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Biharmonic Riemannian maps

by Bayram Şahin (Malatya)

Abstract. We give necessary and sufficient conditions for Riemannian maps to be biharmonic. We also define pseudo-umbilical Riemannian maps as a generalization of pseudo-umbilical submanifolds and show that such Riemannian maps put some restrictions on the target manifolds.

1. Introduction. Smooth maps between Riemannian manifolds are useful for comparing geometric structures between two manifolds. Isometric immersions (Riemannian submanifolds) are basic such maps between Riemannian manifolds and they are characterized by the Riemannian metrics and Jacobian matrices. More precisely, a smooth map $F : (M_1, g_1) \to (M_2, g_2)$ between Riemannian manifolds (M_1, g_1) and (M_2, g_2) is called an *isometric immersion* if F_* is injective and

$$g_2(F_*X, F_*Y) = g_1(X, Y) \tag{1.1}$$

for any vector fields X, Y tangent to M_1, where F_* denotes the derivative map.

On the other hand, the study of Riemannian submersions between Riemannian manifolds was initiated by B. O'Neill [O] and A. Gray [G] (see also [FIP] and [YK]). A smooth map $F : (M_1, g_1) \to (M_2, g_2)$ is called a *Riemannian submersion* if F_* is onto and (1.1) holds for vector fields tangent to the horizontal space $(\ker F_*)^\perp$. For Riemannian submersions between various manifolds, see [FIP] and [YK].

In 1992, Fischer [F] introduced Riemannian maps between Riemannian manifolds as a generalization of isometric immersions and Riemannian submersions. Let $F : (M_1, g_1) \to (M_2, g_2)$ be a smooth map between Riemannian manifolds such that $0 < \operatorname{rank} F < \min\{m, n\}$, where $\dim M_1 = m$ and $\dim M_2 = n$. We denote the kernel space of F_* by $\ker F_*$ and consider the orthogonal complementary space $\mathcal{H} = (\ker F_*)^\perp$ to $\ker F_*$. Then the tangent

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bundle of M_1 has the decomposition

$$TM_1 = \ker F_* \oplus \mathcal{H}.$$

We denote the range of F_* by $\operatorname{range} F_*$ and consider its orthogonal complement $(\operatorname{range} F_*)^\perp$ in the tangent bundle TM_2. Since $\operatorname{rank} F < \min\{m, n\}$, we always have $(\operatorname{range} F_*)^\perp \neq \{0\}$. Thus TM_2 has the following decomposition:

$$TM_2 = (\operatorname{range} F_*) \oplus (\operatorname{range} F_*)^\perp.$$

Now, a smooth map $F : (M_1^m, g_1) \to (M_2^n, g_2)$ is called a *Riemannian map* at $p_1 \in M_1$ if the horizontal restriction $F^{\mathrm{h}}_{*p_1} : (\ker F_{*p_1})^\perp \to (\operatorname{range} F_{*p_1})$ is a linear isometry between the inner product spaces $((\ker F_{*p_1})^\perp, g_1(p_1)|_{(\ker F_{*p_1})^\perp})$ and $(\operatorname{range} F_{*p_1}, g_2(p_2)|_{\operatorname{range} F_{*p_1}})$, $p_2 = F(p_1)$. Fischer stated in [F] that a Riemannian map is a map which is as isometric as it can be. In other words, F_* satisfies (1.1) for vector fields X, Y tangent to $\mathcal{H} = (\ker F_*)^\perp$. It follows that isometric immersions and Riemannian submersions are particular Riemannian maps with $\ker F_* = \{0\}$ and $(\operatorname{range} F_*)^\perp = \{0\}$. It is known that a Riemannian map is a subimmersion. One of the main properties of Riemannian maps is that they satisfy the eikonal equation which is a link between geometric optics and physical optics. For Riemannian maps and their applications, see [GK].

