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Positive solutions and eigenvalue intervals of a singular third-order boundary value problem

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Abstract. This paper studies positive solutions and eigenvalue intervals of a nonlinear third-order two-point boundary value problem. The nonlinear term is allowed to be singular with respect to both the time and space variables. By constructing a proper cone and applying the Guo–Krasnosel'skiĭ fixed point theorem, the eigenvalue intervals for which there exist one, two, three or infinitely many positive solutions are obtained.

1. Introduction. Because of extensive applications to elastic mechanics and fluid mechanics (see [2, 8, 14]), nonlinear three-order ordinary differential equations have attracted wide attention in the recent years; for example, see [1, 3–7, 9–13, 15–18] and the references therein.

Let λ be a positive parameter. We consider the following nonlinear third-order two-point boundary value problem:

$$(P)\qquad \begin{cases} u'''(t)+\lambda[h(t)f(t,u(t))+g(t,u(t))]=0, & 0<t<1,\\ u(0)=u'(0)=u''(1)=0. \end{cases}$$

In this paper, $f(t,u)$ is a continuous function and is called the *continuous part* of the nonlinear term $h(t)f(t,u)+g(t,u)$; $g(t,u)$ may be singular at $t=0$, $t=1$, $u=0$ and is called the *singular part* of the nonlinear term.

Let $0<\sigma<1$ be a constant. In real problems, we can choose σ depending on the properties of the functions $h(t)$ and $f(t,u)$. Let $C[0,1]$ be the Banach space with the norm $\|u\| = \max_{0\le t\le 1}|u(t)|$. Throughout this paper, we assume that $h:(0,1)\to[0,\infty)$ is continuous and

$$0<\int_\sigma^1 h(t)\,dt \le \int_0^1 h(t)\,dt<\infty.$$

So $h(t)$ may be singular at $t=0$ and $t=1$.

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The solvability of the problem (P) has been studied by several authors; for example, see [5, 11–13, 15]. As is well known, only positive solutions are significant in many real problems. In this paper, we study the eigenvalue intervals in which every eigenvalue guarantees the existence of at least one positive solution. Here, the solution $u^*(t)$ of (P) is called *positive* if $u^*(t) > 0$ for $0 < t \leq 1$.

More recently, Li [12] proved the following theorems on positive solutions and eigenvalue intervals.

THEOREM 1.1 ([12, Theorem 3.1]) *Assume that*

(a1) $f(t,u) = f(u)$, $f : [0,\infty) \to [0,\infty)$ *is continuous and* $g(t,u) \equiv 0$.
(a2) *There exist positive numbers* $a < b$ *such that one of the following conditions is satisfied*:

(i) $\lambda \in \left[\dfrac{b}{B \min\{f(u) : \frac{1}{2}\sigma^2 b \leq u \leq b\}}, \dfrac{a}{A \max\{f(u) : 0 \leq u \leq a\}} \right]$.

(ii) $\lambda \in \left[\dfrac{a}{B \min\{f(u) : \frac{1}{2}\sigma^2 a \leq u \leq a\}}, \dfrac{b}{A \max\{f(u) : 0 \leq u \leq b\}} \right]$.

Then the problem (P) *has at least one positive solution* $u^* \in C[0,1]$ *such that* $\min_{\sigma \leq t \leq 1} u^*(t) \geq \frac{1}{2}\sigma^2 \|u^*\|$ *and* $a \leq \|u^*\| \leq b$.

THEOREM 1.2 ([12, Theorems 4.1 and 4.2]). *Assume that*

(b1) $f(t,u) = f(u)$, $f : [0,\infty) \to [0,\infty)$ *is continuous and* $g(t,u) \equiv 0$.
(b2) *One of the following conditions is satisfied*:

(i) $\lim_{u \to +0} f(u)/u = \lim_{u \to \infty} f(u)/u = \infty$ *and*

$$\lambda \in \left(0, \sup_{r>0} \frac{r}{A \max\{f(u) : 0 \leq u \leq r\}} \right).$$

(ii) $\lim_{u \to +0} f(u)/u = \lim_{u \to \infty} f(u)/u = 0$ *and*

$$\lambda \in \left(\inf_{r>0} \frac{r}{B \min\{f(u) : \frac{1}{2}\sigma^2 r \leq u \leq r\}}, \infty \right).$$

Then the problem (P) *has at least two positive solutions.*

For the constants A, B in Theorems 1.1 and 1.2, see Section 3.

When $h(t) \equiv 1$, $g(t,u) \equiv 0$ and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous, we have obtained similar results in [15].

If $f(t,u) = f(u)$ and $g(t,u) \equiv 0$, Theorems 1.1 and 1.2 are effective tools for the problem (P). It is worth mentioning that, in Theorems 1.1 and 1.2, $h(t)$ may be singular at $t = 0$ and $t = 1$, but the singularity of $f(u)$ at $u = 0$ is not allowed.

