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Multiplicity results for a class of concave-convex elliptic systems involving sign-changing weight functions

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Abstract. Our main purpose is to establish the existence of weak solutions of second order quasilinear elliptic systems

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = f_{1\lambda_1}(x)|u|^{q-2}u + \dfrac{2\alpha}{\alpha+\beta} g_\mu |u|^{\alpha-2}u|v|^\beta, & x \in \Omega, \\ -\Delta_p v + |v|^{p-2}v = f_{2\lambda_2}(x)|v|^{q-2}v + \dfrac{2\beta}{\alpha+\beta} g_\mu |u|^{\alpha}|v|^{\beta-2}v, & x \in \Omega, \\ u = v = 0, \quad x \in \partial\Omega, \end{cases}$$

where $1 < q < p < N$ and $\Omega \subset \mathbb{R}^N$ is an open bounded smooth domain. Here $\lambda_1, \lambda_2, \mu \geq 0$ and $f_{i\lambda_i}(x) = \lambda_i f_{i+}(x) + f_{i-}(x)$ $(i = 1, 2)$ are sign-changing functions, where $f_{i\pm}(x) = \max\{\pm f_i(x), 0\}$, $g_\mu(x) = a(x) + \mu b(x)$, and $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ denotes the p-Laplace operator. We use variational methods.

1. Introduction. In this paper we consider some new results concerning the existence of solutions for quasilinear problems of the type

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = f_{1\lambda_1}(x)|u|^{q-2}u + \dfrac{2\alpha}{\alpha+\beta} g_\mu |u|^{\alpha-2}u|v|^\beta, & x \in \Omega, \\ -\Delta_p v + |v|^{p-2}v = f_{2\lambda_2}(x)|v|^{q-2}v + \dfrac{2\beta}{\alpha+\beta} g_\mu |u|^{\alpha}|v|^{\beta-2}v, & x \in \Omega, \\ u = v = 0, \quad x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $1 < q < p < N$, $\alpha, \beta > 1$ satisfy $p < \alpha + \beta \leq p^*$ and $p^* = pN/(N-p)$ denotes the critical Sobolev exponent. $\Omega \subset \mathbb{R}^N$ is an open bounded smooth domain. Moreover, $\lambda_1, \lambda_2, \mu \geq 0$ and $f_{i\lambda_i}(x) = \lambda_i f_{i+}(x) + f_{i-}(x)$ $(i = 1, 2)$ are sign-changing functions, where $f_{i\pm}(x) = \pm\max\{\pm f_i(x), 0\}$, $g_\mu(x) = a(x) + \mu b(x)$, and $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ denotes the p-Laplace operator.

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When we set $f_{1\lambda_1} = f_{2\lambda_2} = f$, $g_\mu = g$, $\alpha = \beta$, $\alpha + \beta = s$ and $u = v$, system (1.1) reduces to the semilinear scalar quasilinear elliptic equation

$$(1.2)\qquad \begin{cases} -\Delta_p u + |u|^{p-2}u = f(x)|u|^{q-2}u + g(x)|u|^{s-2}u, & x \in \Omega, \\ u = 0, \quad x \in \partial\Omega. \end{cases}$$

It has been studied extensively since Ambrosetti, Brézis and Cerami [ABC] considered the equation

$$(1.3)\qquad \begin{cases} -\Delta u = \lambda u^{q-1} + u^{s-1}, & x \in \Omega, \\ u > 0, \quad x \in \Omega, \\ u \in H_0^1(\Omega), \end{cases}$$

where $1 < q < 2 < s \le 2^*$, $\lambda > 0$ and Ω is a bounded domain in $\mathbb{R}^N$. They found that there exists $\lambda_0 > 0$ such that problem (1.3) admits at least two positive solutions for $\lambda \in (0, \lambda_0)$, a positive solution for $\lambda = \lambda_0$ and no positive solution for $\lambda > \lambda_0$.

When $f(x)$, $g(x)$ are some positive constants, $p = 2$, $2 < s \le 2^*$ and $q > 1$, problem (1.2) was considered in [AGP], [BW], [CFP] and the references therein. Subsequently, in [GMP], [GP], problem (1.2) was studied when $1 < p < N$ and $1 < q < p$. The results obtained were similar to the results of [ABC], but only for some ranges of the exponents p, q. Prashanth–Sreenadh [PS] have studied problem (1.2) in the unit ball $B^N(0;1) \subset \mathbb{R}^N$ when $s = p^*$ and $g(x) \equiv 1$. Recently, T. F. Wu [W1] studied (1.2) when $p = 2$ and $f(x) = \lambda f_+(x) + f_-(x)$ is sign-changing and $g(x) = a(x) + \mu b(x)$; he obtained multiple positive solutions for (1.2) in $\mathbb{R}^N$ by variational methods.

In recent years, much attention has been paid to the existence of solutions for elliptic systems, in particular, for the system

$$(1.4)\qquad \begin{cases} -\Delta_p u = \lambda |u|^{q-2}u + \dfrac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^\beta, & x \in \Omega, \\ -\Delta_p v = \theta |v|^{q-2}v + \dfrac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v, & x \in \Omega, \\ u = v = 0, \quad x \in \partial\Omega, \end{cases}$$

where $\alpha + \beta = p^*$. When $p = 2$ and $q = 2$, Alves et al. [AMS] proved the existence of least energy solutions of (1.4) for any $\lambda, \theta \in (0, \lambda_1)$, where λ_1 denotes the first eigenvalue of the operator $-\Delta$. Subsequently, Han [H1], [H2] considered the existence of multiple positive solutions for (1.4), and T. S. Hsu [H3] studied (1.4) when $1 < q < p < N$, $\alpha + \beta = p^*$. Before T. S. Hsu's work, T. F. Wu [W2] considered the following semilinear elliptic system with sign-changing weight functions:

$$
(1.5)\qquad \begin{cases} -\Delta u = \lambda f(x)\lambda|u|^{q-2}u + \dfrac{\alpha}{\alpha+\beta}h(x)|u|^{\alpha-2}u|v|^{\beta}, & x \in \Omega, \\ -\Delta v = \mu g(x)\theta|v|^{q-2}v + \dfrac{\beta}{\alpha+\beta}h(x)|u|^{\alpha}|v|^{\beta-2}v, & x \in \Omega, \\ u = v = 0, \quad x \in \partial\Omega. \end{cases}
$$

He proved problem (1.5) has at least two positive solutions when (λ, μ) belongs to a certain subset of $\mathbb{R}^2$. More precisely, Costa and Magalhães [CM] considered subquadratic perturbations of semilinear elliptic systems by minimization methods. Cao and Tang [CT] considered a class of superlinear elliptic systems by variational methods. Bartsch and Clapp [BC] studied an elliptic system by critical point theorems. In [ZW] and [DSZ], multiplicity results for elliptic systems were obtained by using an abstract linking theorem and the decomposition of the Nehari manifold respectively.

However, as far as we know, there are few results on problem (1.1) with concave-convex nonlinearities. Motivated by [H3], [W1] and [W2], we shall extend the above results to problem (1.1).

In this paper we assume that the functions $f_{i\lambda_i}, g_\mu$ with $\lambda_i, \mu \geq 0$ $(i=2)$ satisfy the following conditions:

- (C_1) $f_i \in L^{q^*}(\Omega)$, $f_{i\lambda_i}(x) = \lambda_i f_{i+}(x) + f_{i-}(x)$ with $f_{i\pm}(x) = \pm\max\{\pm f_i(x), 0\}$ for $i = 1, 2$ and $q^* = \frac{\alpha+\beta}{\alpha+\beta-q}$;
- (C_2) $g_\mu(x) = a(x) + \mu b(x) \in C(\bar{\Omega})$, where $a(x), b(x)$ are nonnegative continuous functions with $a(x) \leq 1$;
- (C_3) there exists an open set $\Omega' \subset \Omega$ containing 0 such that $f_i(x) > 0$, i.e., $f_{i\lambda_i}(x) = \lambda_i f_{i+}(x)$ $(i = 1, 2)$ in Ω';
- (C_4) $b(x) > 0$ in Ω', and μ is large enough such that $g_\mu(x) \geq 1$ in Ω'.