A map between Riemannian manifolds is harmonic if the divergence of its differential vanishes. Harmonic maps between Riemannian manifolds provide a rich display of both differential geometric and analytic phenomena, and they are closely related to the theory of stochastic processes and to the theory of liquid crystals in material science. On the other hand, biharmonic maps are critical points of the bienergy functional and, from this point of view, they generalize harmonic maps. The notion of biharmonic map was suggested by Eells and Sampson [ES]. The first variation formula, and thus the Euler–Lagrange equation associated to the bienergy was obtained by Jiang [J1], [J2]. Biharmonic maps have been extensively studied in the last decade and there are two main research directions. In differential geometry, many authors have obtained classification results and constructed many examples. Biharmonicity of immersions was obtained in [CI], [CM], [OC] and biharmonic Riemannian submersions were studied in [OC]; for a survey on biharmonic maps, see [MO]. From the analytic point of view, biharmonic maps are solutions of fourth order strongly elliptic semilinear partial differential equations. It is known that plane elastic problems can be expressed in terms of the biharmonic equation. On the other hand, wave maps are harmonic maps on Minkowski spaces and biwave maps are biharmonic maps on Minkowski spaces. Wave maps arise in the analysis of the more difficult hyperbolic Yang–Mills equations either as special cases or as equations for certain families of gauge transformations. Such equations arise in general relativity for spacetimes with two Killing vector fields. Bi-Yang–Mills fields, which generalize Yang–Mills fields, have recently been introduced by Bejan and Urakawa [BU]. For relations between biwave maps and the bi-Yang–Mills equations, see [IIU] and [Ch]. Moreover, in geometric optics [D], one can obtain the eikonal equation by using the wave equation.

In this paper, we mainly investigate the biharmonicity of Riemannian maps from Riemannian manifolds to space forms. In Section 2, we introduce notation and give fundamental formulas for the bitension field. Then we obtain some preparatory results on Riemannian maps in Section 3. We also define pseudo-umbilical Riemannian maps as a generalization of pseudo-umbilical isometric immersions, obtain a necessary and sufficient condition for a Riemannian map to be pseudo-umbilical and give a method of constructing pseudo-umbilical Riemannian maps. In Section 4, we find necessary and sufficient conditions for Riemannian maps to be harmonic and observe that pseudo-umbilical Riemannian maps from Riemannian manifolds M_1 to space forms $M_2(c)$ with additional conditions are either harmonic or have $c > 0$.

2. Preliminaries. In this section we recall some basic material from [BW] and [MO]. Let (M, g_M) be a Riemannian manifold and $\mathcal{V}$ be a q-dimensional distribution on M. Denote its orthogonal distribution $\mathcal{V}^\perp$ by $\mathcal{H}$. Then we have

$$TM = \mathcal{V} \oplus \mathcal{H}. \tag{2.1}$$

$\mathcal{V}$ is called the *vertical distribution* and $\mathcal{H}$ the *horizontal distribution*. We use the same letters to denote the orthogonal projections onto these distributions.

By the *unsymmetrized second fundamental form* of $\mathcal{V}$, we mean the tensor field $A^\mathcal{V}$ defined by

$$A^\mathcal{V}_E F = \mathcal{H}(\nabla_{\mathcal{V}E}\mathcal{V}F), \qquad E, F \in \Gamma(TM), \tag{2.2}$$

where ∇ is the Levi-Civita connection on M. The *symmetrized second fundamental form* $B^\mathcal{V}$ of $\mathcal{V}$ is given by

$$B^\mathcal{V}(E,F) = \tfrac{1}{2}\{A^\mathcal{V}_E F + A^\mathcal{V}_F E\} = \tfrac{1}{2}\{\mathcal{H}(\nabla_{\mathcal{V}E}\mathcal{V}F) + \mathcal{H}(\nabla_{\mathcal{V}F}\mathcal{V}E)\} \tag{2.3}$$

for any $E, F \in \Gamma(TM)$. The *integrability tensor* of $\mathcal{V}$ is the tensor field $I^\mathcal{V}$ given by

$$I^\mathcal{V}(E,F) = A^\mathcal{V}_E F - A^\mathcal{V}_F E - \mathcal{H}([\mathcal{V}E, \mathcal{V}F]). \tag{2.4}$$

Moreover, the *mean curvature vector field* of $\mathcal{V}$ is defined by

$$\mu^{\mathcal{V}} = \frac{1}{q}\,\mathrm{Trace}\,B^{\mathcal{V}} = \frac{1}{q}\sum_{r=1}^{q}\mathcal{H}(\nabla_{e_r}e_r), \tag{2.5}$$

where $\{e_1,\dots,e_q\}$ is a local frame of $\mathcal{V}$. By reversing the roles of $\mathcal{V}$, $\mathcal{H}$, one can define $B^{\mathcal{H}}$, $A^{\mathcal{H}}$ and $I^{\mathcal{H}}$ similarly. For instance, $B^{\mathcal{H}}$ is defined by

$$B^{\mathcal{H}}(E,F) = \tfrac{1}{2}\{\mathcal{V}(\nabla_{\mathcal{H}E}\mathcal{H}F) + \mathcal{V}(\nabla_{\mathcal{H}F}\mathcal{H}E)\}, \tag{2.6}$$

and hence we have

$$\mu^{\mathcal{H}} = \frac{1}{m-q}\,\mathrm{Trace}\,B^{\mathcal{H}} = \frac{1}{m-q}\sum_{s=1}^{m-q}\mathcal{V}(\nabla_{E_s}E_s), \tag{2.7}$$

where $E_1,\dots,E_{m-q}$ is a local frame of $\mathcal{H}$. A distribution $\mathcal{D}$ on M is said to be *minimal* if, for each $x\in M$, the mean curvature vector field vanishes.

