A natural occurrence of shift equivalence

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Abstract. A natural occurrence of shift equivalence in a purely algebraic setting is exhibited.

1. Introduction. Group endomorphisms $\alpha : G \to G$ and $\beta : H \to H$ are said to be conjugate if there exists an isomorphism $\theta : G \to H$ such that $\theta \circ \alpha = \beta \circ \theta$, and shift equivalent if there exist group endomorphisms $\varphi : G \to H$ and $\psi : H \to G$ and $n \in \mathbb{Z}^+$ such that

$$\varphi \circ \alpha = \beta \circ \varphi, \quad \psi \circ \varphi = \alpha^n, \quad \psi \circ \beta = \alpha \circ \psi, \quad \varphi \circ \psi = \beta^n,$$

that is, the diagrams

\[
\begin{array}{ccc}
G & \xrightarrow{\alpha} & G \\
\downarrow \varphi & & \downarrow \varphi \\
H & \xrightarrow{\beta} & H \\
\end{array} \quad \begin{array}{ccc}
G & \xrightarrow{\alpha} & G \\
\uparrow \psi & & \uparrow \psi \\
H & \xrightarrow{\beta} & H \\
\end{array} \quad \begin{array}{ccc}
G & \xrightarrow{\alpha^n} & G \\
\varphi & \xrightarrow{} & \varphi \\
\psi & \xleftarrow{} & \psi \\
\end{array}
\]

commute. In the latter case we say that $\varphi, \psi$ effect a shift equivalence of $\alpha$ to $\beta$ of lag $n \in \mathbb{Z}^+$.

The concept of shift equivalence was introduced by R. F. Williams [W1], [W2] in the context of topological dynamics. The fact that shift equivalence is an equivalence relation among group endomorphisms can be demonstrated by a straightforward argument [T1].

Clearly both conjugacy and shift equivalence can be defined in any category and the former constitutes a special case of the latter in two ways:

- A shift equivalence with lag 0 is a conjugacy.
- A shift equivalence between two automorphisms is a conjugacy.

The simple result presented here was independently observed by Yu. I. Ustinov [U]. In our opinion this is the most straightforward and natural

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occurrence of shift equivalence as a complete invariant. Although by no means entirely novel, we feel that this elegant result deserves to be available to a wider public in the form of an independent exposition.

Another very natural occurrence of shift equivalence arises in shape and homotopy theory \[T2\].

2. Statement and proof of the main result. Given a group endomorphism \( \alpha : G \to G \) the simple direct limit of \( \alpha \), denoted by \( \mathcal{G} = \lim_{\to} (G, \alpha) \), is the set of equivalence classes in \( G \times \mathbb{Z}^+ \) under the equivalence relation \( \sim \) where

\[
(g, n) \sim (g', n') \iff \alpha^{N-n}(g) = \alpha^{N-n'}(g') \text{ for some } N \geq n, n'.
\]

This can be easily checked to be an equivalence relation. The set \( \mathcal{G} \) has a natural group structure with respect to the binary operation

\[
(g, n)(g', n') = (\alpha^{n'}(g)\alpha^{n}(g'), n + n')
\]

where, by abuse of notation, we let \((g, n)\) stand for the equivalence class it represents. Again it can be routinely checked that this is a well-defined operation satisfying all group axioms. There are two natural isomorphisms on \( \mathcal{G} \): Firstly,

\[
\tilde{\alpha} : \mathcal{G} \to \mathcal{G}, \quad \tilde{\alpha}((g, n)) = (\alpha(g), n),
\]

secondly,

\[
s_{\alpha} : \mathcal{G} \to \mathcal{G}, \quad s_{\alpha}((g, n)) = (g, n + 1)
\]

(which we like to call the “coshift”).

Again, these can be checked to be well-defined homomorphisms. To see that they are isomorphisms it is enough to observe that

\[
\tilde{\alpha} \circ s_{\alpha} = s_{\alpha} \circ \tilde{\alpha} = \text{Id}(\mathcal{G}).
\]

**Theorem 2.1.** Let \( G \) and \( H \) be finitely generated groups, \( \alpha : G \to G \) and \( \beta : H \to H \) group endomorphisms, and \( \mathcal{G} = \lim_{\to} (G, \alpha) \), \( \mathcal{H} = \lim_{\to} (H, \beta) \). The isomorphisms \( s_{\alpha} : \mathcal{G} \to \mathcal{G} \) and \( s_{\beta} : \mathcal{H} \to \mathcal{H} \) are conjugate iff \( \alpha \) and \( \beta \) are shift equivalent.