For a bounded smooth open set $\Omega \subset \mathbb{R}^N$, we denote by $\|\cdot\|$, $\|\cdot\|_{L^p}$ the norm of $W_0^{1,p}(\Omega)$ and $L^p(\Omega)$ respectively, that is,

$$
\|u\| = \Big(\int_\Omega (|\nabla u|^p + |u|^p)\,dx\Big)^{1/p}, \qquad \|u\|_{L^p} = \Big(\int_\Omega |u|^p\,dx\Big)^{1/p}.
$$

Obviously, $H := W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ is a Banach space. Let H' be the dual of H, and $\langle\,,\rangle$ the duality paring between H' and H. The norm on H is given by

$$
\|z\| = \|(u,v)\| = (\|u\|^p + \|v\|^p)^{1/p}
$$

and the norm on $L^p(\Omega) \times L^p(\Omega)$ by

$$
\|z\|_{L^p} = \|(u,v)\|_{L^p} = (\|u\|_{L^p}^p + \|v\|_{L^p}^p)^{1/p},
$$

where $z = (u, v) \in H$.

Then we have the following results:

THEOREM 1.1. *Assume conditions* (C_1) *and* (C_2) *hold, and* $p < \alpha + \beta \leq p^*$. *Then there exists* $\Lambda_0 > 0$ *such that when*

$$0 < (\lambda_1 \|f_{1+}\|_{L^{q^*}} + \lambda_2 \|f_{2+}\|_{L^{q^*}})^{\alpha+\beta-p}(1 + \mu\|b\|_\infty)^{p-q} < \Lambda_0,$$

system (1.1) *has at least one positive solution in* H.

THEOREM 1.2. *Assume conditions* (C_1)–(C_4) *hold, and* $p < \alpha + \beta \leq p^*$. *Then there exists* $\Lambda_1 > 0$ *such that when*

$$0 < (\lambda_1 \|f_{1+}\|_{L^{q^*}} + \lambda_2 \|f_{2+}\|_{L^{q^*}})^{\alpha+\beta-p}(1 + \mu\|b\|_\infty)^{p-q} < \Lambda_1,$$

system (1.1) *has at least two positive solutions in* H.

We will show the existence and multiplicity of nontrivial solutions of (1.1) by looking for critical points of the associated functional

$$J(u,v) = \frac{1}{p}(\|u\|^p + \|v\|^p) - \frac{1}{q}\int_\Omega (f_{1\lambda_1}|u|^q + f_{2\lambda_2}|v|^q)\,dx - \frac{2}{\alpha+\beta}\int_\Omega g_\mu |u|^\alpha |v|^\beta\,dx. \tag{1.6}$$

This paper is organized as follows. In Section 2, we give some notation and preliminaries. In Section 3, we prove Theorems 1.1 and 1.2.

2. Notation and preliminaries. We define the Palais–Smale sequence ((PS)-sequence), (PS)-value, and (PS)-conditions in H for J as follows.

DEFINITION 2.1.

(I) For $c \in \mathbb{R}$, a sequence $\{z_n\} \subset H$ is a $(PS)_c$*-sequence* for J if $J(z_n) = c + o(1)$ and $J'(z_n) = o(1)$ strongly in H' as $n \to \infty$;

(II) $c \in \mathbb{R}$ is a (PS)*-value* in H for J if there exists a $(\mathrm{PS})_c$-sequence in H for J;

(III) J satisfies the $(PS)_c$*-condition* in H if every $(\mathrm{PS})_c$-sequence in H for J contains a convergent subsequence.

Throughout this paper, we denote weak convergence by $\rightharpoonup$, and strong convergence by $\to$.

We define

$$S_{\alpha+\beta} = \inf_{u \in W^{1,p}(\Omega)\setminus\{0\}} \frac{\|u\|^p}{(\int_\Omega |u|^{\alpha+\beta}\,dx)^{\frac{p}{\alpha+\beta}}}, \tag{2.1}$$

$$S_{\alpha\beta} = \inf_{z \in H\setminus\{0\}} \frac{\|z\|^p}{(\int_\Omega |u|^\alpha |v|^\beta\,dx)^{\frac{p}{\alpha+\beta}}}. \tag{2.2}$$

Clearly, we have $\int_\Omega |u|^\alpha |v|^\beta\,dx \leq S_{\alpha\beta}^{-(\alpha+\beta)/p}\|z\|^{\alpha+\beta}$.

LEMMA 2.2. *Assume $\alpha, \beta > 1$ and $\alpha + \beta \le p^*$, and let $\Omega \subset \mathbb{R}^N$ $(N \ge 3)$ be a domain (not necessarily bounded). Then*

$$S_{\alpha\beta} = \left[\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\alpha}{\beta}\right)^{-\frac{\alpha}{\alpha+\beta}}\right] S_{\alpha+\beta}.$$

Proof. The proof is essentially given in [AMS] when $p = 2$. Modifying that proof, we can deduce our result. For the reader's convenience, we give a sketch here.

Suppose $\{w_n\}$ is a minimizing sequence for $S_{\alpha+\beta}$, and let $u_n = sw_n$ and $v_n = tw_n$, where $s, t > 0$ will be chosen later. Then from (2.2), we infer that

$$(2.3)\qquad S_{\alpha\beta} \le \frac{s^p + t^p}{(s^\alpha t^\beta)^{\frac{p}{\alpha+\beta}}} \frac{\|w_n\|^p}{(\int_\Omega |w_n|^{\alpha+\beta}\,dx)^{\frac{p}{\alpha+\beta}}}$$
$$= \left[\left(\frac{s}{t}\right)^{\frac{p\beta}{\alpha+\beta}} + \left(\frac{s}{t}\right)^{\frac{p\alpha}{\alpha+\beta}}\right] \frac{\|w_n\|^p}{(\int_\Omega |w_n|^{\alpha+\beta}\,dx)^{\frac{p}{\alpha+\beta}}}.$$

Define

$$h(x) = x^{\frac{p\beta}{\alpha+\beta}} + x^{-\frac{p\alpha}{\alpha+\beta}}, \qquad x > 0.$$

By a direct calculation, the minimum of h is achieved at $x_0 = (\alpha/\beta)^{1/p}$ with the minimum value

$$h(x_0) = \left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\alpha}{\beta}\right)^{-\frac{\alpha}{\alpha+\beta}}.$$

Thus, choosing $s, t > 0$ in (2.3) such that $s/t = (\alpha/\beta)^{1/p}$, we obtain

$$S_{\alpha\beta} \le \left[\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\alpha}{\beta}\right)^{-\frac{\alpha}{\alpha+\beta}}\right] S_{\alpha+\beta}.$$

To complete the proof, let $z_n = (u_n, v_n)$ be a minimizing sequence for $S_{\alpha\beta}$. Define $\omega_n = t_n v_n$ for some $t_n > 0$ such that

$$\int_\Omega |u_n|^{\alpha+\beta}\,dx = \int_\Omega |\omega_n|^{\alpha+\beta}\,dx.$$

Then

$$\int_\Omega |u_n|^\alpha |\omega_n|^\beta\,dx \le \frac{\alpha}{\alpha+\beta}\int_\Omega |u_n|^{\alpha+\beta}\,dx + \frac{\beta}{\alpha+\beta}\int_\Omega |\omega_n|^{\alpha+\beta}\,dx$$
$$= \int_\Omega |u_n|^{\alpha+\beta}\,dx = \int_\Omega |\omega_n|^{\alpha+\beta}\,dx.$$

We deduce from the above inequality that

$$\begin{aligned}\frac{\|z_n\|^p}{(\int_\Omega |u_n|^\alpha |v_n|^\beta\,dx)^{\frac{p}{\alpha+\beta}}} &= t_n^{\frac{p\beta}{\alpha+\beta}} \frac{\|z_n\|^p}{(\int_\Omega |u_n|^\alpha |\omega_n|^\beta\,dx)^{\frac{p}{\alpha+\beta}}} \\ &\geq t_n^{\frac{p\beta}{\alpha+\beta}} \frac{\|u_n\|^p}{(\int_\Omega |u_n|^{\alpha+\beta}\,dx)^{\frac{p}{\alpha+\beta}}} \\ &\quad + t_n^{\frac{p\beta}{\alpha+\beta}-p} \frac{\|\omega_n\|^p}{(\int_\Omega |\omega_n|^{\alpha+\beta}\,dx)^{\frac{p}{\alpha+\beta}}} \\ &\geq h(t_n)S_{\alpha+\beta} \geq h(t_0)S_{\alpha+\beta}.\end{aligned}$$

Passing to the limit in the above inequality, we obtain

$$S_{\alpha\beta} \geq \left[\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\alpha}{\beta}\right)^{-\frac{\alpha}{\alpha+\beta}}\right] S_{\alpha+\beta}. \ \blacksquare$$

As the energy functional J defined in (1.6) is not bounded below on H, it is useful to consider the functional on the *Nehari manifold*

$$\mathbf{N} = \{z \in H \setminus \{0\} \mid \langle J'(z), z\rangle = 0\}.$$

Thus $z = (u,v) \in \mathbf{N}$ if and only if

$$\langle J'(z), z\rangle = \|z\|^p - \int_\Omega (f_{1\lambda_1}|u|^q + f_{2\lambda_2}|v|^q)\,dx - 2\int_\Omega g_\mu |u|^\alpha |v|^\beta\,dx = 0. \tag{2.4}$$

Note that $\mathbf{N}$ contains every nonzero solution of problem (1.1). Furthermore, we have the following result.