Let (M,g_M) and (N,g_N) be Riemannian manifolds and suppose that $\varphi: M\to N$ is a smooth map between them. Then the differential φ_* of φ can be viewed as a section of the bundle $\mathrm{Hom}(TM,\varphi^{-1}TN)\to M$, where $\varphi^{-1}TN$ is the pullback bundle which has fibres $(\varphi^{-1}TN)_p = T_{\varphi(p)}N$, $p\in M$. $\mathrm{Hom}(TM,\varphi^{-1}TN)$ has a connection ∇ induced from the Levi-Civita connection ∇^M and the pullback connection. Then the second fundamental form of φ is given by

$$(\nabla\varphi_*)(X,Y) = \nabla^{\varphi}_X\varphi_*(Y) - \varphi_*(\nabla^M_X Y) \tag{2.8}$$

for $X,Y\in\Gamma(TM)$. It is known that the second fundamental form is symmetric.

Now assume M is compact. Then the *energy* of the map φ is

$$E(\varphi) = \int_M e(\varphi)v_g = \frac{1}{2}\int_M |d\varphi|^2 v_g.$$

The critical points of E are called *harmonic maps*. Standard arguments yield the associated Euler–Lagrange equation, the vanishing of the *tension field* $\tau(\varphi) = \mathrm{trace}(\nabla\varphi_*)$. The *bienergy* of φ is defined by

$$E^2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 v_g.$$

Critical points of the functional E^2 are called *biharmonic maps* and its associated Euler–Lagrange equation is the vanishing of the *bitension field*

$$\tau^2(\varphi) = -\Delta^{\varphi}\tau(\varphi) - \mathrm{trace}_{g_M} R^N(d\varphi,\tau(\varphi))d\varphi, \tag{2.9}$$

where $\Delta^{\varphi}\tau(\varphi) = -\,\mathrm{trace}_{g_M}(\nabla^{\varphi}\nabla^{\varphi} - \nabla^{\varphi}_{\nabla})$ is the Laplacian on the sections of $\varphi^{-1}(TN)$ and R^N is the Riemann curvature operator on (N,g_N). A map between two Riemannian manifolds is said to be *proper biharmonic* if it is a non-harmonic biharmonic map.

3. Riemannian maps. In this section, we obtain some new results which will be used in the next section. First note that in [S2] we showed that the second fundamental form $(\nabla F_*)(X,Y)$, for $X,Y \in \Gamma((\ker F_*)^\perp)$, of a Riemannian map has no components in range F_*. Here we give an elementary proof of that result.

LEMMA 3.1. *Let F be a Riemannian map from a Riemannian manifold (M_1, g_1) to a Riemannian manifold (M_2, g_2). Then*

$$g_2((\nabla F_*)(X,Y), F_*(Z)) = 0, \quad \forall X,Y,Z \in \Gamma((\ker F_*)^\perp). \tag{3.1}$$

Proof. Since F is a Riemannian map, from (2.8) we have

$$g_2((\nabla F_*)(X,Y), F_*(Z)) = g_2(\nabla^F_X F_*Y, F_*Z) - g_1(\nabla^1_X Y, Z). \tag{3.2}$$

On the other hand, since ∇^1 is a Levi-Civita connection, from the Koszul identity we have

$$\begin{aligned} g_1(\nabla^1_X Y, Z) = \tfrac{1}{2}\{&Xg_1(Y,Z) + Yg_1(X,Z) - Zg_1(X,Y) \\ &+ g_1([X,Y],Z) + g_1([Z,X],Y) - g_1([Y,Z],X)\}. \end{aligned}$$

