\,dz\Big).$$

THEOREM 3.5. *Suppose that:*

(a) *$f(\cdot,\cdot,0) : E \to \mathbb{R}$ is continuous, bounded and $f(t,x,\cdot) : W^{1,\infty}(E_{t,x}) \to \mathbb{R}$ for $(t,x) \in E$.*

(b) *There is a constant L such that*
$$|f(t,x,w) - f(t,x,v)| \le L\|w - v\|_{W^{1,\infty}(E_{t,x})} \quad \textit{for } (t,x) \in E.$$

Then there exists exactly one solution of (3.3) *in the class $W^{1,\infty}([0,t_0]\times\mathbb{R})$ for some $t_0 \in [0,T]$.*

4. Proofs of theorems. We present the proofs of the theorems from Section 3. Since the integrand of the right-hand side of (3.2) contains the derivative $D_x u$ and possibly $D_t u$, we have to consider an extended system of integral equations

$$\begin{aligned} u(t,x) &= (Su)(t,x) := \int_0^t \int_{\eta^{t,x}(s)}^{\theta^{t,x}(s)} F_0(s,y,t,x,u)\,dy\,ds,\\ D_t u(t,x) &= (Su)_1(t,x) := \int_0^t G_1(s,t,x,u)\,ds\\ &\qquad - \int_0^t \int_{\eta^{t,x}(s)}^{\theta^{t,x}(s)} F_1(s,y,t,x,u)\,dy\,ds,\\ D_x u(t,x) &= (Su)_2(t,x) := \int_0^t G_2(s,t,x,u)\,ds\\ &\qquad - \int_0^t \int_{\eta^{t,x}(s)}^{\theta^{t,x}(s)} F_2(s,y,t,x,u_{|E_{s,y}})\,dy\,ds, \end{aligned} \tag{4.1}$$

where

$$
\begin{aligned}
F_0(s,y,t,x,u) &= \frac{1}{2}\,\frac{f(s,y,u_{|E_{s,y}}) - A(s,y)D_x u(s,y)}{B(s,y,t,x)},\\
F_1(s,y,t,x,u) &= \frac{1}{2}\,\frac{f(s,y,u_{|E_{s,y}}) - A(s,y)D_x u(s,y)}{B^2(s,y,t,x)}\,\frac{\partial B}{\partial t}(s,y,t,x),\\
F_2(s,y,t,x,u) &= \frac{1}{2}\,\frac{f(s,y,u_{|E_{s,y}}) - A(s,y)D_x u(s,y)}{B^2(s,y,t,x)}\,\frac{\partial B}{\partial x}(s,y,t,x),\\
G_1(s,t,x,u) &= \frac{1}{2}\,\frac{f(s,y,u_{|E_{s,y}}) - A(s,y)D_x u(s,y)}{B(s,y,t,x)}\theta_2(s;t,x)\Big|_{y=\theta^{t,x}(s)}\\
&\quad - \frac{1}{2}\,\frac{f(s,y,u_{|E_{s,y}}) - A(s,y)D_x u(s,y)}{B(s,y,t,x)}\eta_2(s;t,x)\Big|_{y=\eta^{t,x}(s)},\\
G_2(s,t,x,u) &= \frac{1}{2}\,\frac{f(s,y,u_{|E_{s,y}}) - A(s,y)D_x u(s,y)}{B(s,y,t,x)}\theta_3(s;t,x)\Big|_{y=\theta^{t,x}(s)}\\
&\quad - \frac{1}{2}\,\frac{f(s,y,u_{|E_{s,y}}) - A(s,y)D_x u(s,y)}{B(s,y,t,x)}\eta_3(s;t,x)\Big|_{y=\eta^{t,x}(s)}.
\end{aligned}
$$

Proof of Theorem 3.2. Consider the sequences u^n, $D_t u^n$, $D_x u^n$ given by $u^0(t,x) = 0$, $D_x u^0(t,x) = 0$, $D_t u^0(t,x) = 0$ and

$$
\begin{cases}
u^{n+1}(t,x) = (Su^n)(t,x),\\
D_t u^{n+1}(t,x) = (Su^n)_1(t,x),\\
D_x u^{n+1}(t,x) = (Su^n)_2(t,x).
\end{cases}
\tag{4.2}
$$

We show that they converge to u, $D_t u$, $D_x u$, respectively, where u is the solution of problem (3.1). For an arbitrary point $(t_0,x_0) \in E$ define seminorms $\|\cdot\|_t^0$ by the formula

$$
\|v\|_t^0 = \sup_{\substack{(s,y)\in E_{t_0,x_0}\\ s\le t}} \|v\|_{W^{1,\infty}(E_{s,y})}.
$$