Lemma 2.3. *The energy functional J is coercive and bounded below on* $\mathbf{N}$.

Proof. Assume $z = (u,v) \in \mathbf{N}$. Then

$$J(u,v) = \left(\frac{1}{p} - \frac{1}{\alpha+\beta}\right)\|z\|^p - \left(\frac{1}{q} - \frac{1}{\alpha+\beta}\right)\int_\Omega (f_{1\lambda_1}|u|^q + f_{2\lambda_2}|v|^q)\,dx.$$

By the Hölder inequality and the Sobolev embedding theorem,

$$\begin{aligned}J(u,v) &\geq \frac{\alpha+\beta-p}{p(\alpha+\beta)}\|z\|^p \\ &\quad - \frac{\alpha+\beta-q}{q(\alpha+\beta)}(\lambda_1\|f_{1+}\|_{L^{q^*}}\|u\|^q_{L^{\alpha+\beta}} + \lambda_2\|f_{2+}\|_{L^{q^*}}\|v\|^q_{L^{\alpha+\beta}}) \\ &\geq \frac{\alpha+\beta-p}{p(\alpha+\beta)}\|z\|^p - \frac{\alpha+\beta-q}{q(\alpha+\beta)}(\lambda_1\|f_{1+}\|_{L^{q^*}} + \lambda_2\|f_{2+}\|_{L^{q^*}})S^{-q/p}_{\alpha+\beta}\|z\|^q \\ &= \frac{\alpha+\beta-p}{p(\alpha+\beta)}\|z\|^p - c_1(\lambda_1\|f_{1+}\|_{L^{q^*}} + \lambda_2\|f_{2+}\|_{L^{q^*}})\|z\|^q,\end{aligned}$$

where $c_1 = \frac{\alpha+\beta-q}{q(\alpha+\beta)}S^{-q/p}_{\alpha+\beta}$. Thus J is coercive.

By the Young inequality, we have

$$J(u,v) \geq \frac{\alpha+\beta-p}{p(\alpha+\beta)}\|z\|^p - \epsilon\|z\|^p - c_\epsilon(\lambda_1\|f_{1+}\|_{L^{q^*}} + \lambda_2\|f_{2+}\|_{L^{q^*}})^{\frac{p}{p-q}}.$$

Set $\epsilon = \frac{\alpha+\beta-p}{p(\alpha+\beta)}$. Then

$$J(u,v) \geq -c_0(\lambda_1\|f_{1+}\|_{L^{q^*}} + \lambda_2\|f_{2+}\|_{L^{q^*}})^{\frac{p}{p-q}},$$

where c_0 is a positive constant depending on $\alpha, \beta, p, q, S_{\alpha+\beta}$. ∎

Set

$$c_\infty = \frac{2(\alpha+\beta-p)}{p(\alpha+\beta)}\left(\frac{S_{\alpha\beta}}{2}\right)^{\frac{\alpha+\beta}{\alpha+\beta-p}} - c_0(\lambda_1\|f_{1+}\|_{L^{q^*}} + \lambda_2\|f_{2+}\|_{L^{q^*}})^{\frac{p}{p-q}}.$$

Then we have the following result.

LEMMA 2.4. *J satisfies the $(PS)_c$-condition on* $\mathbf{N}$ *for all c satisfying*

$$-\infty < c < c_\infty. \tag{2.5}$$

Proof. Let $\{z_n\} \subset \mathbf{N}$ be a $(\mathrm{PS})_c$-sequence for J with $c \in (-\infty, c_\infty)$. Write $z_n = (u_n, v_n)$. We know from Lemma 2.3 that z_n is bounded on $\mathbf{N}$, and so $z_n \rightharpoonup z = (u,v)$ up to a subsequence, where z is a critical point of J. Furthermore, we may assume

$$\begin{cases} u_n \rightharpoonup u,\ v_n \rightharpoonup v, & x \in W_0^{1,p}(\Omega),\\ u_n \to u,\ v_n \to v, & \text{a.e. in } \Omega,\\ u_n \to u,\ v_n \to v, & \text{in } L^s(\Omega) \text{ for } 1 \leq s < p^*. \end{cases}$$

Hence $J'(z) = 0$ and

$$\int_\Omega (f_{1\lambda_1}|u_n|^q + f_{2\lambda_2}|v_n|^q)\,dx = \int_\Omega (f_{1\lambda_1}|u|^q + f_{2\lambda_2}|v|^q)\,dx + o(1).$$

Let $\overline{u}_n = u_n - u$, $\overline{v}_n = v_n - v$, $\overline{z}_n = (\overline{u}_n, \overline{v}_n)$. Then by the Brézis–Lieb lemma [BL], we have

$$\|\overline{z}_n\|^p = \|z_n\|^p - \|z\|^p + o(1),$$

and by an argument of Han [H2, Lemma 2.1], we obtain

$$\int_\Omega g_\mu|\overline{u}_n|^\alpha|\overline{v}_n|^\beta\,dx = \int_\Omega g_\mu|u_n|^\alpha|v_n|^\beta\,dx - \int_\Omega g_\mu|u|^\alpha|v|^\beta\,dx + o(1).$$

Since $J(z_n) = c + o(1)$ and $J'(z_n) = 0$, we deduce that

$$\frac{1}{p}\|\overline{z}_n\|^p - \frac{2}{\alpha+\beta}\int_\Omega g_\mu|\overline{u}_n|^\alpha|\overline{v}_n|^\beta\,dx = c - J(z) + o(1) \tag{2.6}$$

and

$$\|\overline{z}_n\|^p - 2\int_\Omega g_\mu|\overline{u}_n|^\alpha|\overline{v}_n|^\beta\,dx = o(1). \tag{2.7}$$

Assume $\|\overline{z}_n\|^p \to m$, so $2\int_\Omega g_\mu |\overline{u}_n|^\alpha |\overline{v}_n|^\beta\,dx \to m$. If $m=0$, the proof is complete. Assume $m>0$. From (2.7) we have

$$\left(\frac{m}{2}\right)^{\frac{p}{\alpha+\beta}} = \lim_{n\to\infty}\Big(\int_\Omega g_\mu |\overline{u}_n|^\alpha |\overline{v}_n|^\beta\,dx\Big)^{\frac{p}{\alpha+\beta}} \le S_{\alpha\beta}^{-1}\|\overline{z}_n\|^p = S_{\alpha\beta}^{-1} m.$$

Thus $m \ge 2(S_{\alpha\beta}/2)^{\frac{\alpha+\beta}{\alpha+\beta-p}}$. From Lemma 2.3, (2.6) and (2.7) we obtain

$$\begin{aligned} c &= \left(\frac{1}{p}-\frac{1}{\alpha+\beta}\right)m + J(z)\\ &\ge \frac{2(\alpha+\beta-p)}{p(\alpha+\beta)}\left(\frac{S_{\alpha\beta}}{2}\right)^{\frac{\alpha+\beta}{\alpha+\beta-p}} - c_0(\lambda_1\|f_{1+}\|_{L^{q^*}} + \lambda_2\|f_{2+}\|_{L^{q^*}})^{\frac{p}{p-q}}. \end{aligned}$$