Since $F_*([X,Y]) = [F_*X, F_*Y]$, using $g_1(X,Y) = g_2(F_*X, F_*Y)$, we obtain

$$\begin{aligned} g_1(\nabla^1_X Y, Z) = \tfrac{1}{2}\{&Xg_2(F_*Y, F_*Z) + Yg_2(F_*X, F_*Z) - Zg_2(F_*X, F_*Y) \\ &+ g_2([F_*X, F_*Y], F_*Z) + g_2([F_*Z, F_*X], F_*Y) \\ &- g_2([F_*Y, F_*Z], F_*X)\}. \end{aligned}$$

Since ∇^2 is also a Levi-Civita connection, we have

$$g_1(\nabla^1_X Y, Z) = g_2(\nabla^F_X F_*Y, F_*Z). \tag{3.3}$$

Thus the assertion follows from (3.2) and (3.3). ∎

As a result of Lemma 3.1, we have

$$(\nabla F_*)(X,Y) \in \Gamma((\operatorname{range} F_*)^\perp), \quad \forall X,Y \in \Gamma((\ker F_*)^\perp). \tag{3.4}$$

Also from [S1], we have the following.

LEMMA 3.2. *Let $F : (M, g_M) \to (N, g_N)$ be a Riemannian map. Then the tension field τ of F is*

$$\tau = -m_1 F_*(\mu^{\ker F_*}) + m_2 H_2, \tag{3.5}$$

where $m_1 = \dim(\ker F_*)$, $m_2 = \operatorname{rank} F$, $\mu^{\ker F_*}$ and H_2 are the mean curvature vector fields of the distributions of $\ker F_*$ *and* $\operatorname{range} F_*$, *respectively.*

From now on, for simplicity, we denote by ∇^2 both the Levi-Civita connection of (M_2, g_2) and its pullback along F. Then according to [N], for any vector field X on M_1 and any section V of $(\operatorname{range} F_*)^\perp$, where $(\operatorname{range} F_*)^\perp$ is the subbundle of $F^{-1}(TM_2)$ with fibre $(F_*(T_pM))^\perp$, the orthogonal complement of $F_*(T_pM)$ for g_2 over p, we have $\nabla^{F\perp}_X V$ which is the orthogonal projection of $\nabla^2_X V$ on $(F_*(TM))^\perp$. In [N], the author also showed that $\nabla^{F\perp}$

is a linear connection on $(F_*(TM))^\perp$ such that $\nabla^{F\perp} g_2 = 0$. We now define A_V as

$$\nabla^2_{F_*X} V = -A_V F_* X + \nabla^{F\perp}_X V, \tag{3.6}$$

where $A_V F_* X$ is the tangential component (a vector field along F) of $\nabla^2_{F_*X} V$. It is easy to see that $A_V F_* X$ is bilinear in V and $F_* X$, and at p it depends only on V_p and $F_{*p} X_p$. By direct computations, we obtain

$$g_2(A_V F_* X, F_* Y) = g_2(V, (\nabla F_*)(X, Y)) \tag{3.7}$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma((\operatorname{range} F_*)^\perp)$. Since ∇F_* is symmetric, it follows that A_V is a symmetric linear transformation of range F_*.

We now define pseudo-umbilical Riemannian maps as a generalization of pseudo-umbilical isometric immersions. Pseudo-umbilical Riemannian maps will be useful when we deal with the biharmonicity of Riemannian maps.

Definition 3.3. Let $F : (M_1, g_1) \to (M_2, g_2)$ be a Riemannian map. Then we say that F is a *pseudo-umbilical Riemannian map* if

$$A_{H_2} F_*(X) = \lambda F_*(X) \tag{3.8}$$

for $\lambda \in C^\infty(M_1)$ and $X \in \Gamma((\ker F_*)^\perp)$.

Here we present a useful formula for pseudo-umbilical Riemannian maps by using (3.7) and (3.8).

Proposition 3.4. *Let $F : (M_1, g_1) \to (M_2, g_2)$ be a Riemannian map. Then F is pseudo-umbilical if and only if*

$$g_2((\nabla F_*)(X, Y), H_2) = g_1(X, Y) g_2(H_2, H_2) \tag{3.9}$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$.

