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Hydromagnetic stability of stratified shear flows in the presence of cross flow

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Abstract. The hydromagnetic stability of stratified shear flows in the presence of cross flows is discussed. The magnetic field is applied in the direction of the main flow. Some necessary conditions of instability, the growth rate of unstable modes and reduction of the unstable region are discussed.

Introduction. The study of stability of stratified shear flows of an inviscid incompressible fluid is of importance in meteorology and oceanography. The linear stability of a stratified parallel shear flow of inviscid incompressible fluid has been extensively studied by many authors. L. M. Mack (1984) discussed the stability of incompressible boundary layers in the presence of cross flow. C. E. Grosch & T. J. Jackson (1991) analysed the stability of compressible mixing layers in the presence of cross flow and showed that the inclusion of cross flow enhances mixing at supersonic speed. The magnetic field also has great impact on the stability of fluid flows.

M. Padmini & M. Subbiah (1995) studied the effect of the inclusion of cross flow on the stability of shear flows. They extended the physical arguments of S. Chandrasekhar (1961) and J. W. Miles (1961) to obtain two different sufficient conditions for the stability of non-parallel flows. They also obtained some necessary conditions for instability which generalize the results of J. W. Miles (1961), L. N. Howard (1961), G. T. Kochar & R. K. Jain (1983) and M. Subbiah & R. K. Jain (1987) in the parallel flow theory.

In this paper, the work of M. Padmini & M. Subbiah (1995) is extended by considering a magnetic field applied parallel to the main flow. The stability of stratified shear flow of an inviscid, incompressible fluid confined between two rigid planes in the presence of cross flow under a parallel magnetic field is discussed. The magnetic field is assumed to be weak.

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Formulation of the stability problem. The governing equations for the motion of an inviscid, incompressible, stratified fluid confined between two horizontal infinite rigid planes situated at y_1 and y_2 under a horizontal parallel magnetic field are

$$\rho\frac{\partial v_i}{\partial t}+\rho v_j\frac{\partial v_i}{\partial x_j}=-\frac{\partial p}{\partial x_i}+\frac{\mu_e H_i}{4\pi}\frac{\partial H_i}{\partial x_j}-\rho g\lambda_i, \tag{1}$$

$$\frac{\partial H_i}{\partial t}+v_j\frac{\partial H_i}{\partial x_j}=H_j\frac{\partial v_i}{\partial x_j}, \tag{2}$$

$$\frac{\partial \rho}{\partial t}+v_j\frac{\partial \rho}{\partial x_j}=0, \tag{3}$$

$$\frac{\partial v_i}{\partial x_i}=0, \tag{4}$$

$$\frac{\partial H_i}{\partial x_i}=0, \tag{5}$$

where v_i is the velocity, ρ the density, p the pressure, g the acceleration due to gravity, $\lambda_i=(0,1,0)$ and H_i the magnetic field vector. The boundary conditions are that the vertical component of the velocity vanishes on the rigid planes situated at $y=y_1,y_2$.

Let the basic flow be given by $v_i=(U(y),0,W(y))$, $\rho_0=\rho_0(y)$, $H=(H_0,0,0)$ and $p_0=p(y)$ satisfying the governing equations. The boundary conditions provide $p_0'(y)=-\rho_0 g$, where the prime denotes differentiation with respect to y.

Let $U(y),W(y),\rho_0(y)$ and $p_0(y)$ be twice continuously differentiable functions of y in the flow domain. Let the perturbed state be given by $[U(y)+u,v,W(y)],\rho_0(y)+\rho,p_0(y)+p,(H_0+h_x,h_y,h_z)$ where u, v, w, ρ, p and (h_x,h_y,h_z) are functions of x,y,z and t.