**Proof.** Given a subset \( K \) of a group, let \( \langle K \rangle \) denote the subgroup generated by \( K \). There exist finite sets \( A \subseteq G \) and \( B \subseteq H \) such that \( G = \langle A \rangle \) and \( H = \langle B \rangle \). Assume first that \( s_{\alpha} \) and \( s_{\beta} \), or equivalently \( \tilde{\alpha} \) and \( \tilde{\beta} \), are conjugate: there exists an isomorphism \( T : \mathcal{G} \to \mathcal{H} \) such that \( T \circ \tilde{\alpha} = \tilde{\beta} \circ T \). Let

\[
i_{\alpha} : G \to \mathcal{G} \quad \text{and} \quad i_{\beta} : H \to \mathcal{H}
\]

be the natural injections defined by

\[
i_{\alpha}(g) = (g, 0) \in \mathcal{G} \quad \text{and} \quad i_{\beta}(h) = (h, 0) \in \mathcal{H}.
\]
We have \[ T \circ i_\alpha(G) \subseteq \langle T \circ i_\alpha(A) \rangle. \]
Clearly \( T \circ i_\alpha(A) \) is a finite subset of \( \mathcal{F} \). Hence there exists \( k \in \mathbb{Z}^+ \) such that \( T \circ i_\alpha(G) \subseteq \langle T \circ i_\alpha(A) \rangle \subseteq H \times \{k\}. \)
Therefore, \( \beta^k \circ T \circ i_\alpha(G) \subseteq H \times \{0\}. \)
We define \[ \varphi = i_{\beta}^{-1} \circ \beta^k \circ T \circ i_\alpha : G \to H. \]
Similarly, there exists a sufficiently large \( l \in \mathbb{Z}^+ \) such that \[ \psi = i_{\alpha}^{-1} \circ \alpha^l \circ T \circ i_\alpha : H \to G \]
is a well-defined homomorphism. We claim that \( \varphi \) and \( \psi \) effect a shift equivalence of \( \alpha \) to \( \beta \) with lag \( k + l \in \mathbb{Z}^+ \).
Clearly \[ \varphi \circ \alpha = \beta \circ \varphi \quad \text{and} \quad \psi \circ \beta = \alpha \circ \psi. \]
Moreover, \[ \psi \circ \varphi = i_{\alpha}^{-1} \circ \alpha^l \circ T^{-1} \circ i_\beta \circ i_{\beta}^{-1} \circ \beta^k \circ T \circ i_\alpha = \alpha^{k+l}. \]
Similarly, \[ \varphi \circ \psi = \beta^{k+l}. \]
Conversely, assume that there exist \( \varphi : G \to H, \psi : H \to G \) and \( n \in \mathbb{Z}^+ \) such that \( \varphi \circ \alpha = \beta \circ \phi, \psi \circ \beta = \alpha \circ \psi, \psi \circ \varphi = \alpha^n \) and \( \phi \circ \psi = \beta^n \). Consider the map \[ E : \mathcal{G} \to \mathcal{F}, \quad E((g, m)) = (\varphi(g), m). \]
Note that \( E \) is well-defined: if \( \alpha^{l-m}(g) = \alpha^{l-m'}(g') \), then \[ \varphi \circ \alpha^{l-m}(g) = \varphi \circ \alpha^{l-m'}(g'), \]
hence \[ \beta^{l-m} \circ \varphi(g) = \beta^{l-m'} \circ \varphi(g'). \]
We also have \( E \circ \alpha = \beta \circ E \) owing to \( \varphi \circ \alpha = \beta \circ \varphi \), once again. Similarly define \[ F : \mathcal{F} \to \mathcal{G}, \quad F((h, m)) = (\psi(h), m). \]
We observe \[ F \circ E((g, m)) = F((\varphi(g), m)) = (\psi \circ \varphi(g), m) = (\alpha^n(g), m) = \tilde{\alpha}^n(g, m). \]
Thus \( F \circ E = \tilde{\alpha}^n \). The right hand side is an isomorphism, so \( E \) is an isomorphism, which commutes with \( \tilde{\alpha} \) and \( \beta \).

References


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