We know that

$$
\theta^{t,x}(s) - \eta^{t,x}(s) = \int_s^t [a(z,\theta^{t,x}(z)) + a(z,\eta^{t,x}(z))]\,dz.
\tag{4.3}
$$

From (4.3) and continuity of $a(\cdot)$, it is seen that there is a positive constant C_0 depending on t_0, x_0 such that

$$
\theta^{t,x}(s) - \eta^{t,x}(s) \le C_0(t-s) \quad \text{for } 0 \le s \le t \text{ and } (t,x) \in E_{t_0,x_0}.
$$

Since $a(t,x) > 0$, it follows that $\theta^{t,x}(s) - \eta^{t,x}(s) \ge 0$ for $0 \le s \le t$. Fix $(t_0,x_0) \in E$. Then the continuous functions a, a_1 and a_2 are bounded on the bounded set E_{t_0,x_0}. Suppose that $(t,x) \in E_{t_0,x_0}$, or equivalently $E_{t,x} \subset E_{t_0,x_0}$ and $k = 1,2,\ldots$. Then

$$
\begin{aligned}
|u^{k+1}(t,x) - u^k(t,x)| &= |(Su^k)(t,x) - (Su^{k-1})(t,x)| \\
&\le \frac{1}{2}\int_0^t \int_{\eta^{t,x}(s)}^{\theta^{t,x}(s)} \Bigg[\frac{|f(s,y,u^k_{|E_{s,y}}) - f(s,y,u^{k-1}_{|E_{s,y}})|}{|B(s,y,t,x)|} \\
&\qquad + \frac{|A(s,y)|\cdot|D_x u^k(s,y) - D_x u^{k-1}(s,y)|}{|B(s,y,t,x)|}\Bigg]\,dy\,ds \\
&\le \frac{1}{2}\int_0^t \int_{\eta^{t,x}(s)}^{\theta^{t,x}(s)} \Bigg[\frac{L(s,y)\|u^k - u^{k-1}\|_{W^{1,\infty}(E_{s,y})}}{|B(s,y,t,x)|} \\
&\qquad + \frac{|A(s,y)|\cdot|D_x u^k(s,y) - D_x u^{k-1}(s,y)|}{|B(s,y,t,x)|}\Bigg]\,dy\,ds \\
&\le \int_0^t K_1 \|u^k - u^{k-1}\|_s^0\,ds,
\end{aligned}
$$

where

$$
K_1 = K_1(t_0,x_0) = C_0 \sup_{E_{s,y}\subset E_{t,x}\subset E_{t_0,x_0}} \frac{L(s,y) + |A(s,y)|}{2|B(s,y,t,x)|}.
$$

Moreover

$$
\begin{aligned}
&|D_t u^{k+1}(t,x) - D_t u^k(t,x)| \\
&\quad\le \frac{1}{2}\int_0^t \Bigg[\frac{|f(s,y,u^k_{|E_{s,y}}) - f(s,y,u^{k-1}_{|E_{s,y}})|}{|B(s,y,t,x)|} \\
&\qquad\quad + \frac{|A(s,y)|\cdot|D_x u^k(s,y) - D_x u^{k-1}(s,y)|}{|B(s,y,t,x)|}\Bigg] |\theta_2(s;t,x)|\Big|_{y=\theta^{t,x}(s)}\,ds \\
&\quad + \frac{1}{2}\int_0^t \Bigg[\frac{|f(s,y,u^k_{|E_{s,y}}) - f(s,y,u^{k-1}_{|E_{s,y}})|}{|B(s,y,t,x)|} \\
&\qquad\quad + \frac{|A(s,y)|\cdot|D_x u^k(s,y) - D_x u^{k-1}(s,y)|}{|B(s,y,t,x)|}\Bigg] |\eta_2(s;t,x)|\Big|_{y=\eta^{t,x}(s)}\,ds \\
&\quad + \frac{1}{2}\int_0^t \int_{\eta^{t,x}(s)}^{\theta^{t,x}(s)} \Bigg[\frac{|f(s,y,u^k_{|E_{s,y}}) - f(s,y,u^{k-1}_{|E_{s,y}})|}{|B(s,y,t,x)|} \left|\frac{\partial B}{\partial t}(s,y,t,x)\right| dy\,ds \\
&\qquad\quad + \frac{|A(s,y)|\cdot|D_x u^k(s,y) - D_x u^{k-1}(s,y)|}{|B(s,y,t,x)|}\Bigg] \left|\frac{\partial B}{\partial t}(s,y,t,x)\right| dy\,ds \\
&\quad\le \int_0^t K_2 \|u^k - u^{k-1}\|_s^0\,ds,
\end{aligned}
$$

where

$$\begin{aligned}K_2 &= K_2(t_0,x_0)\\ &= \sup_{E_{s,y}\subset E_{t,x}\subset E_{t_0,x_0}} \frac{L(s,y)+|A(s,y)|}{2|B(s,y,t,x)|}\bigg(C_0\bigg|\frac{\partial B}{\partial t}(s,y,t,x)\bigg|\\ &\qquad\qquad + |\theta_2(s;t,x)| + |\eta_2(s;t,x)|\bigg).\end{aligned}$$