This contradicts $c < c_\infty$. ∎

The Nehari manifold $\mathbf{N}$ is closely linked to the behavior of the function $h_z : t \mapsto J(tz)$ for $t>0$. Such maps are known as fibering maps and were introduced by Drábek and Pohozaev in [DP] and also discussed by Brown and Zhang [BZ] and Brown and Wu [BW1], [BW2]. If $z=(u,v)\in W^{1,p}(\Omega)\times W^{1,p}(\Omega)$, we have

$$\begin{aligned} h_z(t) &= \frac{t^p}{p}\|z\|^p - \frac{t^q}{q}\int_\Omega (f_{1\lambda_1}|u|^q + f_{2\lambda_2}|v|^q)\,dx - \frac{2t^{\alpha+\beta}}{\alpha+\beta}\int_\Omega g_\mu |u|^\alpha |v|^\beta\,dx,\\ h_z'(t) &= t^{p-1}\|z\|^p - t^{q-1}\int_\Omega (f_{1\lambda_1}|u|^q + f_{2\lambda_2}|v|^q)\,dx - 2t^{\alpha+\beta-1}\int_\Omega g_\mu |u|^\alpha |v|^\beta\,dx,\\ h_z''(t) &= (p-1)t^{p-2}\|z\|^p - (q-1)t^{q-2}\int_\Omega (f_{1\lambda_1}|u|^q + f_{2\lambda_2}|v|^q)\,dx\\ &\quad - 2(\alpha+\beta-1)t^{\alpha+\beta-2}\int_\Omega g_\mu |u|^\alpha |v|^\beta\,dx. \end{aligned}$$

It is easy to see that

$$th_z'(t) = \|tz\|^p - \int_\Omega (f_{1\lambda_1}|tu|^q + f_{2\lambda_2}|tv|^q)\,dx - 2\int_\Omega g_\mu |tu|^\alpha |tv|^\beta\,dx = \langle J'(tz), tz\rangle.$$

So for $z\in W^{1,p}(\Omega)\times W^{1,p}(\Omega)\setminus\{(0,0)\}$ and $t>0$, we have $tz\in\mathbf{N}$ if and only if $h_z'(t)=0$, i.e., positive critical points of h_z correspond to points on the Nehari manifold. In particular, $h_z'(1)=0$ if and only if $z\in\mathbf{N}$. Thus it is natural to split $\mathbf{N}$ into three parts corresponding to local minima, local maxima, and points of inflection. Accordingly, we define

$$\begin{aligned} \mathbf{N}^+ &= \{z\in\mathbf{N} \mid h_z''(1) > 0\},\\ \mathbf{N}^0 &= \{z\in\mathbf{N} \mid h_z''(1) = 0\},\\ \mathbf{N}^- &= \{z\in\mathbf{N} \mid h_z''(1) < 0\}. \end{aligned}$$

We now derive some basic properties of $\mathbf{N}^+$, $\mathbf{N}^0$ and $\mathbf{N}^-$.

LEMMA 2.5. *Suppose that z_0 is a local minimizer for J on $\mathbf{N}$, and that $z_0 \notin \mathbf{N}^0$. Then $J'(z_0) = 0$ in $H'(\Omega)$.*

Proof. The proof is almost the same as in Brown and Zhang [BZ, Theorem 2.3] (or see Binding et al. [BDH]). ∎

We can easily see that for each $z \in \mathbf{N}$,

$$\begin{aligned} h_z''(1) &= (p-1)\|z\|^p - (q-1)\int_\Omega (f_{1\lambda_1}|u|^q + f_{2\lambda_2}|v|^q)\,dx \\ &\quad - 2(\alpha+\beta-1)\int_\Omega g_\mu |u|^\alpha |v|^\beta\,dx \\ &= p\|z\|^p - q\int_\Omega (f_{1\lambda_1}|u|^q + f_{2\lambda_2}|v|^q)\,dx - 2(\alpha+\beta)\int_\Omega g_\mu |u|^\alpha |v|^\beta\,dx. \end{aligned}$$

Then

$$h_z''(1) = (p-q)\|z\|^p - 2(\alpha+\beta-q)\int_\Omega g_\mu |u|^\alpha |v|^\beta\,dx, \tag{2.8}$$

$$h_z''(1) = (p-\alpha-\beta)\|z\|^p - (q-\alpha-\beta)\int_\Omega (f_{1\lambda_1}|u|^q + f_{2\lambda_2}|v|^q)\,dx. \tag{2.9}$$

So we have the following result.

LEMMA 2.6.

(i) *For any $z \in \mathbf{N}^+ \cup \mathbf{N}^0$, we have*

$$\int_\Omega (f_{1\lambda_1}|u|^q + f_{2\lambda_2}|v|^q)\,dx > 0.$$

(ii) *For any $z \in \mathbf{N}^-$, we have $\int_\Omega g_\mu |u|^\alpha |v|^\beta\,dx > 0$.*

Proof. The result follows immediately from (2.8) and (2.9). ∎

If we assume

$$\Lambda_0 = \left(\frac{p-q}{2}\right)^{p-q} \frac{(\alpha+\beta-p)^{\alpha+\beta-p}}{(\alpha+\beta-q)^{\alpha+\beta-q}} \left[\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\alpha}{\beta}\right)^{-\frac{\alpha}{\alpha+\beta}}\right]^{\frac{(\alpha+\beta)(p-q)}{p}} S_{\alpha+\beta}^{\alpha+\beta-q},$$

then we have the following result.

LEMMA 2.7. *If*

$$(\lambda_1 \|f_{1+}\|_{L^{q^*}} + \lambda_2 \|f_{2+}\|_{L^{q^*}})^{\alpha+\beta-p} (1+\mu\|b\|_\infty)^{p-q} < \Lambda_0, \tag{2.10}$$

then $\mathbf{N}^0 = \emptyset$.

Proof. Suppose the contrary. If there exist $\lambda_1, \lambda_2 > 0$ and $\mu \geq 0$ such that (2.10) holds and $\mathbf{N}^0 \neq \emptyset$, then for any $z \in \mathbf{N}^0$, from (2.8) we have

$$\begin{aligned}\|z\|^p &= \frac{2(\alpha+\beta-q)}{p-q} \int_\Omega g_\mu |u|^\alpha |v|^\beta \, dx \\ &= \frac{2(\alpha+\beta-q)}{p-q} \int_\Omega (a(x)+\mu b(x)) |u|^\alpha |v|^\beta \, dx \\ &\leq \frac{2(\alpha+\beta-q)}{p-q} (1+\mu\|b\|_\infty) S_{\alpha\beta}^{-(\alpha+\beta)/p} \|z\|^{\alpha+\beta}.\end{aligned}$$

So we obtain

$$\left[\frac{(p-q) S_{\alpha\beta}^{(\alpha+\beta)/p}}{2(\alpha+\beta-q)(1+\mu\|b\|_\infty)}\right]^{\frac{1}{\alpha+\beta-p}} \leq \|z\|. \tag{2.11}$$

Similarly, from (2.9) we have

$$\begin{aligned}\|z\|^p &= \frac{q-\alpha-\beta}{p-\alpha-\beta} \int_\Omega (f_{1\lambda_1}|u|^q + f_{2\lambda_2}|v|^q) \, dx \\ &\leq \frac{q-\alpha-\beta}{p-\alpha-\beta} (\lambda_1 \|f_{1+}\|_{L^{q^*}} + \lambda_2 \|f_{2+}\|_{L^{q^*}}) S_{\alpha+\beta}^{-q/p} \|z\|^q.\end{aligned}$$

Then

$$\|z\| \leq \left[\frac{q-\alpha-\beta}{p-\alpha-\beta} (\lambda_1 \|f_{1+}\|_{L^{q^*}} + \lambda_2 \|f_{2+}\|_{L^{q^*}}) S_{\alpha+\beta}^{-q/p}\right]^{\frac{1}{p-q}}. \tag{2.12}$$

From (2.11) and (2.12), we see that

$$\begin{aligned}&\left[\frac{(p-q) S_{\alpha\beta}^{(\alpha+\beta)/p}}{2(\alpha+\beta-q)(1+\mu\|b\|_\infty)}\right]^{\frac{1}{\alpha+\beta-p}} \\ &\qquad \leq \left[\frac{q-\alpha-\beta}{p-\alpha-\beta} (\lambda_1 \|f_{1+}\|_{L^{q^*}} + \lambda_2 \|f_{2+}\|_{L^{q^*}}) S_{\alpha+\beta}^{-q/p}\right]^{\frac{1}{p-q}},\end{aligned}$$

which implies

$$\begin{aligned}&(\lambda_1 \|f_{1+}\|_{L^{q^*}} + \lambda_2 \|f_{2+}\|_{L^{q^*}})^{\alpha+\beta-p} (1+\mu\|b\|_\infty)^{p-q} \\ &\geq \left(\frac{p-q}{2}\right)^{p-q} \frac{(\alpha+\beta-p)^{\alpha+\beta-p}}{(\alpha+\beta-q)^{\alpha+\beta-q}} \left[\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\alpha}{\beta}\right)^{-\frac{\alpha}{\alpha+\beta}}\right]^{\frac{(\alpha+\beta)(p-q)}{p}} S_{\alpha+\beta}^{\alpha+\beta-q} \\ &= \Lambda_0,\end{aligned}$$

contradicting (2.10). ∎

In order to get a better understanding of the Nehari manifold and fibering maps, we consider the function $m_z : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$m_z(t) = t^{p-q}\|z\|^p - 2t^{\alpha+\beta-q} \int_\Omega g_\mu |u|^\alpha |v|^\beta \, dx. \tag{2.13}$$

Clearly, $tz \in \mathbf{N}$ if and only if

$$m_z(t) = \int_\Omega (f_{1\lambda_1}|u|^q + f_{2\lambda_2}|v|^q)\,dx. \tag{2.14}$$

Moreover,

$$m_z'(t) = (p-q)t^{p-q-1}\|z\|^p - 2(\alpha+\beta-q)t^{\alpha+\beta-q-1}\int_\Omega g_\mu|u|^\alpha|v|^\beta\,dx.$$

So it is easy to see that if $tz \in \mathbf{N}$, then

$$t^{q-1}m_z'(t) = h_z''(t).$$

Hence, $tz \in \mathbf{N}^+$ ($\mathbf{N}^-$) if and only if $m_z' > 0$ (< 0).

