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A natural occurrence of shift equivalence

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Abstract. A natural occcurrence of shift equivalence in a purely algebraic setting is exhibited.

1. Introduction. Group endomorphisms $\alpha : G \to G$ and $\beta : H \to H$ are said to be *conjugate* if there exists an isomorphism $\theta : G \to H$ such that $\theta \circ \alpha = \beta \circ \theta$, and *shift equivalent* if there exist group endomorphisms $\varphi : G \to H$ and $\psi : H \to G$ and $n \in \mathbb{Z}^+$ such that

$$\varphi \circ \alpha = \beta \circ \varphi, \quad \psi \circ \varphi = \alpha^n,$$
$$\psi \circ \beta = \alpha \circ \psi, \quad \varphi \circ \psi = \beta^n,$$

that is, the diagrams

$$\begin{array}{ccc} G & \xrightarrow{\alpha} & G \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ H & \xrightarrow{\beta} & H \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{\alpha} & G \\ {\scriptstyle\psi}\uparrow & & \uparrow{\scriptstyle\psi} \\ H & \xrightarrow{\beta} & H \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{\alpha^n} & G \\ {\scriptstyle\varphi}\downarrow & \nearrow{\scriptstyle\psi} & \downarrow{\scriptstyle\varphi} \\ H & \xrightarrow{\beta^n} & H \end{array}$$

commute. In the latter case we say that φ, ψ effect a shift equivalence of α to β of lag $n \in \mathbb{Z}^+$.

The concept of shift equivalence was introduced by R. F. Williams [W1], [W2] in the context of topological dynamics. The fact that shift equivalence is an equivalence relation among group endomorphisms can be demostrated by a straightforward argument [T1].

Clearly both conjugacy and shift equivalence can be defined in any category and the former constitutes a special case of the latter in two ways:

- A shift equivalence with lag 0 is a conjugacy.
- A shift equivalence between two automorphisms is a conjugacy.

The simple result presented here was independently observed by Yu. I. Ustinov [U]. In our opinion this is the most straightforward and natural

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occurrence of shift equivalence as a complete invariant. Although by no means entirely novel, we feel that this elegant result deserves to be available to a wider public in the form of an independent exposition.

Another very natural occurrence of shift equivalence arises in shape and homotopy theory [T2].

2. Statement and proof of the main result. Given a group endomorphism $\alpha : G \to G$ the *simple direct limit* of α, denoted by $\mathfrak{G} = \lim_{\to}(G,\alpha)$, is the set of equivalence classes in $G \times \mathbb{Z}^+$ under the equivalence relation $\sim$ where

$$(g,n) \sim (g',n') \quad \text{iff} \quad \alpha^{N-n}(g) = \alpha^{N-n'}(g') \text{ for some } N \geq n, n'.$$

This can be easily checked to be an equivalence relation. The set $\mathfrak{G}$ has a natural group structure with respect to the binary operation

$$(g,n)(g',n') = (\alpha^{n'}(g)\alpha^{n}(g'), n+n')$$

where, by abuse of notation, we let (g,n) stand for the equivalence class it represents. Again it can be routinely checked that this is a well-defined operation satisfying all group axioms. There are two natural isomorphisms on $\mathfrak{G}$: Firstly,

$$\breve{\alpha} : \mathfrak{G} \to \mathfrak{G}, \qquad \breve{\alpha}((g,n)) = (\alpha(g), n),$$

secondly,

$$s_\alpha : \mathfrak{G} \to \mathfrak{G}, \qquad s_\alpha((g,n)) = (g, n+1)$$

(which we like to call the "coshift").