In the same manner we see that

$$|D_x u^{k+1}(t,x) - D_x u^k(t,x)| \le \int_0^t K_3\|u^k - u^{k-1}\|_s^0\,ds,$$

where

$$\begin{aligned}K_3 &= K_3(t_0,x_0)\\ &= \sup_{E_{s,y}\subset E_{t,x}\subset E_{t_0,x_0}} \frac{L(s,y)+|A(s,y)|}{2|B(s,y,t,x)|}\bigg(C_0\bigg|\frac{\partial B}{\partial x}(s,y,t,x)\bigg|\\ &\qquad\qquad + |\theta_3(s;t,x)| + |\eta_3(s;t,x)|\bigg).\end{aligned}$$

Let $K = K_1 + K_2 + K_3$. By the above inequalities, we get

$$\|u^{k+1} - u^k\|_t^0 \le \int_0^t K\|u^k - u^{k-1}\|_s^0\,ds. \tag{4.4}$$

If we apply induction to (4.4), we see that

$$\|u^{k+1} - u^k\|_t^0 \le \frac{\tilde{C}}{k!}(tK)^k,$$

where $\tilde{C} = \|u^1 - u^0\|_{W^{1,\infty}(E_{t_0,x_0})}$. Thus

$$\|u^{k+1} - u^k\|_{W^{1,\infty}(E_{t_0,x_0})} \le \frac{\tilde{C}(t_0K)^k}{k!}.$$

Consequently, u^n, $D_t u^n$ and $D_x u^n$ converge uniformly on E_{t_0,x_0}. Taking $n\to\infty$ in (4.2) we get

$$u^n \to u, \quad D_t u^n \to D_t u \quad\text{and}\quad D_x u^n \to D_x u \quad \text{on } E_{t_0,x_0},$$

where u is the solution of (3.1). We claim that the solution exists on E. Indeed, the solution exists on $E_{T,x}$ for each $x\in\mathbb{R}$. The function η is increasing and θ is decreasing, hence the set $E_{T,x_1}\cap E_{T,x_2}$ is not empty if the distance between x_1 and x_2 is small enough. By uniqueness, there is only one solution on $E_{T,x_1}\cap E_{T,x_2}$. Thus we get the unique global solution on E.

This completes the proof of Theorem 3.2. ■

Proof of Theorem 3.4. Let $u_1 = D_t u$ and $u_2 = D_x u$. Problem (3.1) is equivalent to the fixed-point equation $u = Su$ where the operator S is defined by (4.1). Fix $(t_0,x_0)\in E$ and consider the set $\Omega^0_{\rho,\omega}$ containing all functions $u\in W^{1,\infty}(E_{t_0,x_0})$ satisfying the inequalities

$$|u(t,x)| \le \rho(t), \quad |u_1(t,x)| \le \rho(t), \quad |u_2(t,x)| \le \rho(t),$$
$$|u_1(t+\epsilon,x) - u_1(t,x)| \le \omega(t,\epsilon), \quad |u_2(t+\epsilon,x) - u_2(t,x)| \le \omega(t,\epsilon),$$
$$|u_1(t,x+\epsilon) - u_1(t,x)| \le \omega(t,\epsilon), \quad |u_2(t,x+\epsilon) - u_2(t,x)| \le \omega(t,\epsilon),$$

for $(t,x),(t+\epsilon,x),(t,x+\epsilon) \in E_{t_0,x_0}$ with $\epsilon \ge 0$. Here ρ and ω are functions which will be specified later. The set $\Omega^0_{\rho,\omega}$ is closed and convex.

