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A criterion for pure unrectifiability of sets (via universal vector bundle)

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Abstract. Let m, n be positive integers such that $m < n$ and let $G(n, m)$ be the Grassmann manifold of all m-dimensional subspaces of $\mathbb{R}^n$. For $V \in G(n, m)$ let π_V denote the orthogonal projection from $\mathbb{R}^n$ onto V. The following characterization of purely unrectifiable sets holds. Let A be an $\mathcal{H}^m$-measurable subset of $\mathbb{R}^n$ with $\mathcal{H}^m(A) < \infty$. Then A is purely m-unrectifiable if and only if there exists a null subset Z of the universal bundle $\{(V, v) \mid V \in G(n, m),\ v \in V\}$ such that, for all $P \in A$, one has $\mathcal{H}^{m(n-m)}(\{V \in G(n, m) \mid (V, \pi_V(P)) \in Z\}) > 0$. One can replace "for all $P \in A$" by "for $\mathcal{H}^m$-a.e. $P \in A$".

1. Introduction. Given a couple of positive integers n and m, with $m < n$, let $G(n, m)$ be the Grassmann manifold of all m-dimensional subspaces of $\mathbb{R}^n$. Then let $E(n, m)$ denote the corresponding universal vector bundle, i.e.

$$E(n, m) := \{(V, v) \mid V \in G(n, m),\ v \in V\}$$

(see [Mil, §5]). Recall that $G(n, m)$ is a manifold of dimension $k := m(n-m)$ (see [Fed, 3.2.28], [Mil, Lemma 5.1]), hence $E(n, m)$ has dimension $k + m = m(n + 1 - m)$. Thus we say that a subset of $E(n, m)$ is "null" when its $\mathcal{H}^{k+m}$-measure is zero.

Consider the natural projection map

$$\Pi : G(n, m) \times \mathbb{R}^n \to E(n, m), \qquad (V, P) \mapsto (V, \pi_V(P)),$$

where π_V is the orthogonal projection from $\mathbb{R}^n$ onto V. For $P \in \mathbb{R}^n$ define $\Pi_P : G(n, m) \to E(n, m)$ by

$$\Pi_P(V) := \Pi(V, P) = (V, \pi_V(P)), \qquad V \in G(n, m).$$

Given a subset Z of $E(n, m)$, let

$$\Sigma_Z := \{P \in \mathbb{R}^n \mid \mathcal{H}^k(\Pi_P^{-1}(Z)) > 0\}.$$

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This paper is devoted to proving the following characterization of pure unrectifiability.

THEOREM 1.1. *Let A be an $\mathcal{H}^m$-measurable subset of $\mathbb{R}^n$ with $\mathcal{H}^m(A) < \infty$. Then the following statements are equivalent.*

(1) *The set A is purely m-unrectifiable.*
(2) *There exists a null subset Z of $E(n,m)$ such that $A \subset \Sigma_Z$.*
(3) *There exists a null subset Z of $E(n,m)$ such that $\mathcal{H}^m(A \setminus \Sigma_Z) = 0$.*

We will show that (1)⇒(2) follows from an easy argument based on the celebrated criterion for pure unrectifiability, the Besicovitch–Federer projection theorem [Mat, Theorem 18.1]. So the main part of our work will be proving (3)⇒(1) and more precisely the following result.

THEOREM 1.2. *If Z is a null subset of $E(n,m)$ then Σ_Z is purely m-unrectifiable, that is,*

$$\mathcal{H}^m(\Sigma_Z \cap E) = 0$$

for every m-rectifiable subset E of $\mathbb{R}^n$.

Other characterizations of pure unrectifiability, besides the Besicovitch–Federer projection theorem mentioned above, can be found in [Mat, Corollary 15.20, Theorem 17.6] and in [KMM, Theorem 2.1].