Suppose $z = (u,v) \in W^{1,p}(\Omega)\times W^{1,p}(\Omega)\setminus\{(0,0)\}$. Then m_z has a unique critical point at $t = t_{\max}(z)$, where

$$t_{\max}(z) = \left[\frac{(p-q)\|z\|^p}{2(\alpha+\beta-q)\int_\Omega g_\mu|u|^\alpha|v|^\beta\,dx}\right]^{\frac{1}{\alpha+\beta-p}} > 0,$$

and clearly m_z is strictly increasing on $(0, t_{\max}(z))$ and strictly decreasing on $(t_{\max}(z), \infty)$ with $\lim_{t\to\infty} m_z(t) = -\infty$. Moreover, if (2.10) holds, then

$$\begin{aligned}
m_z(t_{\max}(z)) &= \frac{\alpha+\beta-p}{\alpha+\beta-q}\left[\frac{p-q}{2(\alpha+\beta-q)}\right]^{\frac{p-q}{\alpha+\beta-p}}\|z\|^{\frac{p(\alpha+\beta-q)}{\alpha+\beta-p}} \\
&\quad\times\Big(\int_\Omega g_\mu|u|^\alpha|v|^\beta\,dx\Big)^{\frac{q-p}{\alpha+\beta-p}} \\
&\ge \frac{\alpha+\beta-p}{\alpha+\beta-q}\left[\frac{p-q}{2(\alpha+\beta-q)}\right]^{\frac{p-q}{\alpha+\beta-p}}\|z\|^q \\
&\quad\times[(1+\mu\|b\|_\infty)S_{\alpha\beta}^{-(\alpha+\beta)/p}]^{\frac{q-p}{\alpha+\beta-p}} \\
&\ge \frac{\alpha+\beta-p}{\alpha+\beta-q}\left[\frac{p-q}{2(\alpha+\beta-q)}\,\frac{1}{(1+\mu\|b\|_\infty)S_{\alpha\beta}^{-(\alpha+\beta)/p}}\right]^{\frac{p-q}{\alpha+\beta-p}} \\
&\quad\times\frac{\int_\Omega(\lambda_1 f_{1+}|u|^q + \lambda_2 f_{2+}|v|^q)\,dx}{(\lambda_1\|f_{1+}\|_{L^{q^*}} + \lambda_2\|f_{2+}\|_{L^{q^*}})S_{\alpha+\beta}^{-q/p}} \\
&\ge \frac{\alpha+\beta-p}{\alpha+\beta-q}\left[\frac{p-q}{2(\alpha+\beta-q)}\right]^{\frac{p-q}{\alpha+\beta-p}} \\
&\quad\times\frac{S_{\alpha+\beta}^{q/p}S_{\alpha\beta}^{\frac{(\alpha+\beta)(q-p)}{p(\alpha+\beta-p)}}\int_\Omega(\lambda_1 f_{1+}|u|^q + \lambda_2 f_{2+}|v|^q)\,dx}{(\lambda_1\|f_{1+}\|_{L^{q^*}} + \lambda_2\|f_{2+}\|_{L^{q^*}})(1+\mu\|b\|_\infty)^{\frac{p-q}{\alpha+\beta-p}}} \\
&\ge \int_\Omega(f_{1\lambda_1}|u|^q + f_{2\lambda_2}|v|^q)\,dx.
\end{aligned}$$

Then we have the following result.

LEMMA 2.8. *Fix* $z=(u,v)\in W^{1,p}(\Omega)\times W^{1,p}(\Omega)\setminus\{(0,0)\}$.

(i) *If* $\int_\Omega (f_{1\lambda_1}|u|^q+f_{2\lambda_2}|v|^q)\,dx\le 0$, *then there is a unique* $t^-=t^-(z)>t_{\max}(z)$ *such that* $t^-z\in\mathbf{N}^-$. *Moreover,*
$$J(t^-z)=\sup_{t\ge 0}J(tz).$$

(ii) *If* $\int_\Omega (f_{1\lambda_1}|u|^q+f_{2\lambda_2}|v|^q)\,dx> 0$, *then there are unique* $t^-=t^-(z)>t_{\max}(z)>t^+(z)=t^+>0$ *such that* $t^-z\in\mathbf{N}^-$ *and* $t^+z\in\mathbf{N}^+$. *Moreover,*
$$J(t^+z)=\inf_{0\le t\le t_{\max}(z)}J(tz),\qquad J(t^-z)=\sup_{t\ge t^+}J(tz).$$

Proof. (i) Suppose $\int_\Omega (f_{1\lambda_1}|u|^q+f_{2\lambda_2}|v|^q)\,dx\le 0$. Then (2.14) has a unique solution $t^->t_{\max}(z)$ such that $m_z'(t^-)<0$ and $h_z'(t^-)=0$. Hence, as $t^{q-1}m_z'(t)=h_z''(t)$, h_z has a unique critical point at $t=t^-$ and $h_z''(t^-)<0$, thus $t^-z\in\mathbf{N}^-$ and
$$J(t^-z)=\sup_{t\ge 0}J(tz).$$

(ii) Suppose $\int_\Omega (f_{1\lambda_1}|u|^q+f_{2\lambda_2}|v|^q)\,dx> 0$. Since
$$m_z(t_{\max}(z))>\int_\Omega (f_{1\lambda_1}|u|^q+f_{2\lambda_2}|v|^q)\,dx,$$
(2.14) has exactly two solutions $t^+<t_{\max}(z)<t^-$ such that $m_z(t^+)>0$ and $m_z(t^-)<0$. Hence, there are exactly two multiples of z lying in $\mathbf{N}$, namely $t^+z\in\mathbf{N}^+$ and $t^-z\in\mathbf{N}^-$. Thus, as $t^{q-1}m_z'(t)=h_z''(t)$, h_z has exactly two critical points at $t=t^+$ and $t=t^-$ with $h_z''(t^+)>0$ and $h_z''(t^-)<0$. Thus, h_z is decreasing on $(0,t^+)$ and on (t^-,∞), and increasing on (t^+,t^-). Therefore,
$$J(t^+z)=\inf_{0\le t\le t_{\max}(z)}J(tz),\qquad J(t^-z)=\sup_{t\ge t^+}J(tz).\ \blacksquare$$

REMARK 2.9. If $\lambda_1=\lambda_2=0$, then by Lemma 2.8(i), we have $\mathbf{N}^+=\emptyset$, so $\mathbf{N}=\mathbf{N}^-$ for all $\mu\ge 0$.

Now we write $\mathbf{N}=\mathbf{N}^+\cup\mathbf{N}^-$ and define
$$\theta=\inf_{z\in\mathbf{N}}J(z),\qquad \theta^+=\inf_{z\in\mathbf{N}^+}J(z),\qquad \theta^-=\inf_{z\in\mathbf{N}^-}J(z).$$
Then we have the following result.

THEOREM 2.10.

(i) *If*
$$0<(\lambda_1\|f_{1+}\|_{L^{q^*}}+\lambda_2\|f_{2+}\|_{L^{q^*}})^{\alpha+\beta-p}(1+\mu\|b\|_\infty)^{p-q}<\Lambda_0,$$
then $\theta\le\theta^+<0$.