The linearised perturbation equations for infinitesimal normal modes of the form $f(y)e^{i(kx+lz-kct)}$ are given by

$$iku+v'+ilw=0, \tag{6}$$

$$ikh_x+h_y'+ilh_z=0, \tag{7}$$

$$i\{k(U-c)+lw\}\rho+\rho_0'v=0, \tag{8}$$

$$\rho_0[i\{k(U-c)+lw\}u+U'v]=-ikp+\frac{\mu_e}{4\pi}(ikH_0h_x), \tag{9}$$

$$\rho_0[i\{k(U-c)+lw\}v]=-p'-\rho g+\frac{\mu_e}{4\pi}(ikH_0h_y), \tag{10}$$

$$\rho_0[i\{k(U-c)+lw\}w+W'v]=-ilp+\frac{\mu_e}{4\pi}(ikH_0h_z), \tag{11}$$

$$ik(U-c)h_x+ilWh_x=ikH_0u+h_yU', \tag{12}$$

$$ik(U-c)h_y+ilWh_y=ikH_0v, \tag{13}$$

$$ik(U-c)h_z+ilWh_z=ikH_0w. \tag{14}$$

Eliminating u, w, h_x, h_y, h_z and p with the help of equations (6) to (14) we get the following stability equation:

$$\begin{aligned}(15)\qquad &[\rho_0\{k(U-c)+lw\}v' - \rho_0(kU'+lW')v]' \\ &\qquad - \rho_0(k^2+l^2)\{k(U-c)+lw\}v + \frac{\rho_0(k^2+l^2)N^2v}{k(U-c)+lW} \\ &\qquad = \rho_0 S\bigg[\bigg(\frac{v}{k(U-c)+lW}\bigg)'' - \frac{(k^2+l^2)v}{k(U-c)+lW}\bigg],\end{aligned}$$

where $N^2(y) = -g\rho_0'/\rho_0$ is the square of the Brunt–Vaisala frequency and $S = \mu_E H_0^2/(4\pi\rho_0)$. The associated boundary conditions are

$$(16)\qquad v = 0 \quad \text{at } y = y_1, y_2.$$

Case of weak applied magnetic field. For weak applied magnetic field ($S \ll 1$) and large wave number, equation (15) reduces to

$$\begin{aligned}(17)\qquad &[\rho_0\{k(U-c)+lw\}v' - \rho_0(kU'+lW')v]' \\ &\qquad - \rho_0(k^2+l^2)\{k(U-c)+lw\}v + \frac{\rho_0(k^2+l^2)N^2v}{k(U-c)+lW} \\ &\qquad + \rho_0 S\bigg\{\frac{(k^2+l^2)v}{k(U-c)+lW}\bigg\} = 0.\end{aligned}$$

We now discuss some theorems:

THEOREM 1. *Corresponding to an unstable parallel flow velocity* $(U(y), 0, 0)$ *and Brunt–Vaisala frequency* N *there exists an unstable cross flow with velocity* $(U(y)/2, 0, U(y)/2)$*, Brunt–Vaisala frequency* $N/\sqrt{2}$ *and magnetic field* $S/2$.

Proof. We may consider two-dimensional disturbances for a parallel shear flow with velocity$(U(y), 0, 0)$ and applied magnetic field. Therefore the eigenvalue problem governing the stability of the parallel flow reduces to

$$(18)\qquad (\rho_0 v')' - \rho_0 k^2 v - \frac{(\rho_0 U')'v}{U-c} + \frac{\rho_0(N^2+Sk^2)v}{(U-c)^2} = 0.$$

The boundary conditions are given by

$$(19)\qquad v = 0 \quad \text{at } y = y_1, y_2.$$

If the parallel flow is unstable then this eigenvalue problem has a complex eigenvalue with $c_i > 0$ for some $k > 0$. For the corresponding cross flow with velocity $(U(y)/2, 0, U(y)/2)$, Brunt–Vaisala frequency $N/\sqrt{2}$, magnetic field $S/2$, and wave vector $(k\sqrt{2}, 0, k\sqrt{2})$, equations (17) and (16) reduce to (18) and (19) which have a complex eigenvalue with $c_i > 0$, i.e. the flow is unstable. ∎

THEOREM 2. *The cross flow with velocity* $(U(y), 0, U(y))$ *supports an internal gravity wave (under Boussinesq approximation) when* $N^2+Sk^2 > 0$.