We choose ρ (for all $u \in \Omega^0_{\rho,\omega}$) to satisfy the following inequalities:

$$\begin{aligned}
|(Su)(t,x)| &\le \int_0^t \int_{\eta^{t,x}(s)}^{\theta^{t,x}(s)} |F_0(s,y,t,x,u)|\,dy\,ds \\
&\le \int_0^t \int_{\eta^{t,x}(s)}^{\theta^{t,x}(s)} \frac{L + |A(s,y)|}{2B(s,y,t,x)}(1+4\rho(s))\,ds\,dy\,ds \\
&\le K\int_0^t C_0(t-s)(1+4\rho(s))\,ds \le r(t), \\
|(S_1u)(t,x)| &\le \int_0^t |G_1(s,t,x,u)|\,ds + \int_0^t \int_{\eta^{t,x}(s)}^{\theta^{t,x}(s)} |F_1(s,y,t,x,u)|\,dy\,ds \\
&\le 2K\int_0^t (1+4\rho(s))\,ds + KC_0T\int_0^t (1+4\rho(s))\,ds \le \rho(t), \\
|(S_2u)(t,x)| &\le \rho(t).
\end{aligned}$$

With an appropriate choice of ω, we prove that the operator S maps the set $\Omega^0_{\rho,\omega}$ into itself. For this purpose, we estimate increments of the functions S_1u and S_2u:

$$\begin{aligned}
&|(S_1u)(t,x+\epsilon) - (S_1u)(t,x)| \\
&\quad \le \int_0^t |G_1(s,t,x+\epsilon,u) - G_1(s,t,x,u)|\,ds \\
&\quad\quad + \left| \int_0^t \int_{\eta^{t,x+\epsilon}(s)}^{\theta^{t,x+\epsilon}(s)} F_1(s,y,t,x+\epsilon,u)\,dy\,ds - \int_0^t \int_{\eta^{t,x}(s)}^{\theta^{t,x}(s)} F_1(s,y,t,x,u)\,dy\,ds \right|.
\end{aligned}$$

We estimate the first term of the above sum:

$$\begin{aligned}
&\int_0^t |G_1(s,t,x+\epsilon,u) - G_1(s,t,x,u)|\,ds \\
&\quad \le \int_0^t \left| \frac{f(s,y,u_{|E_{s,y}}) - A(s,y)D_x u(s,y)}{B(s,y,t,x+\epsilon)} \theta_2(s;t,x+\epsilon) \right|_{y=\theta^{t,x+\epsilon}(s)}
\end{aligned}$$

$$- \frac{f(s,y,u_{|E_{s,y}}) - A(s,y)D_x u(s,y)}{B(s,y,t,x)}\theta_2(s;t,x)\Big|_{y=\theta^{t,x}(s)}\Big| ds$$

$$+ \int_0^t \Big| \frac{f(s,y,u_{|E_{s,y}}) - A(s,y)D_x u(s,y)}{B(s,y,t,x+\epsilon)}\eta_2(s;t,x+\epsilon)\Big|_{y=\eta^{t,x+\epsilon}(s)}$$

$$- \frac{f(s,y,u_{|E_{s,y}}) - A(s,y)D_x u(s,y)}{B(s,y,t,x)}\eta_2(s;t,x)\Big|_{y=\eta^{t,x}(s)}\Big| ds$$

$$\le \int_0^t \big(K\delta_2(\epsilon) + KN_f(s, \operatorname{dist}_{K|\theta^{t,x+\epsilon}(s)-\theta^{t,x}(s)|}(u_{|E_{s,\theta^{t,x+\epsilon}(s)}}, u_{|E_{s,\theta^{t,x}(s)}}))$$

$$+ KM_f(s, \operatorname{dist}_{K|\theta^{t,x+\epsilon}(s)-\theta^{t,x}(s)|}(Du_{|E_{s,\theta^{t,x+\epsilon}(s)}}, Du_{|E_{s,\theta^{t,x}(s)}}))$$

$$+ K\omega(s, |\theta^{t,x+\epsilon}(s) - \theta^{t,x}(s)|) + K\delta_3(\epsilon)\big) ds$$

$$+ \int_0^t \big(K\delta_4(\epsilon) + KN_f(s, \operatorname{dist}_{K|\eta^{t,x+\epsilon}(s)-\eta^{t,x}(s)|}(u_{|E_{s,\eta^{t,x+\epsilon}(s)}}, u_{|E_{s,\eta^{t,x}(s)}}))$$

$$+ KM_f(s, \operatorname{dist}_{K|\eta^{t,x+\epsilon}(s)-\eta^{t,x}(s)|}(Du_{|E_{s,\eta^{t,x+\epsilon}(s)}}, Du_{|E_{s,\eta^{t,x}(s)}}))$$

$$+ K\omega(s, |\eta^{t,x+\epsilon}(s) - \eta^{t,x}(s)|) + K\delta_5(\epsilon)\big) ds.$$