(ii) *If*

$$0 < (\lambda_1 \|f_{1+}\|_{L^{q^*}} + \lambda_2 \|f_{2+}\|_{L^{q^*}})^{\alpha+\beta-p}(1+\mu\|b\|_\infty)^{p-q} < \frac{q}{p}\Lambda_0,$$

then $d < \theta^-$ for some positive constant d depending on p, q, α, β, $f_{1\lambda_1}, f_{2\lambda_2}$ and Ω.

Proof. (i) Let $z \in \mathbf{N}^+$. By (2.8) we have

$$\frac{p-q}{2(\alpha+\beta-q)}\|z\|^p > \int_\Omega g_\mu |u|^\alpha |v|^\beta \, dx,$$

hence

$$\begin{aligned} J(z) &= \left(\frac{1}{p} - \frac{1}{q}\right)\|z\|^p + 2\left(\frac{1}{q} - \frac{1}{\alpha+\beta}\right)\int_\Omega g_\mu |u|^\alpha |v|^\beta \, dx \\ &< \left[\left(\frac{1}{p} - \frac{1}{q}\right) + 2\left(\frac{1}{q} - \frac{1}{\alpha+\beta}\right)\frac{p-q}{2(\alpha+\beta-q)}\right]\|z\|^p \\ &= \left[\frac{p-q}{q(\alpha+\beta)} - \frac{p-q}{pq}\right]\|z\|^p. \end{aligned}$$

So we deduce that $\theta \le \theta^+ < 0$ by the definition of θ, θ^+.

(ii) Set $z \in \mathbf{N}^-$. By (2.8) and (2.9) we have

$$\frac{p-q}{2(\alpha+\beta-q)}\|z\|^p < \int_\Omega g_\mu |u|^\alpha |v|^\beta \, dx,$$

$$\frac{p-\alpha-\beta}{q-\alpha-\beta}\|z\|^p > \int_\Omega (f_{1\lambda_1}|u|^q + f_{2\lambda_2}|v|^q) \, dx.$$

Moreover, by the Hölder inequality and the Sobolev embedding theorem,

$$\int_\Omega g_\mu |u|^\alpha |v|^\beta \, dx \le (1+\mu\|b\|_\infty) S_{\alpha\beta}^{-(\alpha+\beta)/p}\|z\|^{\alpha+\beta}.$$

This implies

$$\|z\| > \left[\frac{(p-q)S_{\alpha\beta}^{(\alpha+\beta)/p}}{2(\alpha+\beta-q)(1+\mu\|b\|_\infty)}\right]^{\frac{1}{\alpha+\beta-p}}.$$

Thus

$$\begin{aligned} J(z) &\ge \frac{\alpha+\beta-p}{p(\alpha+\beta)}\|z\|^p - \frac{\alpha+\beta-q}{q(\alpha+\beta)}(\lambda_1\|f_{1+}\|_{L^{q^*}} + \lambda_2\|f_{2+}\|_{L^{q^*}})S_{\alpha+\beta}^{-q/p}\|z\|^q \\ &= \|z\|^q\left[\frac{\alpha+\beta-p}{p(\alpha+\beta)}\|z\|^{p-q} - \frac{\alpha+\beta-q}{q(\alpha+\beta)}(\lambda_1\|f_{1+}\|_{L^{q^*}} + \lambda_2\|f_{2+}\|_{L^{q^*}})S_{\alpha+\beta}^{-q/p}\right] \end{aligned}$$

$$> \left[\frac{(p-q)S_{\alpha\beta}^{(\alpha+\beta)/p}}{2(\alpha+\beta-q)(1+\mu\|b\|_\infty)}\right]^{\frac{q}{\alpha+\beta-p}}$$
$$\times \left\{\frac{\alpha+\beta-p}{p(\alpha+\beta)}\left[\frac{(p-q)S_{\alpha\beta}^{(\alpha+\beta)/p}}{2(\alpha+\beta-q)(1+\mu\|b\|_\infty)}\right]^{\frac{p-q}{\alpha+\beta-p}}\right.$$
$$\left.-\frac{\alpha+\beta-q}{q(\alpha+\beta)}(\lambda_1\|f_{1+}\|_{L^{q^*}}+\lambda_2\|f_{2+}\|_{L^{q^*}})S_{\alpha+\beta}^{-q/p}\right\}.$$

So if $0 < (\lambda_1\|f_{1+}\|_{L^{q^*}}+\lambda_2\|f_{2+}\|_{L^{q^*}})^{\alpha+\beta-p}(1+\mu\|b\|_\infty)^{p-q} < (q/p)\Lambda_0$, then $J(z) > d$ for all $z \in \mathbf{N}^-$, where $d > 0$ depends on $p, q, \alpha, \beta, f_{1\lambda_1}, f_{2\lambda_2}$ and Ω. ∎

3. Proofs of main results. In this section we prove Theorems 1.1 and 1.2.

First, we establish the following result.

PROPOSITION 3.1.

(i) *If*
$$0 < (\lambda_1\|f_{1+}\|_{L^{q^*}}+\lambda_2\|f_{2+}\|_{L^{q^*}})^{\alpha+\beta-p}(1+\mu\|b\|_\infty)^{p-q} < \Lambda_0,$$
then there exists a $(PS)_\theta$*-sequence* $\{z_n\} \subset \mathbf{N}$ *in* H *for* J.

(ii) *If*
$$0 < (\lambda_1\|f_{1+}\|_{L^{q^*}}+\lambda_2\|f_{2+}\|_{L^{q^*}})^{\alpha+\beta-p}(1+\mu\|b\|_\infty)^{p-q} < \frac{q}{p}\Lambda_0,$$
then there exists a $(PS)_{\theta^-}$*-sequence* $\{z_n\} \subset \mathbf{N}$ *in* H *for* J.

Proof. The proof is almost the same as that in [W3, Proposition 9]. ∎

Now we establish the existence of a local minimum for J on $\mathbf{N}^+$.

THEOREM 3.2. *If*
$$0 < (\lambda_1\|f_{1+}\|_{L^{q^*}}+\lambda_2\|f_{2+}\|_{L^{q^*}})^{\alpha+\beta-p}(1+\mu\|b\|_\infty)^{p-q} < \Lambda_0,$$
then there exists a minimizer $z^0 \in \mathbf{N}^+$ *of* J *and it satisfies:*

(i) $J(z^0) = \theta = \theta^+ < 0$;
(ii) z^0 *is a positive solution of* (1.1);
(iii) $\|z^0\| \to 0$ *as* $\lambda_1 \to 0$ *and* $\lambda_2 \to 0$ *at the same time.*

Proof. By Proposition 3.1(i), there exists a minimizing sequence $\{z_n\} \subset \mathbf{N}$ for J such that

$$J(z_n) = \theta + o(1), \qquad J'(z_n) = o(1). \tag{3.1}$$

Thus by Lemma 2.3, $\{z_n\}$ is bounded in H. Then there exists a subsequence (still denoted by $\{z_n\}$) and $z^0 = (u^0, v^0) \in H$ such that

$$\begin{cases} u_n \rightharpoonup u^0,\ v_n \rightharpoonup v^0, & x \in W_0^{1,p}(\Omega), \\ u_n \to u^0,\ v_n \to v^0, & \text{a.e. in } \Omega, \\ u_n \to u^0,\ v_n \to v^0, & \text{in } L^s(\Omega) \text{ for } 1 \le s < p^*. \end{cases} \tag{3.2}$$

Then as $n \to \infty$,

$$\int_\Omega (f_{1\lambda_1}|u_n|^q + f_{2\lambda_2}|v_n|^q)\,dx = \int_\Omega (f_{1\lambda_1}|u^0|^q + f_{2\lambda_2}|v^0|^q)\,dx + o(1). \tag{3.3}$$

First, we claim that z^0 is a nontrivial solution of (1.1). By (3.1) and (3.2), it is easy to verify that z^0 is a weak solution of (1.1). Combining (3.2) and $z_n \in \mathbf{N}$, we deduce that

$$\int_\Omega (f_{1\lambda_1}|u_n|^q + f_{2\lambda_2}|v_n|^q)\,dx = \frac{q}{p}\frac{\alpha+\beta-p}{\alpha+\beta-q}\|z_n\|^p - \frac{q(\alpha+\beta)}{\alpha+\beta-q}J(z_n). \tag{3.4}$$

From $\theta < 0$, we get

$$\int_\Omega (f_{1\lambda_1}|u_n|^q + f_{2\lambda_2}|v_n|^q)\,dx \ge -\frac{q(\alpha+\beta)}{\alpha+\beta-q}\theta > 0. \tag{3.5}$$

Thus z^0 is a nontrivial solution of (1.1).