Proof. For the velocity field $(U(y), 0, U(y))$ and $1 = -k$ the stability equation (17) (under Boussinesq approximation) reduces to

$$v'' - k^2 v + \frac{2(\beta g + Sk^2)v}{c^2} = 0. \tag{20}$$

Equation (20) together with boundary conditions (16) has the solution

$$v = \sin\left[\frac{n\pi(y-y_1)}{y_2-y_1}\right], \qquad n = 1, 2, \tag{21}$$

with

$$c^2 = \frac{2(N^2+Sk^2)}{2k^2 + \frac{n^2\pi^2}{(y_2-y_1)^2}}$$

for $N^2 + Sk^2 > 0$. Here the real value of c implies the stability of the fluid flow. ∎

THEOREM 3. *The cross flow with velocity* $(U(y), 0, U(y))$ *is unstable (under Boussinesq approximation) when* $N^2 + Sk^2 < 0$.

Proof. For the velocity field $(U(y), 0, U(y))$ and $1 = -k$ the stability equation (17) (under Boussinesq approximation) reduces to (20). When $N^2+Sk^2 < 0$, equation (20) together with boundary conditions (16) has complex eigenvalue c given by

$$c = \pm\frac{i\sqrt{N^2+Sk^2}}{\sqrt{2k^2 + \frac{n^2\pi^2}{(y_2-y_1)^2}}}.$$

This implies the instability of the fluid flow. ∎

For $S = 0$ these results reduce to that obtained by M. Padmini & M. Subbiah (1995).

Reduction of the stability problem. With the help of the transformations $k_1^2 = k^2 + l^2$, $k_1 U_1 = kU + lW$ and $k_1 c_1 = kc$ equation (17) reduces to

$$[\rho_0(U_1 - c_1)v' - \rho_0 U_1' v]' - \rho_0 k_1^2 (U_1 - c_1)v + \frac{\rho_0(N^2 + Sk_1^2)v}{U_1 - c_1} = 0. \tag{22}$$

Equation (22) is similar to the corresponding equation of hydromagnetic stratified parallel flows.

Thus the problem of stability of cross flow with velocity $(U(y), 0, W(y))$ is reduced to the problem of stability of parallel flow with velocity $(U_1(y), 0, 0)$.

Therefore we have the following results:

THEOREM 4. *A necessary condition for instability is that*

$$N^2 + Sk_1^2 < \frac{1}{4}U_1'^2 = \frac{1}{4}(U'^2 + W'^2)$$

at least at one point in the flow domain.

Proof. For unstable modes ($c_i > 0$) the transformation $v = (U_1 - c_1)^{1/2} G$ reduces equation (22) to

$$[\rho_0(U_1 - c_1)G']' - \frac{1}{2}(\rho_0 U_1')'G - \frac{\rho_0 U_1'^2 G}{4(U_1 - c_1)} - k_1^2(U_1 - c_1)G + \frac{\rho_0(N^2 + Sk_1^2)G}{U_1 - c_1} = 0. \tag{23}$$

The corresponding boundary conditions are

$$G = 0 \quad \text{at } y = y_1, y_2. \tag{24}$$

Multiplying (23) by $\overline{G}$, the complex conjugate of G, integrating the resulting equation over the flow domain and using the boundary conditions (24) we get

$$\int \rho_0(U_1 - c_1)[|G'|^2 + k_1^2|G|^2] + \frac{1}{2}\int (\rho_0 U_1')'|G|^2 + \int \rho_0\left[\frac{U_1'^2}{4} - N^2 - Sk_1^2\right]\frac{|G|^2}{U_1 - c_1} = 0. \tag{25}$$

The real and imaginary parts of (25) for unstable modes ($c_i > 0$) give

$$\int \rho_0(U_1 - c_r)[|G'|^2 + k_1^2|G|^2] + \frac{1}{2}\int (\rho_0 U_1')'|G|^2 + \int \rho_0\left[\frac{U_1'^2}{4} - N^2 - Sk_1^2\right]\frac{(U_1 - c_r)|G|^2}{|U_1 - c_1|^2} = 0 \tag{26}$$

and

$$\int \rho_0[|G'|^2 + k_1^2|G|^2] - \int \rho_0\left[\frac{U_1'^2}{4} - N^2 - Sk_1^2\right]\frac{|G|^2}{|U_1 - c_1|^2} = 0. \tag{27}$$