In this paper, we study the general singular problem (P). More precisely, we study the problem (P) under the following assumptions.

(H1) $f : [0,1] \times [0,\infty) \to [0,\infty)$ and $g : (0,1) \times (0,\infty) \to [0,\infty)$ are continuous.

(H2) For every pair of positive numbers $r_2 > r_1 > 0$, there is a nonnegative function $j_{r_1}^{r_2} \in C(0,1)$ such that $\int_0^1 j_{r_1}^{r_2}(t)\,dt < \infty$ and

$$g(t,u) \le j_{r_1}^{r_2}(t), \quad 0 < t < 1,\ \tfrac{1}{2}r_1 t^2 \le u \le r_2.$$

Under the assumptions (H1) and (H2), $g(t,u)$ may be singular at $t = 0$, $t = 1$ and $u = 0$. If $g(t,u)$ is continuous with respect to the time variable t and the space variable u, then it satisfies (H2).

In this paper, we do not require the existence of upper and lower solutions and do not assume that $g(t,u)$ is nonincreasing. To the best of our knowledge, there is no literature concerned with positive solutions and eigenvalue intervals of the problem (P) under the assumptions (H1) and (H2).

This paper is organized as follows. In Section 2, we construct an appropriate cone and prove complete continuity of the associated integral operator. In Section 3, we introduce three control functions in order to describe the growth behavior of the nonlinear term on some bounded sets. Further, we obtain the eigenvalue intervals for which there exist one, two, three or infinitely many positive solutions. The main tool is the Guo–Krasnosel'skiĭ fixed point theorem of cone expansion-compression type. In Section 4, we show that Theorems 1.1 and 1.2 are simple corollaries of our main results and give two examples to illustrate some applications of the present work. In addition, we correct an error in [12].

2. Preliminaries. Define a cone K in $C[0,1]$ as follows:

$$K = \{u \in C[0,1] : u(t) \ge \tfrac{1}{2}\|u\|t^2,\ 0 \le t \le 1\}.$$

For $r > 0$, write $\Omega(r) = \{u \in K : \|u\| < r\}$, $\partial\Omega(r) = \{u \in K : \|u\| = r\}$.

Let $G(t,s)$ be the Green function of the homogeneous linear problem

$$-u'''(t) = 0, \quad 0 \le t \le 1, \quad u(0) = u'(0) = u''(1) = 0,$$

that is,

$$G(t,s) = \begin{cases} \frac{1}{2}t^2 - \frac{1}{2}(t-s)^2, & 0 \le s \le t \le 1, \\ \frac{1}{2}t^2, & 0 \le t \le s \le 1. \end{cases}$$

Thus $G(t,s) \ge 0$ for $0 \le t, s \le 1$.

For $u \in K \setminus \{0\}$, define an operator T as follows:

$$(Tu)(t) = \lambda \int_0^1 G(t,s)[h(s)f(s,u(s)) + g(s,u(s))]\,ds, \quad 0 \le t \le 1.$$

Lemma 2.1.

(1) $\max_{0 \le t \le 1} G(t,s) = G(1,s) = \frac{1}{2}s(2-s)$ *for* $0 \le s \le 1$.

(2) $\frac{1}{2}t^2 G(1,s) \le G(t,s) \le G(1,s)$ *for* $0 \le t, s \le 1$.

Proof. Simple computations give that

$$\max_{0\le t\le 1} G(t,s) = G(1,s) = \tfrac{1}{2}s(2-s), \quad 0 \le s \le 1.$$

Clearly, $\frac{1}{2}s(2-s) \le 1$ for $0 \le s \le 1$. If $0 \le t \le s \le 1$, then

$$G(t,s) = \tfrac{1}{2}t^2 \ge \tfrac{1}{4}t^2 s(2-s) = \tfrac{1}{2}t^2 G(1,s).$$

It is easy to see that $2t - s \ge \frac{1}{2}t^2(2-s)$ for $0 \le s \le t \le 1$. If $0 \le s \le t \le 1$, then

$$G(t,s) = \tfrac{1}{2}t^2 - \tfrac{1}{2}(t-s)^2 = \tfrac{1}{2}s(2t-s) \ge \tfrac{1}{4}t^2 s(2-s) = \tfrac{1}{2}t^2 G(1,s). \ \blacksquare$$

LEMMA 2.2. *Suppose that* (H1) *and* (H2) *hold and* $0 < r_1 < r_2$. *Then* $T : \overline{\Omega(r_2)} \setminus \Omega(r_1) \to K$.