Next, we prove $z_n \to z^0$ in H and $J(z^0) = \theta$. For any $z \in \mathbf{N}$, by (3.4) we have

$$J(z) = \frac{\alpha+\beta-p}{p(\alpha+\beta)}\|z\|^p - \frac{\alpha+\beta-q}{q(\alpha+\beta)}\int_\Omega (f_{1\lambda_1}|u|^q + f_{2\lambda_2}|v|^q)\,dx.$$

By Fatou's lemma,

$$\begin{aligned} \theta \le J(z^0) &= \frac{\alpha+\beta-p}{p(\alpha+\beta)}\|z^0\|^p - \frac{\alpha+\beta-q}{q(\alpha+\beta)}\int_\Omega (f_{1\lambda_1}|u^0|^q + f_{2\lambda_2}|v^0|^q)\,dx \\ &\le \liminf_{n\to\infty}\left[\frac{\alpha+\beta-p}{p(\alpha+\beta)}\|z_n\|^p - \frac{\alpha+\beta-q}{q(\alpha+\beta)}\int_\Omega (f_{1\lambda_1}|u_n|^q + f_{2\lambda_2}|v_n|^q)\,dx\right] \\ &= \liminf_{n\to\infty} J(z_n) = \theta. \end{aligned}$$

That is, $J(z^0) = \theta$, and by (3.3) we also have $\|z_n\|^p = \|z^0\|^p + o(1)$. If we let $\overline{z}_n = z_n - z^0$, then by the Brézis–Lieb lemma [BL],

$$\|\overline{z}_n\|^p = \|z_n\|^p - \|z^0\|^p + o(1).$$

Thus we get $z_n \to z^0$ in H.

Finally, we claim that $z^0 \in \mathbf{N}^+$. On the contrary, if $z^0 \in \mathbf{N}^-$, then by Lemma 2.8, there exist unique t^+ and t^- such that $t^+z^0 \in \mathbf{N}^+$ and $t^-z^0 \in \mathbf{N}^-$. Again by Lemma 2.8, we have $t^+ < t^- = 1$. Then $h'_{z^0}(t^+) =$

$h'_{z^0}(t^-) = 0$, $h''_{z^0}(t^+) = (t^+)^{q-1} m'_{z^0}(t^+) > 0$ and $h''_{z^0}(t^-) = (t^-)^{q-1} m'_{z^0}(t^-) < 0$. From the proof of Lemma 2.8(ii), we know h_z is increasing on $[t^+, t^-]$, so

$$J(t^+ z^0) < J(z^0) = \theta = \inf_{z \in \mathbf{N}^+} J(z),$$

which is a contradiction.

Since $J(z^0) = J(|z^0|)$ and $|z^0| \in \mathbf{N}^+$, Lemma 2.5 shows that z^0 is a nontrivial nonnegative solution of (1.1). Moreover, if $\lambda_1 > 0$ or $\lambda_2 > 0$ and $\mu \geq 0$, by the maximum principle we conclude that $z^0 = (u^0, v^0)$ is a positive solution of (1.1).

(iii) For $z^0 \in \mathbf{N}^+$, by (2.9), the Hölder and the Sobolev inequalities we get

$$\|z^0\|^{p-q} \leq \frac{q-\alpha-\beta}{p-\alpha-\beta} (\lambda_1 \|f_{1+}\|_{L^{q^*}} + \lambda_2 \|f_{2+}\|_{L^{q^*}}) S_{\alpha+\beta}^{-q/p},$$

and so $\|z^0\| \to 0$ as $\lambda_1 \to 0$ and $\lambda_2 \to 0$ at the same time. ∎

Proof of Theorem 1.1. From Theorem 3.2, we obtain Theorem 1.1 immediately. ∎

To prove Theorem 1.2, we need to find another positive solution of (1.1), and motivated by Theorem 3.2, we need to establish the existence of a local minimum for J on $\mathbf{N}^-$. Since the functional J defined in (1.6) satisfies the $(\mathrm{PS})_c$-condition for any $c \in \mathbb{R}$ in the subcritical case ($\alpha + \beta < p^*, \alpha, \beta > 1$), we only need to consider the critical case: $\alpha + \beta = p^*$. We have the following lemma.

LEMMA 3.3. *Assume (C1)–(C4) hold, $\alpha + \beta = p^*$, $\alpha, \beta > 1$. Then there exists $\Lambda^* > 0$ such that when*

$$0 < (\lambda_1 \|f_{1+}\|_{L^{q^*}} + \lambda_2 \|f_{2+}\|_{L^{q^*}})^{\alpha+\beta-p} (1 + \mu \|b\|_\infty)^{p-q} < \Lambda^*,$$

we have $\theta^- < c_\infty$, where c_∞ is given in Lemma 2.4.

Proof. First, we consider the functional $I : H \to \mathbb{R}$ defined by

$$I(z) = \frac{1}{p}(\|u\|^p + \|v\|^p) - \frac{2}{\alpha+\beta} \int_\Omega g_\mu |u|^\alpha |v|^\beta \, dx$$

for all $z = (u, v) \in H$. Define

$$u_\varepsilon(x) = \frac{b \varepsilon^{\frac{N-p}{p^2}}}{(\varepsilon + |x|^{\frac{p}{p-1}})^{\frac{N-p}{p}}},$$

where $b > 0$ is a constant. By the results of [DP], we have

$$\|\nabla u_\varepsilon\|_p^p = \|u_\varepsilon\|_{p^*}^{p^*} = S^{N/p},$$

where S is the best Sobolev constant of $W_0^{1,p} \hookrightarrow L^{p^*}$. Since $0 \in \Omega'$, we can choose $r > 0$ such that $B_{2r}(0) \subset \Omega'$, where $B_{2r}(0)$ is the ball centered at the origin and of radius $2r$. Let $\psi \in C_0^\infty$ be such that $\psi(x) \equiv 1$ if $|x| \le r$, $\psi(x) \equiv 0$ if $|x| \ge 2r$, and let $\psi_\varepsilon(x) = \psi(x)u_\varepsilon(x)$. Then we have the following estimates (see [Z] for the details):

$$\int_\Omega |\nabla \psi_\varepsilon|^p\,dx = S^{N/p} + O(\varepsilon^{(N-p)/p}),$$

$$\int_\Omega |\psi_\varepsilon|^{p^*}\,dx = S^{N/p} + O(\varepsilon^{N/p}),$$

$$\int_\Omega |\psi_\varepsilon|^p\,dx = \begin{cases} K_1\varepsilon^{(N-p)/p} + O(\varepsilon^{(N-p)/p}), & p < N < p^2, \\ K_2\varepsilon^{p-1}(\ln\varepsilon) + O(\varepsilon^{p-1}), & N = p^2, \\ K_3\varepsilon^{p-1} + O(\varepsilon^{(N-p)/p}), & N > p^2 \ge 2^{1/2}. \end{cases}$$

Set $\widetilde{u} = \alpha^{1/p}\psi_\varepsilon$, $\widetilde{v} = \beta^{1/p}\psi_\varepsilon$, and $\widetilde{z} = (\widetilde{u}, \widetilde{v}) \in H$. Then

$$\begin{aligned}
\sup_{t\ge0} I(t\widetilde{z}) &= \sup_{t\ge0}\left(\frac{t^p}{p}\|\widetilde{z}\|^p - \frac{2t^{p^*}}{p^*}\alpha^{\alpha/p}\beta^{\beta/p}\int_\Omega g_\mu|\psi_\varepsilon|^{p^*}\,dx\right) \\
&\le \sup_{t\ge0}\left(\frac{t^p}{p}\|\widetilde{z}\|^p - \frac{2t^{p^*}}{p^*}\alpha^{\alpha/p}\beta^{\beta/p}\int_\Omega |\psi_\varepsilon|^{p^*}\,dx\right) \\
&\le \frac{1}{N2^{(N-p)/p}}\left[\left(\frac{\alpha}{\beta}\right)^{\alpha/p^*} + \left(\frac{\beta}{\alpha}\right)^{\beta/p^*}\right]^{N/p}\left[\frac{\int_\Omega |\nabla\psi_\varepsilon|^p\,dx}{(\int_\Omega |\psi_\varepsilon|^{p^*}\,dx)^{p/p^*}}\right]^{N/p} \\
&= \frac{1}{N2^{(N-p)/p}}\left[\left(\frac{\alpha}{\beta}\right)^{\alpha/p^*} + \left(\frac{\beta}{\alpha}\right)^{\beta/p^*}\right]^{N/p}[S + O(\varepsilon^{(N-p)/p})]^{N/p} \\
&\le \frac{1}{N2^{(N-p)/p}}\left[\left(\left(\frac{\alpha}{\beta}\right)^{\alpha/p^*} + \left(\frac{\beta}{\alpha}\right)^{\beta/p^*}\right)S\right]^{N/p} + O(\varepsilon^{(N-p)/p}) \\
&= \frac{2}{N}\left(\frac{S_{\alpha\beta}}{2}\right)^{N/p} + O(\varepsilon^{(N-p)/p}).
\end{aligned}$$