Using the elementary calculus we get

$$\begin{aligned} |u(t_1,x_1) - u(t_2,x_2)| &= |D_t u(\xi_1,x_1)|\cdot|t_1-t_2| + |D_x u(t_2,\xi_2)|\cdot|x_1-x_2| \\ &\le C(|t_1-t_2| + |x_1-x_2|) \end{aligned}$$

$$(t_1 \le \xi_1 \le t_2,\ x_1 \le \xi_2 \le x_2)$$

$$|\theta^{t,x+\epsilon}(s) - \theta^{t,x}(s)| \le \epsilon e^{L_a(t-s)},$$

$$|\eta^{t,x+\epsilon}(s) - \eta^{t,x}(s)| \le \epsilon e^{L_a(t-s)},$$

where L_a is the Lipschitz constant of the function $a(t,x)$ and

$$\operatorname{dist}_{|\theta^{t,x+\epsilon}(s)-\theta^{t,x}(s)|}(u_{|E_{s,\theta^{t,x+\epsilon}(s)}}, u_{|E_{s,\theta^{t,x}(s)}}) \le M\epsilon e^{L_a t},$$

$$\operatorname{dist}_{|\eta^{t,x+\epsilon}(s)-\eta^{t,x}(s)|}(u_{|E_{s,\eta^{t,x+\epsilon}(s)}}, u_{|E_{s,\eta^{t,x}(s)}}) \le M\epsilon e^{L_a t}.$$

Moreover, we have

$$\operatorname{dist}_{|\theta^{t,x+\epsilon}(s)-\theta^{t,x}(s)|}(Du_{|E_{s,\theta^{t,x+\epsilon}(s)}}, Du_{|E_{s,\theta^{t,x}(s)}}) \le \omega(s, K\epsilon e^{L_a t}),$$

$$\operatorname{dist}_{|\eta^{t,x+\epsilon}(s)-\eta^{t,x}(s)|}(Du_{|E_{s,\eta^{t,x+\epsilon}(s)}}, Du_{|E_{s,\eta^{t,x}(s)}}) \le \omega(s, K\epsilon e^{L_a t}).$$

From the above inequalities we get

$$\int_0^t \big|G_1(s,t,x+\epsilon, u_{|E_{t,\theta^{t,x+\epsilon}(s)}}, u_{|E_{t,\eta^{t,x+\epsilon}(s)}}) \tag{4.5}$$

$$- G_1(s,t,x, u_{|E_{t,\theta^{t,x}(s)}}, u_{|E_{t,\eta^{t,x}(s)}})\big| ds$$

$$\leq \tilde{M}\tilde{\delta}(\epsilon) + C\int_0^t [N_f(s, K\epsilon e^{L_a t}) + M_f(s, \omega(s, K\epsilon e^{L_a t})) + \omega(s, \epsilon e^{L_a t})]\, ds$$

$$\leq M\delta(\epsilon) + C\int_0^t [M_f(s, \omega(s, K\epsilon e^{L_a t})) + \omega(s, \epsilon e^{L_a t})]\, ds.$$

We can show that there is a modulus of continuity $\tilde{\delta}$ such that

$$\left| \int_0^t \int_{\eta^{t,x+\epsilon}(s)}^{\theta^{t,x+\epsilon}(s)} F_1(s, y, t, x+\epsilon, u)\, dy\, ds - \int_0^t \int_{\eta^{t,x}(s)}^{\theta^{t,x}(s)} F_1(s, y, t, x, u)\, dy\, ds \right| \leq \tilde{\delta}(\epsilon). \tag{4.6}$$

From (4.5) and (4.6), there is a constant $M > 0$ and a modulus of continuity δ such that

$$|(S_1\tilde{u})(t, x+\epsilon) - (S_1\tilde{u})(t, x)| \leq M\delta(\epsilon) + M\int_0^t [M_f(s, \omega(s, K\epsilon e^{L_a t})) + \omega(s, \epsilon e^{L_a t})]\, ds.$$

Similarly, we get

$$|(S_2\tilde{u})(t, x+\epsilon) - (S_2\tilde{u})(t, x)| \leq M\delta(\epsilon) + M\int_0^t [M_f(s, \omega(s, K\epsilon e^{L_a t})) + \omega(s, \epsilon e^{L_a t})]\, ds.$$

Note that

$$\begin{aligned} |(S_1 u)(t+\epsilon, x) - (S_1 u)(t, x)| &\leq \int_t^{t+\epsilon} |G_1(s, t+\epsilon, x, u)|\, ds \\ &+ \int_0^t |G_1(s, t+\epsilon, x, u) - G_1(s, t, x, u_{|E_{s,\theta^{t,x}(s)}}, u)|\, ds \\ &+ \Bigg| \int_0^{t+\epsilon} \int_{\eta^{t+\epsilon,x}(s)}^{\theta^{t+\epsilon,x}(s)} F_1(s, y, t+\epsilon, x, u_{|E_{s,y}})\, dy\, ds \\ &- \int_0^t \int_{\eta^{t,x}(s)}^{\theta^{t,x}(s)} F_1(s, y, t, x, u)\, dy\, ds \Bigg|. \end{aligned} \tag{4.7}$$