Proof. Let $\{\tilde e_1, \ldots, \tilde e_{m_1}, e_1, \ldots, e_{m_2}\}$ be an orthonormal basis of $\Gamma(TM_1)$ such that $\{\tilde e_1, \ldots, \tilde e_{m_1}\}$ is an orthonormal basis of $\ker F_*$ and $\{e_1, \ldots, e_{m_2}\}$ is an orthonormal basis of $(\ker F_*)^\perp$. Since F is a Riemannian map we have

$$\sum_{i=1}^{m_2} g_2(A_{H_2} F_*(e_i), F_*(e_i)) = m_2 \lambda.$$

Using (3.7), we get

$$\sum_{i=1}^{m_2} g_2\Big(\frac{1}{m_2} (\nabla F_*)(e_i, e_i), H_2\Big) = \lambda.$$

Thus we obtain

$$\lambda = g_2(H_2, H_2). \tag{3.10}$$

Then, from (3.7), (3.8) and (3.10) we obtain (3.9). The converse is clear. ∎

It is known that the composition of a Riemannian submersion and an isometric immersion is a Riemannian map [F]. Using this we have the following.

THEOREM 3.5. *Let $F_1 : (M_1, g_1) \to (M_2, g_2)$ be a Riemannian submersion and $F_2 : (M_2, g_2) \to (M_3, g_3)$ a pseudo-umbilical isometric immersion. Then $F_2 \circ F_1$ is a pseudo-umbilical Riemannian map.*

Proof. From the second fundamental form of $F_2 \circ F_1$ [BW], we have

$$(\nabla(F_2 \circ F_1)_*)(X, Y) = F_{2*}((\nabla F_{1*})(X, Y)) + (\nabla F_{2*})(F_{1*}X, F_{1*}Y)$$

for $X, Y \in \Gamma((\ker F_{1*})^{\perp})$. Then the assertion follows from the definition of pseudo-umbilical submanifolds. ∎

REMARK. We note that the above theorem gives a method to find examples of pseudo-umbilical Riemannian maps. It also tells us that if one has an example of pseudo-umbilical submanifolds, it is possible to find an example of pseudo-umbilical Riemannian maps. For examples of pseudo-umbilical submanifolds, see [C].

4. Biharmonicity of Riemannian maps. In this section we obtain the biharmonicity of Riemannian maps between Riemannian manifolds. We also show that pseudo-umbilical biharmonic Riemannian maps put some restrictions on the target manifold of such maps.

Let $F : (M_1, g_1) \to (M_2, g_2)$ be a map between Riemannian manifolds. Then the adjoint map ${}^*F_*$ of F_* is characterized by $g_1(x, {}^*F_{*p_1}y) = g_2(F_{*p_1}x, y)$ for $x \in T_{p_1}M_1$, $y \in T_{F(p_1)}M_2$ and $p_1 \in M_1$. Considering F_*^{h} at each $p_1 \in M_1$ as a linear transformation

$$F_{*p_1}^{\mathrm{h}} : ((\ker F_*)^{\perp}(p_1), g_{1_{p_1((\ker F_*)^{\perp}(p_1))}}) \to (\operatorname{range} F_*(p_2), g_{2_{p_2(\operatorname{range} F_*)(p_2))}}),$$

we will denote the adjoint of $F_{*p_1}^{\mathrm{h}}$ by ${}^*F_{*p_1}^{\mathrm{h}}$. Let ${}^*F_{*p_1}$ be the adjoint of $F_{*p_1} : (T_{p_1}M_1, g_{1_{p_2}}) \to (T_{p_2}M_2, g_{2_{p_2}})$. Then the linear transformation

$$({}^*F_{*p_1})^{\mathrm{h}} : \operatorname{range} F_*(p_2) \to (\ker F_*)^{\perp}(p_1)$$

defined by $({}^*F_{*p_1})^{\mathrm{h}}y = {}^*F_{*p_1}y$, where $y \in \Gamma(\operatorname{range} F_{*p_1})$, $p_2 = F(p_1)$, is an isomorphism and $(F_{*p_1}^{\mathrm{h}})^{-1} = ({}^*F_{*p_1})^{\mathrm{h}} = {}^*(F_{*p_1}^{\mathrm{h}})$.

We also recall that the curvature tensor R of a space form $(M(c), g)$ is given by

$$R(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y\}. \tag{4.1}$$

We are now ready to prove the following theorem which gives necessary and sufficient conditions for a Riemannian map to be biharmonic.