Proof. Let $u \in \overline{\Omega(r_2)} \setminus \Omega(r_1)$. Then $r_1 \le \|u\| \le r_2$. So

$$\tfrac{1}{2} r_1 t^2 \le \tfrac{1}{2}\|u\| t^2 \le u(t) \le r_2, \quad 0 \le t \le 1.$$

By the assumption (H2), there exists a nonnegative function $j_{r_1}^{r_2} \in C(0,1)$ such that $\int_0^1 j_{r_1}^{r_2}(t)\,dt < \infty$ and

$$g(t,u(t)) \le \max\{g(t,u) : \tfrac{1}{2}r_1 t^2 \le u \le r_2\} \le j_{r_1}^{r_2}(t), \quad 0 < t < 1.$$

Since $\int_0^1 h(t)\,dt < \infty$, one has

$$\int_0^1 [h(s)f(s,u(s)) + g(s,u(s))]\,ds \le \max_{0\le s\le 1} f(s,u(s)) \int_0^1 h(s)\,ds + \int_0^1 j_{r_1}^{r_2}(s)\,ds < \infty.$$

So, $(Tu)(t)$ is well defined and $Tu \in C[0,1]$.

On the other hand, by Lemma 2.1, for $0 \le t \le 1$,

$$\begin{aligned}
(Tu)(t) &= \lambda \int_0^1 G(t,s)[h(s)f(s,u(s)) + g(s,u(s))]\,ds \\
&\ge \tfrac{1}{2}\lambda t^2 \int_0^1 G(1,s)[h(s)f(s,u(s)) + g(s,u(s))]\,ds \\
&\ge \tfrac{1}{2}\lambda t^2 \max_{0\le t\le 1} \int_0^1 G(t,s)[h(s)f(s,u(s)) + g(s,u(s))]\,ds = \tfrac{1}{2}t^2\|Tu\|.
\end{aligned}$$

Consequently, $T : \overline{\Omega(r_2)} \setminus \Omega(r_1) \to K$. $\blacksquare$

LEMMA 2.3. *Suppose that* (H1) *and* (H2) *hold, and* $0 < r_1 < r_2$. *Then* $T : \overline{\Omega(r_2)} \setminus \Omega(r_1) \to K$ *is completely continuous.*

Proof. Define operators $\tilde{T}$ and $\bar{T}$ as follows:

$$(\tilde{T}u)(t) = \lambda \int_0^1 G(t,s)h(s)f(s,u(s))\,ds, \quad 0 \le t \le 1,\ u \in K,$$

$$(\bar{T}u)(t) = \lambda \int_0^1 G(t,s)g(s,u(s))\,ds, \qquad 0 \le t \le 1,\ u \in K \setminus \{0\}.$$

By the continuity of $f(t,u)$ and the Arzelà–Ascoli theorem, $\tilde{T} : \overline{\Omega(r_2)} \setminus \Omega(r_1) \to C[0,1]$ is completely continuous.

Let $j_{r_1}^{r_2} \in C(0,1)$ be the function given in (H2) and let, for $k = 3,4,\ldots,$

$$[j_{r_1}^{r_2}]_k(t) = \begin{cases} \min\{j_{r_1}^{r_2}(t), ktj_{r_1}^{r_2}(1/k)\}, & 0 \le t \le 1/k, \\ j_{r_1}^{r_2}(t), & 1/k \le t \le (k-1)/k, \\ \min\{j_{r_1}^{r_2}(t), k(1-t)j_{r_1}^{r_2}((k-1)/k)\}, & (k-1)/k \le t \le 1. \end{cases}$$

Then $[j_{r_1}^{r_2}]_k \in C[0,1]$ and $[j_{r_1}^{r_2}]_k(0) = [j_{r_1}^{r_2}]_k(1) = 0$. Since $\int_0^1 j_{r_1}^{r_2}(t)\,dt < \infty$, one has

$$\lim_{k\to\infty} \int_0^1 \{j_{r_1}^{r_2}(t) - [j_{r_1}^{r_2}]_k(t)\}\,dt = 0.$$

Define

$$g_k(t,u) = \begin{cases} \min\{g(t,u), [j_{r_1}^{r_2}]_k(t)\}, & \frac{1}{2}r_1t^2 \le u < \infty, \\ \min\{g(t, \frac{1}{2}r_1t^2), [j_{r_1}^{r_2}]_k(t)\}, & 0 \le u \le \frac{1}{2}r_1t^2. \end{cases}$$

Then $g_k : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous. Define an operator $\bar{T}_k$ by

$$(\bar{T}_k u)(t) = \lambda \int_0^1 G(t,s)g_k(s,u(s))\,ds, \quad 0 \le t \le 1.$$

Then $\bar{T}_k : \overline{\Omega(r_2)} \setminus \Omega(r_1) \to C[0,1]$ is completely continuous by the Arzelà–Ascoli theorem.