There is a modulus of continuity $\tilde{\delta}$ such that

$$\Big|\int_0^{t+\epsilon}\int_{\eta^{t+\epsilon,x}(s)}^{\theta^{t+\epsilon,x}(s)} F_1(s,y,t+\epsilon,x,u)\,dy\,ds - \int_0^{t}\int_{\eta^{t,x}(s)}^{\theta^{t,x}(s)} F_1(s,y,t,x,u)\,dy\,ds\Big| \le \tilde{\delta}(\epsilon).$$

As before, we get

$$\int_0^t |G_1(s,t+\epsilon,x,u)-G_1(s,t,x,u)|\,ds \\ \le M\delta(\epsilon)+M\int_0^t [M_f(s,\omega(s,C\epsilon e^{L_a t}))+\omega(s,C\epsilon e^{L_a t})]\,ds.$$

There is a constant $M>0$ such that

$$\int_t^{t+\epsilon} |G_1(s,t+\epsilon,x,u)|\,ds \le M\epsilon.$$

By the above reasoning, we get

$$|(S_1u)(t+\epsilon,x)-(S_1u)(t,x)| \\ \le M\delta(\epsilon)+M\int_0^t [M_f(s,\omega(s,C\epsilon e^{L_a t}))+\omega(s,C\epsilon e^{L_a t})]\,ds.$$

Summarizing, the operator S maps the set $\Omega^0_{\rho,\omega}$ into itself if ω satisfies the integral inequality

$$\omega(t,\epsilon)\ge M\delta(\epsilon)+M\int_0^t [M_f(s,\omega(s,C\epsilon))+\omega(s,C\epsilon)]\,ds.$$

Now ω is defined as the maximal solution of the integral equation

$$\omega(t,\epsilon)= M\delta(\epsilon)+M\int_0^t [M_f(s,\omega(s,C\epsilon))+\omega(s,C\epsilon)]\,ds. \tag{4.8}$$

This solution is a modulus of continuity, i.e. it is the unique maximal solution and $\lim_{\epsilon\to 0}\omega(t,\epsilon)=0$ (see Lemma 4.1).

Lemma 4.1. *Let M_f satisfy assumption* (b) *of Theorem* 3.4. *Then there is a unique maximal solution $\omega(t,\epsilon)$ of the integral equation* (4.8) *and*

$$\lim_{\epsilon\to 0}\omega(t,\epsilon)=0 \quad \text{for } t\in[0,t_0] \text{ and } \epsilon\ge 0.$$

Proof. Let $\tilde{M}_f(t,r)=M[M_f(t,r)+r]$ for $(t,r)\in[0,t_0]\times\mathbb{R}_+$ and $\tilde{\delta}(\epsilon)=M\delta(\epsilon)$. There is no loss of generality in assuming that the function $\tilde{\delta}$ is bounded and $\tilde{M}_f$ is sublinear, i.e.

$$\tilde{M}(t,r) \le A + Br \quad \text{for all } (t,r) \in [0,t_0] \times \mathbb{R}_+,$$
$$\tilde{\delta}(\epsilon) \le C_\delta \quad \text{for all } \epsilon \in \mathbb{R}_+.$$

We define a functional sequence $\{\omega_k(t,\epsilon)\}$ by the recurrence formulas

$$\omega_0(t,\epsilon) = (C_\delta + At_0)e^{Bt},$$
$$\omega_{k+1}(t,\epsilon) = \tilde{\delta}(\epsilon) + \frac{1}{2^k} + \int_0^t \tilde{M}_f(s,\omega_k(s,C\epsilon))\,ds, \qquad k = 0,1,\ldots.$$

It is seen that $\omega_1(t,\epsilon) \le \omega_0(t,\epsilon)$, and by induction we have

$$\omega_{k+1}(t,\epsilon) = \tilde{\delta}(\epsilon) + \int_0^t \tilde{M}_f(s,\omega_k(s,C\epsilon))\,ds$$
$$\le \tilde{\delta}(\epsilon) + \int_0^t \tilde{M}_f(s,\omega_{k-1}(s,C\epsilon))\,ds = \omega_k(s,\epsilon).$$

Thus $\omega_{k+1}(t,\epsilon) \le \omega_k(t,\epsilon)$ for all $k \in \mathbb{N}$. The monotone sequence $\{\omega_k(\cdot,\epsilon)\}$ is convergent. The function $\omega(t,\epsilon) = \lim_{k\to\infty} \omega_k(t,\epsilon)$ is the unique maximal solution of (4.8). Set $\epsilon = 0$. Because $\tilde{M}_f(t,0) = \tilde{\delta}(0) = 0$ and $\tilde{M}_f(t,r)$ is the Perron function, we see that $\omega(t,0) = 0$ is the unique solution of the integral equation (4.8) with $\epsilon = 0$.