2. Technical lemma

LEMMA 2.1. *Let $u_1, \ldots, u_m$ be linearly independent vectors in $\mathbb{R}^n$ and let*

$$\mathbf{G} := \{V \in G(n,m) \mid \pi_V(u_1) \wedge \cdots \wedge \pi_V(u_m) = 0\}.$$

Then

$$\mathcal{H}^k(\mathbf{G}) = 0 \tag{2.1}$$

where k is the dimension of $G(n,m)$, i.e. $k := m(n-m)$.

Proof. Recall that $G(n,m)$ is a connected real analytic variety and observe that $\mathbf{G}$ is the set of zeros of the analytic function

$$G(n,m) \to \mathbb{R}, \qquad V \mapsto \|\pi_V(u_1) \wedge \cdots \wedge \pi_V(u_m)\|^2,$$

which is nonconstant (it is nonzero on $\operatorname{span}\{u_1, \ldots, u_m\}$, while it takes value zero at any V orthogonal to u_1). Then the conclusion follows easily from the following well-known fact [Fed, 3.1.24]: *If f is an analytic function whose domain is a connected open subset A of $\mathbb{R}^k$, then either $f^{-1}\{0\} = A$ or $L^k(f^{-1}\{0\}) = 0$.* ■

REMARK 2.2. The result stated in Lemma 2.1 can also be derived from general theorems about the dimension of an analytic set (e.g. Propositions 15 and 18 in [Nar, Ch. V] or Proposition 2 in [Lo, Sect. 18]). A sharp dimensional argument can be obtained directly as follows. For $i = 1, \dots, \mu := \min\{m, n-m\}$, let

$$\mathbf{G}_i := \{V \in G(n,m) \mid \dim(V_0 \cap V^\perp) = i\}, \qquad V_0 := \operatorname{span}\{u_1, \dots, u_m\},$$
$$\mathbf{G}_i^\perp := \{V^\perp \mid V \in \mathbf{G}_i\} = \{V^\perp \mid V \in G(n,m),\, \dim(V_0 \cap V^\perp) = i\}$$

and observe that $\mathbf{G}$ is the set of $V \in G(n,m)$ such that $\pi_V(u_1), \dots, \pi_V(u_m)$ are linearly dependent. Hence

$$\begin{aligned}\mathbf{G} &= \Big\{V \in G(n,m) \;\Big|\; \pi_V\Big(\sum_{j=1}^m x_j u_j\Big) = 0 \text{ for some } (x_1, \dots, x_m) \in \mathbb{R}^m \setminus \{0\}\Big\} \\ &= \Big\{V \in G(n,m) \;\Big|\; \sum_{j=1}^m x_j u_j \in V^\perp \text{ for some } (x_1, \dots, x_m) \in \mathbb{R}^m \setminus \{0\}\Big\} \\ &= \{V \in G(n,m) \mid V_0 \cap V^\perp \neq \{0\}\} = \bigcup_{i=1}^{\mu} \mathbf{G}_i.\end{aligned}$$

Since

- $\mathbf{G}_i$ and $\mathbf{G}_i^\perp$ have the same dimension;
- the set $\mathbf{G}_i^\perp$ can be viewed as a bundle over the Grassmannian manifold of all i-dimensional subspaces of V_0 with fiber the Grassmannian manifold of all $(n-m-i)$-dimensional subspaces of $V_0^\perp$,

we get

$$\begin{aligned}\dim(\mathbf{G}_i) &= \dim G(m,i) + \dim G(n-m, n-m-i) \\ &= i(m-i) + (n-m-i)i = i(n-2i),\end{aligned}$$

and an elementary computation shows that $i(n-2i) < k$ for all $i = 1, \dots, \mu$. The subadditivity of $\mathcal{H}^k$ finally yields (2.1).