Let $u \in \overline{\Omega(r_2)} \setminus \Omega(r_1)$. Then $\frac{1}{2}r_1t^2 \le u(t) \le r_2$ for $0 \le t \le 1$. By the definition of $j_{r_1}^{r_2}(t)$, one has $g(t,u(t)) \le j_{r_1}^{r_2}(t)$ for $0 < t < 1$. It follows that

$$\lim_{k\to\infty} \sup_{u \in \overline{\Omega(r_2)} \setminus \Omega(r_1)} \|\bar{T}u - \bar{T}_k u\|$$
$$= \lambda \lim_{k\to\infty} \sup_{u \in \overline{\Omega(r_2)} \setminus \Omega(r_1)} \max_{0 \le t \le 1} \int_0^1 G(t,s)[g(s,u(s)) - g_k(s,u(s))]\,ds$$

$$\le \lambda \lim_{k\to\infty} \max_{0\le t\le 1} \int_0^1 G(t,s)\{j_{r_1}^{r_2}(s) - [j_{r_1}^{r_2}]_k(s)\}\, ds$$

$$\le \lambda \max_{0\le t,s\le 1} G(t,s) \lim_{k\to\infty} \int_0^1 \{j_{r_1}^{r_2}(s) - [j_{r_1}^{r_2}]_k(s)\}\, ds = 0.$$

This shows that the completely continuous operators $\bar{T}_k$, $k = 3, 4, \ldots$, uniformly converge to the operator $\bar{T}$ on the bounded closed set $\overline{\Omega(r_2)} \setminus \Omega(r_1)$. Therefore, $\bar{T} : \overline{\Omega(r_2)} \setminus \Omega(r_1) \to C[0,1]$ is completely continuous.

Since $T = \tilde{T} + \bar{T}$, from Lemma 2.2 we infer that $T : \overline{\Omega(r_2)} \setminus \Omega(r_1) \to K$ is completely continuous. ■

In order to prove our main results, we need the following Guo–Krasnosel'skiĭ fixed point theorem of cone expansion-compression type.

Lemma 2.4 (Guo–Krasnosel'skiĭ). *Let X be a Banach space, $K \subset X$ be a cone, and Ω_1, Ω_2 be bounded open subsets of K with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. Assume that $F : \overline{\Omega}_2 \setminus \Omega_1 \to K$ is a completely continuous operator such that one of the following conditions is satisfied*:

(1) $\|Fx\| \le \|x\|$ *for* $x \in \partial\Omega_1$ *and* $\|Fx\| \ge \|x\|$ *for* $x \in \partial\Omega_2$.
(2) $\|Fx\| \ge \|x\|$ *for* $x \in \partial\Omega_1$ *and* $\|Fx\| \le \|x\|$ *for* $x \in \partial\Omega_2$.

Then F has a fixed point in $\overline{\Omega}_2 \setminus \Omega_1$.

3. Main results. We introduce the following control functions which are basic tools of this paper:

$$\varphi(r) = \max\{f(t,u) : 0 \le t \le 1,\ \tfrac{1}{2}t^2 r \le u \le r\},$$
$$\psi(r) = \min\{f(t,u) : \sigma \le t \le 1,\ \tfrac{1}{2}t^2 r \le u \le r\},$$
$$\mu(r) = \int_0^1 \max\{g(t,u) : \tfrac{1}{2}t^2 r \le u \le r\}\, dt.$$

If the assumptions (H1) and (H2) hold, then $\varphi(r), \psi(r)$ and $\mu(r)$ are nonnegative real numbers for any $r > 0$.

In addition, define the constants

$$A = \max_{0\le t\le 1} \int_0^1 G(t,s)h(s)\, ds\ , \quad B = \max_{0\le t\le 1} \int_\sigma^1 G(t,s)h(s)\, ds,$$
$$C = \max_{0\le t,s\le 1} G(t,s) = 1/2.$$

If $h(t) \equiv 1$, then $A = 1/3$, $B = \max\left\{\frac{1}{2}\sigma^2(1-\sigma), \frac{1}{6}(2 - 3\sigma^2 + \sigma^3)\right\}$.

Our main results are the following existence theorems on positive solutions and eigenvalue intervals.

THEOREM 3.1. *Suppose that* (H1) *and* (H2) *hold, and there exist positive numbers $a < b$ such that one of the following conditions is satisfied*:

(c1) $\lambda \in \left[\frac{b}{B\psi(b)}, \frac{a}{A\varphi(a)+C\mu(a)}\right]$.

(c2) $\lambda \in \left[\frac{a}{B\psi(a)}, \frac{b}{A\varphi(b)+C\mu(b)}\right]$.

Then the problem (P) *has at least one positive solution* $u^* \in K \cap C^2[0,1] \cap C^3(0,1)$ *with* $a \le \|u^*\| \le b$.

Proof. Without loss of generality, we only consider the case (c1).

Since $\lambda \in \left[\frac{b}{B\psi(b)}, \frac{a}{A\varphi(a)+C\mu(a)}\right]$, one has

$$\lambda[A\varphi(a) + C\mu(a)] \le a, \qquad \lambda B\psi(b) \ge b.$$

By Lemma 2.3, the operator $T : \overline{\Omega(b)} \setminus \Omega(a) \to K$ is completely continuous.