Again, these can be checked to be well-defined homomorphisms. To see that they are isomorphisms it is enough to observe that

$$\breve{\alpha} \circ s_\alpha = s_\alpha \circ \breve{\alpha} = \mathrm{Id}(\mathfrak{G}).$$

THEOREM 2.1. *Let G and H be finitely generated groups, $\alpha : G \to G$ and $\beta : H \to H$ group endomorphisms, and $\mathfrak{G} = \lim_{\to}(G,\alpha)$, $\mathfrak{H} = \lim_{\to}(H,\beta)$. The isomorphisms $s_\alpha : \mathfrak{G} \to \mathfrak{G}$ and $s_\beta : \mathfrak{H} \to \mathfrak{H}$ are conjugate iff α and β are shift equivalent.*

Proof. Given a subset K of a group, let $\langle K \rangle$ denote the subgroup generated by K. There exist finite sets $A \subseteq G$ and $B \subseteq H$ such that $G = \langle A \rangle$ and $H = \langle B \rangle$. Assume first that s_α and s_β, or equivalently $\breve{\alpha}$ and $\breve{\beta}$, are conjugate: there exists an isomorphism $T : \mathfrak{G} \to \mathfrak{H}$ such that $T \circ \breve{\alpha} = \breve{\beta} \circ T$. Let

$$i_\alpha : G \to \mathfrak{G} \quad \text{and} \quad i_\beta : H \to \mathfrak{H}$$

be the natural injections defined by

$$i_\alpha(g) = (g,0) \in \mathfrak{G} \quad \text{and} \quad i_\beta(h) = (h,0) \in \mathfrak{H}.$$

We have
$$T \circ i_\alpha(G) \subseteq \langle T \circ i_\alpha(A) \rangle.$$
Clearly $T \circ i_\alpha(A)$ is a finite subset of $\mathfrak{H}$. Hence there exists $k \in \mathbb{Z}^+$ such that
$$T \circ i_\alpha(G) \subseteq \langle T \circ i_\alpha(A) \rangle \subseteq H \times \{k\}.$$
Therefore,
$$\check{\beta}^k \circ T \circ i_\alpha(G) \subseteq H \times \{0\}.$$
We define
$$\varphi = i_\beta^{-1} \circ \check{\beta}^k \circ T \circ i_\alpha : G \to H.$$
Similarly, there exists a sufficiently large $l \in \mathbb{Z}^+$ such that
$$\psi = i_\alpha^{-1} \circ \check{\alpha}^l \circ T \circ i_\alpha : H \to G$$
is a well-defined homomorphism. We claim that φ and ψ effect a shift equivalence of α to β with lag $k + l \in \mathbb{Z}^+$: Clearly
$$\varphi \circ \alpha = \beta \circ \varphi \quad \text{and} \quad \psi \circ \beta = \alpha \circ \psi.$$
Moreover,
$$\psi \circ \varphi = i_\alpha^{-1} \circ \check{\alpha}^l \circ T^{-1} \circ i_\beta \circ i_\beta^{-1} \circ \check{\beta}^k \circ T \circ i_\alpha = \alpha^{k+l}.$$
Similarly,
$$\varphi \circ \psi = \beta^{k+l}.$$
Conversely, assume that there exist $\varphi : G \to H$, $\psi : H \to G$ and $n \in \mathbb{Z}^+$ such that $\varphi \circ \alpha = \beta \circ \phi$, $\psi \circ \beta = \alpha \circ \psi$, $\psi \circ \varphi = \alpha^n$ and $\phi \circ \psi = \beta^n$. Consider the map
$$E : \mathfrak{G} \to \mathfrak{H}, \qquad E((g, m)) = (\varphi(g), m).$$
Note that E is well-defined: if $\alpha^{l-m}(g) = \alpha^{l-m'}(g')$, then
$$\varphi \circ \alpha^{l-m}(g) = \varphi \circ \alpha^{l-m'}(g'),$$
hence
$$\beta^{l-m} \circ \varphi(g) = \beta^{l-m'} \circ \varphi(g').$$
We also have $E \circ \check{\alpha} = \check{\beta} \circ E$ owing to $\varphi \circ \alpha = \beta \circ \varphi$, once again. Similarly define
$$F : \mathfrak{H} \to \mathfrak{G}, \qquad F((h, m)) = (\psi(h), m).$$
We observe
$$F \circ E((g, m)) = F((\varphi(g), m)) = (\psi \circ \varphi(g), m) = (\alpha^n(g), m) = \check{\alpha}^n(g, m).$$
Thus $F \circ E = \check{\alpha}^n$. The right hand side is an isomorphism, so E is an isomorphism, which commutes with $\check{\alpha}$ and $\check{\beta}$. ∎

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