3. Proof of theorems

3.1. Proof of Theorem 1.2. By the definition of rectifiable set, it is enough to show that

$$\mathcal{H}^m(\Sigma_Z \cap \varphi(\Omega)) = 0$$

for every 1-1 map $\varphi : \Omega := (0,1)^m \to \mathbb{R}^n$ of class C^1 with bounded derivatives. Without loss of generality, we can also suppose that

$$J\varphi(\rho) := \|D_1\varphi(\rho) \wedge \dots \wedge D_m\varphi(\rho)\| \neq 0 \tag{3.1}$$

for all $\rho \in \Omega$. Define the map

$$\Phi(V, \rho) := (V, \pi_V(\varphi(\rho)), \rho), \qquad (V, \rho) \in G(n,m) \times \Omega,$$

and the following measurable subsets of $G(n,m)\times\Omega$:

$$\begin{aligned}
E &:= \Phi^{-1}(Z\times\Omega),\\
G &:= \{(V,\rho)\in G(n,m)\times\Omega \mid \pi_V(D_1\varphi(\rho))\wedge\cdots\wedge\pi_V(D_m\varphi(\rho))=0\},\\
H &:= E\setminus G.
\end{aligned}$$

One has $E=\{(V,\rho)\in G(n,m)\times\Omega \mid (V,\pi_V(\varphi(\rho)))\in Z\}$, hence

$$E_\rho := \{V\in G(n,m)\mid (V,\rho)\in E\} = \Pi^{-1}_{\varphi(\rho)}(Z)$$

for all $\rho\in\Omega$. It follows that

$$\begin{aligned}
(3.2)\qquad \int_{\Sigma_Z\cap\varphi(\Omega)} \mathcal{H}^k(\Pi_P^{-1}(Z))\,d\mathcal{H}^m(P) &\le \int_{\varphi(\Omega)} \mathcal{H}^k(\Pi_P^{-1}(Z))\,d\mathcal{H}^m(P)\\
&= \int_{\Omega} \mathcal{H}^k(\Pi^{-1}_{\varphi(\rho)}(Z))J\varphi(\rho)\,d\rho\\
&= \int_{\Omega} \mathcal{H}^k(E_\rho)J\varphi(\rho)\,d\rho.
\end{aligned}$$

Since

$$(\mathcal{H}^k\times L^m)(E)\le(\mathcal{H}^k\times L^m)(G(n,m)\times\Omega)\le\mathcal{H}^k(G(n,m))L^m(\Omega)<\infty$$

Fubini's theorem [Fed, 2.6.2] yields

$$\int_E J\varphi(\rho)\,d(\mathcal{H}^k\times L^m)(V,\rho) = \int_\Omega \mathcal{H}^k(E_\rho)J\varphi(\rho)\,d\rho.$$

By (3.2) we get

$$(3.3)\qquad \int_{\Sigma_Z\cap\varphi(\Omega)} \mathcal{H}^k(\Pi_P^{-1}(Z))\,d\mathcal{H}^m(P) \le \int_E J\varphi(\rho)\,d(\mathcal{H}^k\times L^m)(V,\rho).$$

Observe that, for all $\rho\in\Omega$, one has

$$\begin{aligned}
G_\rho &:= \{V\in G(n,m)\mid (V,\rho)\in G\}\\
&= \{V\in G(n,m)\mid \pi_V(D_1\varphi(\rho))\wedge\cdots\wedge\pi_V(D_m\varphi(\rho))=0\}.
\end{aligned}$$

Since $D_1\varphi(\rho),\dots,D_m\varphi(\rho)$ are linearly independent, by (3.1), we can apply Lemma 2.1 to obtain

$$\mathcal{H}^k(G_\rho)=0 \quad \text{for all } \rho\in\Omega.$$

By invoking Fubini's theorem again, we get

$$(3.4)\qquad (\mathcal{H}^k\times L^m)(G)=0.$$

Now, for $V\in G(n,m)$, define

$$Z_V := \{v\in V\mid (V,v)\in Z\}$$

and

$$\begin{aligned} H_V &:= \{\rho \in \Omega \mid (V,\rho) \in H\} \\ &= \{\rho \in \Omega \mid \pi_V(\varphi(\rho)) \in Z_V,\ \pi_V(D_1\varphi(\rho)) \wedge \cdots \wedge \pi_V(D_m\varphi(\rho)) \neq 0\}. \end{aligned}$$