If $u \in \partial\Omega(a)$, then $\|u\| = a$ and $\frac{1}{2}at^2 \le \frac{1}{2}\|u\|t^2 \le u(t) \le a$ for $0 \le t \le 1$. Thus, $f(t,u(t)) \le \varphi(a)$ for $0 \le t \le 1$ and $\int_0^1 g(t,u(t))\,dt \le \mu(a)$. Applying these facts, we get

$$\begin{aligned}\|Tu\| &= \lambda \max_{0\le t\le 1} \int_0^1 G(t,s)[h(s)f(s,u(s)) + g(s,u(s))]\,ds \\ &\le \lambda \max_{0\le t\le 1} \int_0^1 G(t,s)h(s)f(s,u(s))\,ds + \lambda \max_{0\le t\le 1} \int_0^1 G(t,s)g(s,u(s))\,ds \\ &\le \lambda\varphi(a) \max_{0\le t\le 1} \int_0^1 G(t,s)h(s)\,ds + \lambda \max_{0\le t,s\le 1} G(t,s) \int_0^1 g(s,u(s))\,ds \\ &\le \lambda A\varphi(a) + \lambda C\mu(a) \le a = \|u\|.\end{aligned}$$

If $u \in \partial\Omega(b)$, then $\|u\| = b$ and $\frac{1}{2}bt^2 \le u(t) \le b$ for $0 \le t \le 1$. So $f(t,u(t)) \ge \psi(b)$ for $\sigma \le t \le 1$. It follows that

$$\begin{aligned}\|Tu\| &\ge \lambda \max_{0\le t\le 1} \int_\sigma^1 G(t,s)[h(s)f(s,u(s)) + g(s,u(s))]\,ds \\ &\ge \lambda \max_{0\le t\le 1} \int_\sigma^1 G(t,s)h(s)f(s,u(s))\,ds \\ &\ge \lambda\psi(b) \max_{0\le t\le 1} \int_\sigma^1 G(t,s)h(s)\,ds = \lambda B\psi(b) \ge b = \|u\|.\end{aligned}$$

By Lemma 2.4, the operator T has a fixed point $u^* \in K$ with $a \le \|u^*\| \le b$.

Since $u^* \in K\setminus\{0\}$, we see that $h(t)f(t,u^*(t))+g(t,u^*(t))$ is an integrable function on $[0,1]$. Since $Tu^* = u^*$, one has

$$u^*(t) = \lambda \int_0^1 G(t,s)[h(s)f(s,u^*(s)) + g(s,u^*(s))]\,ds, \qquad 0 \le t \le 1.$$

Successively differentiating both sides of the equality we obtain, for $0 \le t \le 1$,

$$(u^*)^{(i)}(t) = \lambda \int_0^1 \frac{\partial^i}{\partial t^i} G(t,s)[h(s)f(s,u^*(s)) + g(s,u^*(s))]\,ds, \quad i = 1,2.$$

Here, for $0 \le s \le t \le 1$ and $0 \le t \le s \le 1$, respectively,

$$\frac{\partial}{\partial t} G(t,s) = \begin{cases} s, \\ t, \end{cases} \qquad \frac{\partial^2}{\partial t^2} G(t,s) = \begin{cases} 0, \\ 1. \end{cases}$$

It follows that

$$(u^*)''(t) = \lambda \int_t^1 [h(s)f(s,u^*(s)) + g(s,u^*(s))]\,ds, \quad 0 \le t \le 1,$$

and $u^* \in C^2[0,1]$. Since $G(0,s) \equiv 0$, $\frac{\partial}{\partial t} G(0,s) \equiv 0$, $\frac{\partial^2}{\partial t^2} G(1,s) \equiv 0$, we obtain $u^*(0) = (u^*)'(0) = (u^*)''(1) = 0$. Further,

$$(u^*)'''(t) = -\lambda[h(t)f(t,u^*(t)) + g(t,u^*(t))], \qquad 0 < t < 1.$$

Since $u^*(t) \ge \frac{1}{2}at^2 > 0$ for $0 < t \le 1$, we find that $u^* \in K \cap C^2[0,1] \cap C^3(0,1)$ and $u^*(t)$ is a positive solution of the problem (P). ∎

Theorem 3.2. *Suppose that* (H1) *and* (H2) *hold, and there exist positive numbers $a < b < c$ such that one of the following conditions is satisfied*:

(d1) $\lambda \in \left(\frac{b}{B\psi(b)}, \min\left\{\frac{a}{A\varphi(a)+C\mu(a)}, \frac{c}{A\varphi(c)+C\mu(c)}\right\}\right]$.

(d2) $\lambda \in \left[\max\left\{\frac{a}{B\psi(a)}, \frac{c}{B\psi(c)}\right\}, \frac{b}{A\varphi(b)+C\mu(b)}\right)$.