THEOREM 4.1. *Let F be a Riemannian map from a Riemannian manifold $(M_1^{m_1+m_2}, g_1)$ to a space form $(M_2(c), g_2)$. Then F is biharmonic if and only if*

$$\begin{aligned}(4.2)\qquad & m_1 \operatorname{trace} A_{(\nabla F_*)(\cdot,\mu^{\ker F_*})} F_*(\cdot) - m_1 \operatorname{trace} F_*(\nabla_{(\cdot)}\nabla_{(\cdot)}\mu^{\ker F_*}) \\ & - m_2 \operatorname{trace} F_*(\nabla_{(\cdot)}{}^*F_*(A_{H_2}F_*(\cdot))) - m_2 \operatorname{trace} A_{\nabla^{F\perp}_{F_*(\cdot)}H_2} F_*(\cdot) \\ & - m_1 c(m_2-1) F_*(\mu^{\ker F_*}) = 0\end{aligned}$$

and

$$\begin{aligned}(4.3)\qquad & m_1 \operatorname{trace} \nabla^{F\perp}_{F_*(\cdot)}(\nabla F_*)(\cdot,\mu^{\ker F_*}) + m_1 \operatorname{trace}(\nabla F_*)(\cdot,\nabla_{(\cdot)}\mu^{\ker F_*}) \\ & + m_2 \operatorname{trace}(\nabla F_*)(\cdot, {}^*F_*(A_{H_2}F_*(\cdot))) - m_2 \Delta^{R^\perp} H_2 \\ & - m_2^2 c H_2 = 0.\end{aligned}$$

Proof. First of all, from (4.1) and (3.6) we have

$$(4.4)\qquad \operatorname{trace} R^2(F_*(\cdot),\tau(F))F_*(\cdot) = m_1 c(m_2-1)F_*(\mu^{\ker F_*}) - m_2^2 c H_2,$$

where R^2 is the curvature tensor field of M_2. Let $\{\tilde{e}_1,\dots,\tilde{e}_{m_1}, e_1,\dots,e_{m_2}\}$ be a local orthonormal frame on M_1, geodesic at $p \in M_1$, such that $\{\tilde{e}_1,\dots,\tilde{e}_{m_1}\}$ is an orthonormal basis of $\ker F_*$ and $\{e_1,\dots,e_{m_2}\}$ is an orthonormal basis of $(\ker F_*)^\perp$. At p we have

$$\Delta\tau(F) = -\sum_{i=1}^{m_2} \nabla^F_{e_i}\nabla^F_{e_i}\tau(F) = -\sum_{i=1}^{m_2}\nabla^F_{e_i}\{\nabla^F_{e_i}(-m_1F_*(\mu^{\ker F_*}) + m_2H_2)\}.$$

Then using (2.8), (3.4) and (3.6) we get

$$\begin{aligned}\Delta\tau(F) = -\sum_{i=1}^{m_2}\nabla^F_{e_i}\{&-m_1(\nabla F_*)(e_i,\mu^{\ker F_*}) - m_1F_*(\nabla_{e_i}\mu^{\ker F_*}) \\ &+ m_2(-A_{H_2}F_*(e_i) + \nabla^{F\perp}_{F_*(e_i)}H_2)\}.\end{aligned}$$

Using again (2.8), (3.4) and (3.6) we obtain

$$\begin{aligned}\Delta\tau(F) = {} & m_1\sum_{i=1}^{m_2}\{-A_{(\nabla F_*)(e_i,\mu^{\ker F_*})}F_*(e_i) + \nabla^{F\perp}_{F_*(e_i)}(\nabla F_*)(e_i,\mu^{\ker F_*})\} \\ & + m_1\sum_{i=1}^{m_2}\{(\nabla F_*)(e_i,\nabla_{e_i}\mu^{\ker F_*}) + F_*(\nabla_{e_i}\nabla_{e_i}\mu^{\ker F_*})\} \\ & + m_2\sum_{i=1}^{m_2}\nabla^F_{e_i}A_{H_2}F_*(e_i) - m_2\sum_{i=1}^{m_2} -A_{\nabla^{F\perp}_{F_*(e_i)}H_2}F_*(e_i) \\ & + \nabla^{F\perp}_{F_*(e_i)}\nabla^{F\perp}_{F_*(e_i)}H_2.\end{aligned}$$

On the other hand, since $A_{H_2}F_*(e_i) \in \Gamma(F_*((\ker F_*)^\perp))$, we can write

$$F_*(X) = A_{H_2}F_*(e_i)$$

for $X \in \Gamma((\ker F_*)^\perp)$, where

$$X = (F_*)^{-1}(A_{H_2}F_*(e_i)) = {}^*F_*(A_{H_2}F_*(e_i)).$$

Then using (2.8) we have

$$\nabla^F_{e_i} A_{H_2} F_*(e_i) = (\nabla F_*)(e_i, {}^*F_*(A_{H_2} F_*(e_i))) + F_*(\nabla_{e_i} {}^*F_*(A_{H_2} F_*(e_i))).$$