Now for the validity of (27) it is necessary that

$$N^2 + Sk_1^2 < \frac{1}{4}U_1'^2$$

at least at one point in the flow domain. Thus a necessary condition for instability is that

$$N^2 + Sk_1^2 < \frac{1}{4}U_1'^2 = \frac{1}{4}[U'^2 + W'^2]$$

at least at one point in the flow domain. ■

THEOREM 5. *An estimate for the growth rate of an unstable mode is given by*

$$k^2c_i^2 \le \left[\frac{1}{4}(U'^2+W'^2)-N^2-Sk_1^2\right]_{\max}.$$

Proof. From (27) it follows that

$$k^2c_i^2 \le \frac{1}{4}U_1'^2-N^2-Sk_1^2$$

at least at one point in the flow domain. Since $k_1c_1 = kc$ and $U_1'^2 = U'^2+W'^2$, the above inequality implies

$$k^2c_i^2 \le \left[\frac{U'^2+W'^2}{4}-N^2-Sk_1^2\right]_{\max}. \ \blacksquare$$

THEOREM 6. *A necessary condition for the existence of an unstable mode is that*

$$c_i^2 \le \lambda(c_r-a+b)$$

where

$$\lambda = \frac{\max(U_1'^2/4-N^2-Sk_1^2)}{ak_1^2}, \qquad a>0.$$

Proof. Adding $a-b$ times (27) to (26), we get

$$\int \rho_0(U_1-c_r+a-b)Q+\int\frac{(\rho_0U_1')'}{2}|G|^2 + \int\frac{\rho_0(U_1-c_r+b-a)(U_1'^2/4-N^2-Sk_1^2)}{|U_1-c_1|^2}|G|^2=0, \tag{28}$$

where $a = U_{1\min}$, $b = U_{1\max}$ and $Q = |G'|^2+k_1^2|G|^2$.

Now U_1-c_r+a-b is negative throughout the flow domain because $a<c_r<b$. Therefore (28) implies that

$$\int\frac{(\rho_0U_1')'}{2}|G|^2 \ge \int\frac{(a-b-U_1+c_r)(U_1'^2/4-N^2-Sk_1^2)}{|U_1-c_1|^2}|G|^2. \tag{29}$$

Adding c_r times (27) to (26), we get

$$\int\rho_0U_1Q+\int\frac{(\rho_0U_1')'}{2}|G|^2+\int\frac{\rho_0(U_1-2c_r)(U_1'^2/4-N^2-Sk_1^2)}{|U_1-c_1|^2}|G|^2=0. \tag{30}$$

Eliminating $\int\frac{(\rho_0U_1')'}{2}|G|^2$ from (29) and (30) we get

$$\int\rho_0U_1Q+\int\frac{\rho_0(a-b-c_r)(U_1'^2/4-N^2-Sk_1^2)}{|U_1-c_1|^2}|G|^2\le 0.$$

This can be written as

$$\int\rho_0U_1(|G'|^2+k_1^2|G|^2)+\int\frac{\rho_0(a-b-c_r)(U_1'^2/4-N^2-Sk_1^2)}{|U_1-c_1|^2}|G|^2\le 0. \tag{31}$$

For the validity of (31) it is necessary that

$$U_1 k_1^2 \le \frac{(U_1'^2/4 - N^2 - Sk_1^2)(c_r - a + b)}{|U_1 - c_1|^2}$$

at least at one point in the flow domain. This implies that

$$c_i^2 \le \lambda(c_r - a + b) \quad \text{where} \quad \lambda = \frac{\max(U_1'^2/4 - N^2 - Sk_1^2)}{ak_1^2}. \ \blacksquare$$

THEOREM 7. *A necessary condition for the existence of an unstable mode is that*

$$\left(c_r - \frac{a+b}{2}\right)^2 + c_i^2 \le \left(\frac{b-a}{2}\right)^2 - S_{\min}.$$

Proof. This can be proved by following the procedure of S. C. Agrawal & G. S. Agrawal (1969). ■