Then the problem (P) has at least two positive solutions $u_1^*, u_2^* \in K \cap C^2[0,1] \cap C^3(0,1)$ with $a \le \|u_1^*\| < b < \|u_2^*\| \le c$.

Proof. We only consider the case (d1).

Since $\lambda \in \left(\frac{b}{B\psi(b)}, \min\left\{\frac{a}{A\varphi(a)+C\mu(a)}, \frac{c}{A\varphi(c)+C\mu(c)}\right\}\right]$, one has

$$\lambda[A\varphi(a) + C\mu(a)] \le a, \qquad \lambda B\psi(b) > b, \qquad \lambda[A\varphi(c) + C\mu(c)] \le c.$$

Applying the conditions $\lambda[A\varphi(a) + C\mu(a)] \le a$ and $\lambda B\psi(b) > b$ and imitating the proof of Theorem 3.1 (c1), we deduce that the problem (P) has a positive solution $u_1^* \in K \cap C^2[0,1] \cap C^3(0,1)$ with $a \le \|u_1^*\| < b$. Applying the conditions $\lambda B\psi(b) > b$ and $\lambda[A\varphi(c) + C\mu(c)] \le c$ shows that the problem (P) has another positive solution $u_2^* \in K \cap C^2[0,1] \cap C^3(0,1)$ with $b < \|u_2^*\| \le c$. ∎

Similarly, we can prove the following theorem.

Theorem 3.3. *Suppose that* (H1) *and* (H2) *hold, and there exist positive numbers $a < b < c < d$ such that one of the following conditions is satisfied*:

(e1) $\lambda \in \left(\max\left\{\frac{b}{B\psi(b)}, \frac{d}{B\psi(d)}\right\}, \min\left\{\frac{a}{A\varphi(a)+C\mu(a)}, \frac{c}{A\varphi(c)+C\mu(c)}\right\}\right)$.

(e2) $\lambda \in \left(\max\left\{\frac{a}{B\psi(a)}, \frac{c}{B\psi(c)}\right\}, \min\left\{\frac{b}{A\varphi(b)+C\mu(b)}, \frac{d}{A\varphi(d)+C\mu(d)}\right\}\right)$.

Then the problem (P) *has at least three positive solutions* $u_1^*, u_2^*, u_3^* \in K \cap C^2[0,1] \cap C^3(0,1)$ *with* $a < \|u_1^*\| < b < \|u_2^*\| < c < \|u_3^*\| \le d$.

Obviously, a similar result holds for every positive integer n.

Furthermore, we have the following results concerned with the growth behaviors of the nonlinear term on an infinite interval.

THEOREM 3.4. *Suppose that* (H1) *and* (H2) *hold, and one of the following conditions is satisfied*:

(f1) $\lambda \in \left(\liminf_{r\to 0+} \frac{r}{B\psi(r)}, \limsup_{r\to\infty} \frac{r}{A\varphi(r)+C\mu(r)}\right)$.

(f2) $\lambda \in \left(\liminf_{r\to\infty} \frac{r}{B\psi(r)}, \limsup_{r\to 0+} \frac{r}{A\varphi(r)+C\mu(r)}\right)$.

Then the problem (P) *has at least one positive solution* $u^* \in K \cap C^2[0,1] \cap C^3(0,1)$.

Proof. If the condition (f1) is satisfied, then there exist $0 < b < a$ such that $\lambda \in \left[\frac{b}{B\psi(b)}, \frac{a}{A\varphi(a)+C\mu(a)}\right]$. By Theorem 3.1 (b1), the proof is complete. For (f2), the proof is similar. ∎

THEOREM 3.5. *Suppose that* (H1) *and* (H2) *hold, and*

$$\lambda \in \left(\liminf_{r\to\infty} \frac{r}{B\psi(r)}, \limsup_{r\to\infty} \frac{r}{A\varphi(r)+C\mu(r)}\right).$$

Then the problem (P) *has infinitely many positive solutions* $u_k^* \in K \cap C^2[0,1] \cap C^3(0,1)$, $k = 1, 2, \ldots$, *and* $\|u_k^*\| \to \infty$.

Proof. By assumption, there exist sequences $\hat{r}_k \to \infty$ and $\check{r}_k \to \infty$ such that

$$\frac{\hat{r}_k}{B\psi(\hat{r}_k)} \le \lambda \le \frac{\check{r}_k}{A\varphi(\check{r}_k)+C\mu((\check{r}_k)}, \qquad k = 1, 2, \ldots.$$

Without loss of generality, assume that

$$\hat{r}_1 < \check{r}_1 < \hat{r}_2 < \check{r}_2 < \cdots < \hat{r}_k < \check{r}_k < \hat{r}_{k+1} < \check{r}_{k+1} < \cdots.$$

By Theorem 3.1 (c2), for each $k = 1, 2, \ldots$, there exists a positive solution $u_k^* \in K \cap C^2[0,1] \cap C^3(0,1)$ such that $\hat{r}_k \le \|u_k^*\| \le \check{r}_k$. So $\|u_k^*\| \to \infty$. ∎

4. Remarks and examples. In this section, we show that our main results improve Theorems 1.1 and 1.2, and we correct an error in [12].