This completes the proof of Lemma 4.1 and of Theorem 3.4. ∎

Proof of Theorem 3.5. We consider the sequence of successive approximations u^n defined by $u^0(t,x) = 0$ and

$$\begin{cases} D_{tt}u^{n+1}(t,x) - (u^n(t,x)+1)^2 D_{xx}u^{n+1}(t,x) = f(t,x,u^{n+1}_{|E_{t,x}}) \\ \hfill \text{for } (t,x) \in E, \\ u^{n+1}(0,x) = 0, \quad D_t u^{n+1}(0,x) = 0 \quad \text{for } x \in \mathbb{R}. \end{cases} \tag{4.9}$$

We define bicharacteristics η_n, θ_n of the hyperbolic equation (4.9):

$$\eta_n'(s) = u^n(s,\eta_n(s)) + 1, \qquad \eta_n(t) = x,$$
$$\theta_n'(s) = -u^n(s,\theta_n(s)) - 1, \quad \theta_n(t) = x.$$

The existence and uniqueness of solutions of (4.9) follows from Theorem 3.2 with $a(t,x) = u^n(t,x) + 1$.

The Cauchy problem (4.9) is equivalent to the integral fixed-point equation

$$(4.10)\qquad u^{n+1}(t,x)=\frac{1}{2}\int_0^t\int_{\eta_n^{t,x}(s)}^{\theta_n^{t,x}(s)}\frac{f(s,y,u^n_{|E_{s,y}})}{B^n(s,y,t,x)}\,dy\,ds$$
$$-\frac{1}{2}\int_0^t\int_{\eta_n^{t,x}(s)}^{\theta_n^{t,x}(s)}\frac{[D_tu^n(s,y)+(u^n(s,y)+1)D_xu^n(s,y)]D_x^{n+1}u(s,y)}{B^n(s,y,t,x)}\,dy\,ds,$$

where

$$B^n(s,y,t,x)=\big(u^n(\tau,\theta_n^{t,x}(\tau))+1\big)\exp\Big(\int_\tau^s D_xu^n(z,\eta_n^{\tau,\theta_n^{t,x}(\tau)}(z))\,dz\Big).$$

We can use Theorem 3.2 as far as $u^n(t,x)+1>0$. So, there exists a solution of equation (4.9) on some set $[0,t_0]\times\mathbb{R}$, where $t_0>0$ is sufficiently small (see Lemma 4.2).

Define increments $e^n(t,x)=u^{n+1}(t,x)-u^n(t,x)$. Since the bicharacteristics strongly depend on the solution, we have to derive $W^{1,\infty}$ estimates which are global with respect to x. One works out the following estimates by induction on n:

$$\|e^0\|_{W^{1,\infty}(E_{t,x})}\le Ct,$$
$$\|e^n\|_{W^{1,\infty}(E_{t,x})}\le C\int_0^t\|e^{n-1}\|_{W^{1,\infty}(E_s)}\,ds.$$

As a result we get the following estimate of the increments e^n:

$$\|e^n\|_{W^{1,\infty}(E_{t_0,x_0})}\le\frac{(Ct_0)^{n+1}}{(n+1)!}.$$

At each step of the demonstration of these estimates it is important to make sure that the function $a(t,x,u)=u+1$ is strictly positive, which follows from the inequality $\|u^n\|_{W^{1,\infty}(E_{t_0,x_0})}\le e^{Ct_0}-1$, hence $u^n+1\ge 1+(e^{Ct_0}-1)$. So, if we take $t_0\in(0,T]$ so small that $e^{Ct_0}<2$, then the solution of (3.3) exists on the set $[0,t_0]\times\mathbb{R}$.

LEMMA 4.2. *Let f satisfy the assumptions of Theorem 3.5 and let u in $W^{1,\infty}([0,t_0]\times\mathbb{R})$ be such that $u+1\ge K_1$ and $\|u\|_{W^{1,\infty}([0,t_0]\times\mathbb{R})}\le K_2$ for some $K_1\in(0,1)$ and $K_2>0$. Then the solution of the problem*

$$(4.11)\qquad \begin{cases} D_{tt}v(t,x)-(u(t,x)+1)^2D_{xx}v(t,x)=f(t,x,v_{|E_{t,x}}) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } (t,x)\in[0,t_0]\times\mathbb{R}, \\ v(0,x)=0,\quad D_tv(0,x)=0\quad \text{for } x\in\mathbb{R}, \end{cases}$$

is such that $v+1\ge K_1$ and $\|v\|_{W^{1,\infty}([0,t_0]\times\mathbb{R})}\le K_2$.