THEOREM 8. *A necessary condition for the existence of an unstable mode is that*

$$\left(c_r - \frac{a+b}{2}\right)^2 + c_i^2 + \frac{J_0 c_i^4}{[P + \mu^2]^2} \le \left(\frac{b-a}{2}\right)^2 - S_{\min},$$

where

$$J_0 = \min J = \min(\beta g / U_1'^2),$$

$$P^2 = \frac{U_{1\,\max}'^2}{\lambda^2 + k_1^2}\left[\frac{1}{4} - J_0 + \sqrt{\left(\frac{1}{4} - J_0\right)^2 + \frac{4(\lambda^2 + k^2)\mu^2}{U_{1\,\max}'^2}}\right]$$

$$\mu^2 = \left(\frac{9}{4} + \frac{b|U''|_{\max}}{U_{1\,\max}'^2}\right) S_{\max}$$

and λ^2 *is the lower bound of* $\int \rho|G'|^2/\rho|G|^2$.

Proof. This can be proved by following the procedure of G. T. Kochar & R. K. Jain (1983). ■

THEOREM 9. *For* $a > 0$ *and* $\lambda < \lambda_c = (3b - a) - 2\sqrt{b(2b - a) + S_{\min}}$, *the region of unstable modes obtained by S. C. Agrawal & G. S. Agrawal (1969) gets reduced.*

Proof. In Theorems 4 and 3 it is shown that arbitrary unstable modes (if any) lie inside the semi-circle and parabola given by

$$\left(c_r - \frac{a+b}{2}\right)^2 + c_i^2 = \left(\frac{b-a}{2}\right)^2 - S_{\min}, \tag{32}$$

$$c_i^2 = \lambda(c_r - a + b). \tag{33}$$

Equation (33) for given values of λ, a and b represents a parabola in the upper half of the (c_r, c_i)-plane whose axis is the axis of c_r, vertex at $a - b$

and latus rectum equal to λ. Thus for given values of a and b equation (33) gives different parabolas for different values of λ.

Now we calculate the value of $\lambda = \lambda_c$ (say) for which the parabola (33) touches the circle (32).

Eliminating c_i^2 from equations (32) and (33), we get

$$c_r^2 - (a+b-\lambda)c_r + (\lambda b - \lambda a + ab + S_{\min}) = 0. \tag{34}$$

The parabola (33) touches the circle (32) only when the two roots of c_r given by (34) coincide. For equal roots of (34), we have

$$(a+b-\lambda)^2 - 4(\lambda b - \lambda a + ab + S_{\min}) = 0.$$

This implies that

$$\lambda^2 - 2\lambda(3b-a) + (b-a)^2 - 4S_{\min} = 0.$$

This gives

$$\begin{aligned} \lambda &= (3b-a) \pm \sqrt{(3b-a)^2 - (b-a)^2 + 4S_{\min}}, \\ \lambda &= (3b-a) \pm 2\sqrt{b(2b-a) + S_{\min}}. \end{aligned} \tag{35}$$

For the critical value of λ the smaller of two values given by equation (35) is taken. Thus one obtains a criterion for reduction of the semi-circle region: If $\lambda < \lambda_c$, where

$$\lambda_c = (3b-a) - 2\sqrt{b(2b-a) + S_{\min}},$$

then the region of unstable modes gets reduced.

This proves the theorem. ∎

Concluding remarks. In this paper, the stability of stratified shear flow of an inviscid, incompressible fluid confined between two rigid planes in the presence of cross flow under a parallel magnetic field has been investigated. It has been shown that the cross flow with velocity $(U(y), 0, U(y))$ supports the internal gravity wave (under Boussinesq approximation) when $N^2 + Sk^2 > 0$, and the flow is unstable when $N^2 + Sk^2 < 0$. The stability problem for cross flow has also been reduced to the stability of parallel flows by the introduction of suitable transformations. Then some necessary conditions of instability, the growth rate of unstable modes and reduction of the unstable region have been obtained. The result on the reduction of the unstable region can be further improved by considering the semi-ellipse type region of unstable modes.

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