REMARK 4.1. Theorems 1.1 and 1.2 are simple corollaries of Theorems 3.1 and 3.2 respectively.

In fact, by (a1), $f(t,u)=f(u)$, $g(t,u)\equiv 0$ and $f(u)$ is continuous on $[0,\infty)$, so the assumptions (H1) and (H2) are satisfied. Moreover

$$\varphi(r)=\max\{f(u):0\le u\le r\},\qquad \psi(r)=\min\{f(u):\tfrac{1}{2}\sigma^2 r\le u\le r\},$$

and $\mu(r)=0$ for any $r>0$.

It follows that

$$\left[\frac{b}{B\min\{f(u):\frac{1}{2}\sigma^2 b\le u\le b\}},\frac{a}{A\max\{f(u):0\le u\le a\}}\right] = \left[\frac{b}{B\psi(b)},\frac{a}{A\varphi(a)+C\mu(a)}\right],$$

$$\left[\frac{a}{B\min\{f(u):\frac{1}{2}\sigma^2 a\le u\le a\}},\frac{b}{A\max\{f(u):0\le u\le b\}}\right] = \left[\frac{a}{B\psi(a)},\frac{b}{A\varphi(b)+C\mu(b)}\right].$$

Consequently, Theorem 1.1 follows from Theorem 3.1.

On the other hand, if $\lim_{u\to+0} f(u)/u=\lim_{u\to\infty} f(u)/u=\infty$, then

$$\lim_{r\to 0}\frac{r}{\min\{f(u):\frac{1}{2}\sigma^2 r\le u\le r\}}=\lim_{r\to\infty}\frac{r}{\min\{f(u):\frac{1}{2}\sigma^2 r\le u\le r\}}=0.$$

So, the assumption (b2)(i) implies that there exist $0<a<b<c$ such that

$$\lambda\in\left[\max\left\{\frac{a}{B\psi(a)},\frac{c}{B\psi(c)}\right\},\frac{b}{A\varphi(b)+C\mu(b)}\right).$$

Similarly, (b2)(ii) implies that there exist $0<a<b<c$ such that

$$\lambda\in\left(\frac{b}{B\psi(b)},\min\left\{\frac{a}{A\varphi(a)+C\mu(a)},\frac{c}{A\varphi(c)+C\mu(c)}\right\}\right].$$

Hence, Theorem 1.2 follows from Theorem 3.2.

Remark 4.2. In [12], the author assumes that the coefficient $h(t)$ satisfies the following condition:

(LH) $h\in C(0,1)$, $h(t)\ge 0$ for $0<t<1$ and $\int_0^1 G(1,s)h(s)\,ds<\infty$.

Unfortunately, this condition is unsuitable for Theorem 1.1.

In fact, under the conditions (a1), (a2) and (LH) a fixed point of the operator T need not be a solution of problem (P).

For example, let $\lambda=1$, $h(t)=\frac{1}{t\sqrt{1-t}}$, $f(u)\equiv 1$. Then

$$\int_0^1 G(1,s)h(s)\,ds=\frac{1}{2}\int_0^1\frac{2-s}{\sqrt{1-s}}\,ds=\frac{1}{2}\int_0^1\frac{ds}{\sqrt{1-s}}+\frac{1}{2}\int_0^1\sqrt{1-s}\,ds=\frac{4}{3}<\infty.$$

This shows that the condition (LH) is satisfied. Let

$$\begin{aligned}
u^*(t) &= \int_0^1 G(t,s)h(s)\,ds = \frac{1}{2}\int_0^t \frac{2t-s}{\sqrt{1-s}}\,ds + \frac{1}{2}t^2\int_t^1 \frac{ds}{s\sqrt{1-s}} \\
&= \frac{1}{3}(2-5t)\sqrt{1-t} + 2t - \frac{2}{3} + \frac{1}{2}t^2 \ln\frac{1-\sqrt{1-t}}{1+\sqrt{1-t}}.
\end{aligned}$$

Then, for $0 \le t \le 1$,

$$u^*(t) = \int_0^1 G(t,s)h(s)\,ds = \int_0^1 G(t,s)h(s)f(u^*(s))\,ds = (Tu^*)(t),$$

so $u^*(t)$ is a fixed point of the operator T.

However, direct calculations give that

$$(u^*)'(t) = -\frac{5}{3}\sqrt{1-t} + \frac{4t-1}{3\sqrt{1-t}} + 2 + t\ln\frac{1-\sqrt{1-t}}{1+\sqrt{1-t}}, \qquad 0 < t < 1.$$

Thus, $\lim_{t\to 1-}(u^*)'(t) = \infty$ and $(u^*)''(1) \neq 0$. Therefore, $u^*(t)$ is not a solution of the problem (P). This deficiency of the proof in [12] is corrected here by replacing the assumption $\int_0^1 G(1,s)h(s)\,ds < \infty$ with $\int_0^1 h(t)\,dt < \infty$.