Proof. The Cauchy problem (4.11) is equivalent to the integral fixed-point equations

$$
\begin{aligned}
v(t,x) &= \int_0^t \int_{\eta^{t,x}(s)}^{\theta^{t,x}(s)} F_0(s,y,t,x,v)\,dy\,ds, \\
D_x v(t,x) &= \int_0^t G_2(s,t,x,v)\,ds - \int_0^t \int_{\eta^{t,x}(s)}^{\theta^{t,x}(s)} F_2(s,y,t,x,v_{|E_{s,y}})\,dy\,ds,
\end{aligned}
\tag{4.12}
$$

where

$$
\begin{aligned}
F_0(s,y,t,x,v) &= \frac{1}{2}\,\frac{f(s,y,v_{|E_{s,y}}) - A(s,y)D_x v(s,y)}{B(s,y,t,x)}, \\
F_2(s,y,t,x,v) &= \frac{1}{2}\,\frac{f(s,y,v_{|E_{s,y}}) - A(s,y)D_x v(s,y)}{B^2(s,y,t,x)}\,\frac{\partial B}{\partial x}(s,y,t,x), \\
G_2(s,t,x,v) &= \frac{1}{2}\,\frac{f(s,y,v_{|E_{s,y}}) - A(s,y)D_x v(s,y)}{B(s,y,t,x)}\theta_3(s;t,x)\bigg|_{y=\theta^{t,x}(s)} \\
&\quad - \frac{1}{2}\,\frac{f(s,y,v_{|E_{s,y}}) - A(s,y)D_x v(s,y)}{B(s,y,t,x)}\eta_3(s;t,x)\bigg|_{y=\eta^{t,x}(s)}, \\
A(s,y) &= D_t u(s,y) + (u(s,y)+1)D_x u(s,y), \\
B(s,y,t,x) &= (u(\tau,\theta^{t,x}(\tau))+1)\exp\Big(\int_\tau^s u_2(z,\eta^{\tau,\theta^{t,x}(\tau)}(z))\,dz\Big), \\
\eta'(s) &= u(s,\eta(s))+1, \qquad \eta(t)=x, \\
\theta'(s) &= -u(s,\theta(s))-1, \qquad \theta(t)=x.
\end{aligned}
$$

From the assumptions of the lemma and from (4.12), we deduce that there exists $A>0$ such that

$$
\begin{aligned}
|v(t,x)| &\le \int_0^t \int_{\eta^{t,x}(s)}^{\theta^{t,x}(s)} A(1+|D_x v(s,y)|)\,dy\,ds, \\
|D_x v(t,x)| &\le \int_0^t A\Big(1+\sup_{y\in[\eta^{t,x}(s),\theta^{t,x}(s)]}|D_x v(s,y)|\Big)\,ds \\
&\quad + \int_0^t \int_{\eta^{t,x}(s)}^{\theta^{t,x}(s)} A(1+|D_x v(s,y)|)\,dy\,ds.
\end{aligned}
$$

It is easy to show that there is $B>0$ such that $|\theta^{t,x}(s)-\eta^{t,x}(s)|<B$. So there exists $C>0$ such that

$$
|v(t,x)| + |D_x v(t,x)| \le \int_0^t C\Big(1+\sup_{y\in[\eta^{t,x}(s),\theta^{t,x}(s)]}|D_x v(s,y)|\Big)\,ds. \tag{4.13}
$$

Therefore we get the inequality

$$\sup_{x\in\mathbb{R}} |D_x v(t,x)| \le \int_0^t C\Big(1+\sup_{x\in\mathbb{R}} |D_x v(s,x)|\Big)\, ds.$$

Now Gronwall's lemma tells us that

$$\sup_{x\in\mathbb{R}} |D_x v(t,x)| \le C t_0 e^{Ct}. \tag{4.14}$$

From (4.13) and (4.14) we get

$$\sup_{x\in\mathbb{R}} |v(t,x)| \le C t_0 e^{Ct}.$$

We may choose $t_0 > 0$ such that $v(t,x)+1 \ge 1 - Ct_0 e^{Mt_0} \ge K_1$ (because $K_1 \in (0,1)$) and $\|v\|_{W^{1,\infty}([0,t_0]\times\mathbb{R})} \le K_2$.

This completes the proof of Lemma 4.2 and of Theorem 3.5. ∎

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