From (3.3), (3.4) and again by Fubini's theorem, we obtain

$$\begin{aligned} \int_{\Sigma_Z \cap \varphi(\Omega)} \mathcal{H}^k(\Pi_P^{-1}(Z))\, d\mathcal{H}^m(P) &\le \int_H J\varphi(\rho)\, d(\mathcal{H}^k \times L^m)(V,\rho) \\ &= \int_{G(n,m)} \Big(\int_{H_V} J\varphi(\rho)\, d\rho \Big)\, d\mathcal{H}^k(V). \end{aligned}$$

Then it remains to show that

$$L^m(H_V) = 0 \quad \text{for } \mathcal{H}^k\text{-a.e. } V \in G(n,m). \tag{3.5}$$

To this end, observe that $\mathcal{H}^m(Z_V) = 0$ for $\mathcal{H}^k$-a.e. $V \in G(n,m)$, by Fubini's theorem for Riemannian submersions [Sak, Ch. II, Theorem 5.6]. Then, applying [Fed, 3.2.3] with

$$f := \pi_V \circ \varphi, \quad A := H_V,$$

and recalling that $\pi_V(\varphi(\rho)) \in Z_V$ for all $\rho \in Z_V$, we find that

$$\begin{aligned} \int_{H_V} \|\pi_V(D_1\varphi(\rho)) \wedge \cdots \wedge \pi_V(D_m\varphi(\rho))\|\, d\rho &= \int_{\mathbb{R}^n} N((\pi_V \circ \varphi)|H_V, y)\, d\mathcal{H}^m(y) \\ &= \int_{Z_V} N((\pi_V \circ \varphi)|H_V, y)\, d\mathcal{H}^m(y) = 0 \end{aligned}$$

for $\mathcal{H}^k$-a.e. $V \in G(n,m)$. Hence (3.5) follows immediately from the definition of H_V. ■

3.2. Proof of Theorem 1.1. (1)⇒(2). Consider the following measurable subset of $E(n,m)$:

$$Z := \{(V,v) \mid V \in G(n,m),\ v \in \pi_V(A)\}.$$

Since

$$Z_V := \{v \in V \mid (V,v) \in Z\} = \pi_V(A)$$

for all $V \in G(n,m)$, the Besicovitch–Federer projection theorem [Mat, Theorem 18.1] yields $\mathcal{H}^m(Z_V) = 0$ for $\mathcal{H}^k$-a.e. $V \in G(n,m)$. Hence $\mathcal{H}^{k+m}(Z) = 0$ by Fubini's theorem for Riemannian submersions [Sak, Ch. II, Theorem 5.6]. Moreover, for $P \in A$, one has

$$\begin{aligned} \Pi_P^{-1}(Z) &= \{V \in G(n,m) \mid \pi_V(P) \in Z_V\} \\ &= \{V \in G(n,m) \mid \pi_V(P) \in \pi_V(A)\} = G(n,m). \end{aligned}$$

Hence we finally obtain $\mathcal{H}^k(\Pi_P^{-1}(Z)) > 0$ for all $P \in A$, which concludes the proof of (2).

(2)⇒(3) is trivial.

(3)⇒(1). To see this, consider any m-rectifiable subset E of $\mathbb{R}^n$. We will show that $\mathcal{H}^m(E \cap A) = 0$. Observe first of all that

$$A \cap E = [(A \setminus \Sigma_Z) \cap E] \cup [(A \cap \Sigma_Z) \cap E] \subset (A \setminus \Sigma_Z) \cup (\Sigma_Z \cap E),$$

hence

$$\mathcal{H}^m(A \cap E) \leq \mathcal{H}^m(A \setminus \Sigma_Z) + \mathcal{H}^m(\Sigma_Z \cap E) = \mathcal{H}^m(\Sigma_Z \cap E).$$

The conclusion now follows immediately from Theorem 1.2. ∎

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