Remark 4.3. According to Theorems 3.1–3.5, we can choose the constant σ according to the following rule.

For fixed $r > 0$, if $r/B\psi(r)$ is smaller, the result will be better. Therefore, if $B\psi(r)$ is larger, the result will be better.

We recall the definitions of B and $\psi(r)$:

$$B = B(\sigma) = \max_{0\le t\le 1}\int_\sigma^1 G(t,s)h(s)\,ds,$$

$$\psi(r) = \psi(r,\sigma) = \max\{f(t,u) : \sigma \le t \le 1,\ \tfrac{1}{2}rt^2 \le u \le r\}.$$

This shows that $B = B(\sigma)$ is decreasing in σ, and $\psi(r) = \psi(r,\sigma)$ is increasing in σ. Therefore, we should choose σ depending on the specific properties of the functions $h(t)$ and $f(t,u)$ in real problems. Generally, we can choose $1/4 \le \sigma \le 3/4$. In particular, $\sigma = 1/2$ is an admissible choice.

Example 4.4. Our main results are applicable to the singular problem (P) even if $f(t,u) = f(u)$ and $g(t,u) = g(u)$.

Consider the nonlinear boundary value problem

$$(P1) \qquad \begin{cases} u'''(t) + \dfrac{\lambda}{\sqrt[3]{u(t)}} = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u''(1) = 0. \end{cases}$$

In this problem, $h(t) \equiv 1$. Let

$$f(t,u)=f(u)=\min\left\{1,\frac{1}{\sqrt[3]{u}}\right\},\qquad g(t,u)=g(u)=\max\left\{0,\frac{1}{\sqrt[3]{u}}-1\right\}.$$

Then $\lambda/\sqrt[3]{u(t)}=\lambda[h(t)f(u)+g(u)]$. In addition, let $\sigma=3/4$. Then $A=1/3,\ B=9/128,\ C=1/2$.

It is easy to see that $\psi(r)=1$ if $0<r\le 1$, $\varphi(r)\le 1$ if $r\ge 1$ and $\mu(r)\le 3\sqrt[3]{2}/\sqrt[3]{r}$ for any $0<r<\infty$. From these facts, we get

$$\lim_{r\to 0+}\frac{r}{B\psi(r)}=\lim_{r\to 0+}\frac{128r}{9}=0,$$

$$\lim_{r\to\infty}\frac{r}{A\varphi(r)+C\mu(r)}\ge\lim_{r\to\infty}\frac{6r\sqrt[3]{r}}{2\sqrt[3]{r}+9\sqrt[3]{2}}=\infty.$$

By Theorem 3.4, for any $0<\lambda<\infty$, the problem $(P1)$ has a positive solution $u^*\in K\cap C^2[0,1]\cap C^3(0,1)$. Because of the singularity of $1/\sqrt[3]{u}$, this conclusion cannot be derived from Theorem 1.1.

Example 4.5. Theorem 3.1 extends Theorem 1.1 even if $g(t,u)\equiv 0$.

Consider the continuous boundary value problem

$$(P2)\quad\begin{cases}u'''(t)+\lambda\min\left\{4e^{u(t)},\dfrac{1}{\sqrt{\min\{t,1-t\}}}\right\}=0, & 0\le t\le 1,\\ u(0)=u'(0)=u''(1)=0.\end{cases}$$

Here, $h(t)\equiv 1$, the continuous part $f(t,u)$ is $\min\{4e^u,1/\sqrt{\min\{t,1-t\}}\}$ and the singular part $g(t,u)$ is 0. So $\mu(r)=0$.

Let $\sigma=1/2$. Then $A=1/3,\ B=11/48,\ C=1/2$. Moreover,

$$\varphi(r)=\max\{4e^u: 0\le u\le r\}=4e^r,$$

$$\psi(r)=\min\left\{\frac{1}{\sqrt{\min\{t,1-t\}}}:\tfrac{1}{2}\le t\le 1\right\}=4.$$

It follows that

$$\frac{1}{A\varphi(1)+C\mu(1)}=\frac{3}{4e},\qquad\frac{0.1}{B\psi(0.1)}=\frac{12}{110}.$$

By Theorem 3.1 (c2), for any $12/110\le\lambda\le 3/(4e)$, the problem $(P2)$ has a positive solution $u^*\in K\cap C^2[0,1]\cap C^3(0,1)$ with $0.1\le\|u^*\|\le 1$. Because the function $\min\{4e^u,1/\sqrt{\min\{t,1-t\}}\}$ cannot be decomposed as $h(t)f(u)$, the conclusion cannot be derived from Theorem 1.1.

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