Thus we obtain

$$\begin{aligned}(4.5)\quad \Delta\tau(F) = m_1 \sum_{i=1}^{m_2} \{-A_{(\nabla F_*)(e_i,\mu^{\ker F_*})} F_*(e_i) + \nabla^{F\perp}_{F_*(e_i)} (\nabla F_*)(e_i, \mu^{\ker F_*})\} \\ + m_1 \sum_{i=1}^{m_2} \{(\nabla F_*)(e_i, \nabla_{e_i} \mu^{\ker F_*}) + F_*(\nabla_{e_i} \nabla_{e_i} \mu^{\ker F_*})\} \\ + m_2 \sum_{i=1}^{m_2} \{(\nabla F_*)(e_i, {}^*F_*(A_{H_2} F_*(e_i))) + F_*(\nabla_{e_i} {}^*F_*(A_{H_2} F_*(e_i)))\} \\ - m_2 \sum_{i=1}^{m_2} \{-A_{\nabla^{F\perp}_{F_*(e_i)} H_2} F_*(e_i) + \nabla^{F\perp}_{F_*(e_i)} \nabla^{F\perp}_{F_*(e_i)} H_2\}.\end{aligned}$$

Thus putting (4.4) and (4.5) in (2.9) and then taking the $F_*((\ker F_*)^\perp) = \operatorname{range} F_*$ and $(\operatorname{range} F_*)^\perp$ parts we obtain (4.2) and (4.3). ■

In particular, we have the following.

COROLLARY 4.2. *Let F be a Riemannian map from a Riemannian manifold (M_1, g_1) to a space form $(M_2(c), g_2)$. If the mean curvature vector fields of* range F_* *and* $\ker F_*$ *are parallel, then F is biharmonic if and only if*

$$\begin{aligned}(4.6)\quad m_1 \operatorname{trace} A_{(\nabla F_*)(\cdot,\mu^{\ker F_*})} F_*(\cdot) - m_2 \operatorname{trace} F_*(\nabla_{(\cdot)} {}^*F_*(A_{H_2} F_*(\cdot)) \\ - m_1 c(m_2 - 1) F_*(\mu^{\ker F_*}) = 0\end{aligned}$$

and

$$\begin{aligned}(4.7)\quad m_1 \operatorname{trace} \nabla^{F\perp}_{F_*(\cdot)} (\nabla F_*)(\cdot, \mu^{\ker F_*}) + m_2 \operatorname{trace} (\nabla F_*)(\cdot, {}^*F_*(A_{H_2} F_*(\cdot))) \\ - m_2^2 c H_2 = 0.\end{aligned}$$

We also have the following result for pseudo-umbilical Riemannian maps.

THEOREM 4.3. *Let F be a pseudo-umbilical biharmonic Riemannian map from a Riemannian manifold (M_1, g_1) to a space form $(M_2(c), g_2)$ such that the distribution* $\ker F_*$ *is minimal and the mean curvature vector field H_2 is parallel. Then either F is harmonic or $c = \|H_2\|^2$.*

Proof. First note that it is easy to see that $\|H_2\|^2$ is constant. If F is biharmonic Riemannian map such that $\mu^{\ker F_*} = 0$ and H_2 is parallel, then from (4.3) we have

$$m_2 \sum_{i=1}^{m_2} (\nabla F_*)(e_i, {}^*F_*(A_{H_2} F_*(e_i))) - m_2^2 c H_2 = 0.$$

Since F is pseudo-umbilical, we get

$$m_2\sum_{i=1}^{m_2}(\nabla F_*)(e_i, {}^*F_*(\|H_2\|^2F_*(e_i))) - m_2^2cH_2 = 0.$$

On the other hand, from the linear map ${}^*F_*$ and ${}^*F_*\circ F_* = I$ (identity map), we obtain

$$m_2\sum_{i=1}^{m_2}(\nabla F_*)(e_i, \|H_2\|^2e_i) - m_2^2cH_2 = 0.$$

Since the second fundamental form is also linear in its arguments, it follows that

$$m_2\|H_2\|^2\sum_{i=1}^{m_2}(\nabla F_*)(e_i, e_i) - m_2^2cH_2 = 0.$$

Hence

$$m_2^2\|H_2\|^2H_2 - m_2^2cH_2 = 0,$$

which implies that

$$(\|H_2\|^2 - c)H_2 = 0. \tag{4.8}$$

Thus either $H_2 = 0$ or $\|H_2\|^2 - c = 0$. If $H_2 = 0$, then Lemma 3.2 implies that F is harmonic, thus the proof is complete. ∎

From (4.8), we have the following result which puts some restrictions on $M_2(c)$.