We choose $\delta_1 > 0$ such that when $0 < \lambda_1\|f_{1+}\|_{L^{q^*}} + \lambda_2\|f_{2+}\|_{L^{q^*}} < \delta_1$, we have

$$c_\infty = \frac{2}{N}\left(\frac{S_{\alpha\beta}}{2}\right)^{N/p} - c_0(\lambda_1\|f_{1+}\|_{L^{q^*}} + \lambda_2\|f_{2+}\|_{L^{q^*}})^{\frac{p}{p-q}} > 0,$$

and using the definitions of I and $\widetilde{z}$, we obtain

$$I(t\widetilde{z}) \le \frac{t^p}{p}\|\widetilde{z}\|^p, \quad \forall t > 0 \quad \text{and} \quad \lambda_1, \lambda_2, \mu \ge 0.$$

Then there exists $\widetilde{t} \in (0,1)$ satisfying

$$\sup_{\widetilde{t} \geq t \geq 0} I(t\widetilde{z}) < c_\infty \quad \text{when } 0 < \lambda_1 \|f_{1+}\|_{L^{q^*}} + \lambda_2 \|f_{2+}\|_{L^{q^*}} < \delta_1.$$

Also, we have

$$\begin{aligned} \sup_{t \geq \widetilde{t}} J(t\widetilde{z}) &= \sup_{t \geq \widetilde{t}} \left[I(t\widetilde{z}) - \frac{t^q}{q} \int_\Omega (f_{1\lambda_1} + f_{2\lambda_2}) |\psi_\varepsilon|^q \, dx \right] \\ &\leq \frac{2}{N} \left(\frac{S_{\alpha\beta}}{2} \right)^{N/p} + O(\varepsilon^{(N-p)/p}) \\ &\quad - \frac{\widetilde{t}^q}{q} (\lambda_1 \min_{B_r(0)} f_1 + \lambda_2 \min_{B_r(0)} f_2) \int_{B_r(0)} |\psi_\varepsilon|^q \, dx. \end{aligned}$$

Let $0 < \varepsilon \leq r^{p/(p-1)}$. We have

$$\int_{B_r(0)} |\psi_\varepsilon|^q \, dx \geq c_1,$$

where c_1 depends on N, p, q, r. Then for all $\varepsilon = (\lambda_1 \|f_{1+}\|_{L^{q^*}} + \lambda_2 \|f_{2+}\|_{L^{q^*}})^{\frac{p^2}{N-p}} \in (0, r^{p/(p-1)})$, we obtain

$$\begin{aligned} \sup_{t \geq \widetilde{t}} J(t\widetilde{z}) \leq \frac{2}{N} \left(\frac{S_{\alpha\beta}}{2} \right)^{N/p} &+ O((\lambda_1 \|f_{1+}\|_{L^{q^*}} + \lambda_2 \|f_{2+}\|_{L^{q^*}})^p) \\ &- c_1 \frac{\widetilde{t}^q}{q} (\lambda_1 \min_{B_r(0)} f_1 + \lambda_2 \min_{B_r(0)} f_2). \end{aligned}$$

Hence we can choose $\delta_2 > 0$ such that when $0 < \lambda_1 \|f_{1+}\|_{L^{q^*}} + \lambda_2 \|f_{2+}\|_{L^{q^*}} < \delta_2$, we have

$$\begin{aligned} O((\lambda_1 \|f_{1+}\|_{L^{q^*}} + \lambda_2 \|f_{2+}\|_{L^{q^*}})^p) - c_1 \frac{\widetilde{t}^q}{q} (\lambda_1 \min_{B_r(0)} f_1 + \lambda_2 \min_{B_r(0)} f_2) \\ < -c_0 (\lambda_1 \|f_{1+}\|_{L^{q^*}} + \lambda_2 \|f_{2+}\|_{L^{q^*}})^{p/(p-q)}. \end{aligned}$$

Set $\Lambda^* = \min\{\delta_1^{p^*-p}, \delta_2^{p^*-p}, r^{\frac{p}{p-1}}\} > 0$, $\varepsilon = (\lambda_1 \|f_{1+}\|_{L^{q^*}} + \lambda_2 \|f_{2+}\|_{L^{q^*}})^{\frac{p^2}{N-p}}$. Then if $0 < (\lambda_1 \|f_{1+}\|_{L^{q^*}} + \lambda_2 \|f_{2+}\|_{L^{q^*}})^{\alpha+\beta-p} (1 + \mu \|b\|_\infty)^{p-q} < \Lambda^*$, we have $\sup_{t \geq 0} J(t\widetilde{z}) \leq c_\infty$. Recalling the definition of $\widetilde{z}$, it is easy to see that

$$\int_\Omega (f_{1\lambda_1} |\widetilde{u}|^q + f_{2\lambda_2} |\widetilde{v}|^q) \, dx > 0.$$

Combining this with Lemma 2.8, there exists $t^- > t_{\max}(\widetilde{z})$ such that $t^- \widetilde{z} \in \mathbf{N}^-$, and

$$\theta^- \leq J(t^- \widetilde{z}) \leq \sup_{t \geq 0} J(t\widetilde{z}) < c_\infty$$

whenever

$$0 < (\lambda_1\|f_{1+}\|_{L^{q^*}} + \lambda_2\|f_{2+}\|_{L^{q^*}})^{\alpha+\beta-p}(1+\mu\|b\|_\infty)^{p-q} < \Lambda^*. \ \blacksquare$$

Set $\Lambda_1 = \min\{\Lambda^*, (q/p)\Lambda_0\}$. Then we have

THEOREM 3.4. *If*

$$0 < (\lambda_1\|f_{1+}\|_{L^{q^*}} + \lambda_2\|f_{2+}\|_{L^{q^*}})^{\alpha+\beta-p}(1+\mu\|b\|_\infty)^{p-q} < \Lambda_1,$$

then the functional J has a minimizer z^1 in $\mathbf{N}^-$ and satisfies

(i) $J(z^1) = \theta^-$;

(ii) z^1 *is a positive solution of* (1.1).

Proof. By Proposition 3.1(ii), there exists a minimizing sequence $\{z_n\} \subset \mathbf{N}^-$ for J such that

$$J(z_n) = \theta^- + o(1), \quad J'(z_n) = o(1). \tag{3.6}$$

Thus by Lemma 2.3, $\{z_n\}$ is bounded in H. Then there exists a subsequence (still denoted by $\{z_n\}$) and $z^1 \in \mathbf{N}^-$ such that $z_n \to z^1$ in H, and $J(z^1) = \theta^- > 0$. Next, by using the same argument as in the proof of Theorem 3.2, when $0 < (\lambda_1\|f_{1+}\|_{L^{q^*}} + \lambda_2\|f_{2+}\|_{L^{q^*}})^{\alpha+\beta-p}(1+\mu\|b\|_\infty)^{p-q} < \Lambda_1$, we conclude that z^1 is a positive solution of (1.1). $\blacksquare$

Proof of Theorem 1.2. When

$$0 < (\lambda_1\|f_{1+}\|_{L^{q^*}} + \lambda_2\|f_{2+}\|_{L^{q^*}})^{\alpha+\beta-p}(1+\mu\|b\|_\infty)^{p-q} < \Lambda_1 < \Lambda_0,$$

by Theorem 3.2 there exists a positive solution $z^0 \in \mathbf{N}^+$, and by Theorem 3.4 there exists a positive solution $z^1 \in \mathbf{N}^-$. Since $\mathbf{N}^+ \cap \mathbf{N}^- = \emptyset$, problem (1.1) have two positive solutions for any $\lambda_1, \lambda_2, \mu \geq 0$ satisfying

$$0 < (\lambda_1\|f_{1+}\|_{L^{q^*}} + \lambda_2\|f_{2+}\|_{L^{q^*}})^{\alpha+\beta-p}(1+\mu\|b\|_\infty)^{p-q} < \Lambda_1. \ \blacksquare$$

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