Corollary 4.4. *There exists no proper biharmonic pseudo-umbilical Riemannian map F from a Riemannian manifold to a space form $M_2(c)$ with $c \le 0$ such that the distribution $\ker F_*$ is minimal and the mean curvature vector field H_2 is parallel.*

Remark. In this paper, we investigate the biharmonicity of Riemannian maps between Riemannian manifolds. Our results give some clues to investigate the biharmonicity of arbitrary maps between Riemannian manifolds. They also give a method to investigate the geometry of Riemannian maps. Since Riemannian maps are solutions of the eikonal equations which can be obtained starting from the wave equation, biharmonic maps are solutions of fourth order strongly elliptic semilinear partial differential equations and they are related to the biwave equation and bi-Yang–Mills fields; biharmonic Riemannian maps have potential for further research in partial differential equations, geometric optics and mathematical physics.

References

[BW] P. Baird and J. C. Wood, *Harmonic Morphisms between Riemannian Manifolds*, Clarendon Press, Oxford, 2003.

[BU] C. L. Bejan and H. Urakawa, *Yang–Mills fields analogue of biharmonic maps*, in: Topics in Almost Hermitian Geometry and Related Fields, World Sci., 2005, 41–49.

[CM] R. Caddeo, S. Montaldo and C. Oniciuc, *Biharmonic submanifolds of* S^3, Int. J. Math. 12 (2001), 867–876.

[C] B. Y. Chen, *What can we do with Nash's embedding theorem?*, Soochow J. Math. 30 (2004), 303–338.

[CI] B. Y. Chen and S. Ishikawa, *Biharmonic pseudo-Riemannian submanifolds in pseudo-Euclidean spaces*, Kyushu J. Math. 52 (1998), 167–185.

[Ch] Y. J. Chiang, *Biwave maps into manifolds*, Int. J. Math. Math. Sci. 2009, art. ID 104274.

[D] M. Dietrich, *Light Transmission Optics*, Krieger and Van Nostrand, 1982.

[ES] J. Eells and H. J. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. 86 (1964), 109–160.

[FIP] M. Falcitelli, S. Ianus and A. M. Pastore, *Riemannian Submersions and Related Topics*, World Sci., Singapore, 2004.

[F] A. E. Fischer, *Riemannian maps between Riemannian manifolds*, in: Contemp. Math. 132, Amer. Math. Soc., 1992, 331–366.

[GK] E. Garcia-Rio and D. N. Kupeli, *Semi-Riemannian Maps and Their Applications*, Kluwer, Dordrecht, 1999.

[G] A. Gray, *Pseudo-Riemannian almost product manifolds and submersions.* J. Math. Mech. 16 (1967), 715–737.

[IIU] T. Ichiyama, J. I. Inoguchi and H. Urakawa, *Bi-harmonic maps and bi-Yang–Mills fields*, Note Mat. 1 (2008), suppl. no. 1, 233–275.

[J1] G. Y. Jiang, *2-harmonic isometric immersions between Riemannian manifolds*, Chinese Ann. Math. Ser. A 7 (1986), 130–144.

[J2] —, *2-harmonic maps and their first and second variation formulas*, ibid. 7 (1986), 389–402.

[MO] S. Montaldo and C. Oniciuc, *A short survey on biharmonic maps between Riemannian manifolds*, Rev. Un. Mat. Argentina 47 (2006), no. 2, 1–22.

[N] T. Nore, *Second fundamental form of a map*, Ann. Mat. Pura Appl. 146 (1987), 281–310.

[O] B. O'Neill, *The fundamental equations of a submersion*, Michigan Math. J. 13 (1966), 459–469.

[OC] C. Oniciuc, *Biharmonic maps between Riemannian manifolds*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi Mat. (N.S.) Mat. 48 (2002), 237–248.

[S1] B. Șahin, *Conformal Riemannian maps between Riemannian manifolds, their harmonicity and decomposition theorems*, Acta Appl. Math. 109 (2010), 829–847.

[S2] —, *Invariant and anti-invariant Riemannian maps to Kähler manifolds*, Int. J. Geom. Methods Modern Phys. 7 (2010), 337–355.

[YK] K. Yano and M. Kon, *Structures on Manifolds*, World Sci., Singapore, 1984.

Bayram Șahin
Department of Mathematics
Inonu University
44280, Malatya, Turkey
E-mail: bsahin@inonu.edu.tr

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