ANNALES
POLONICI MATHEMATICI
102.1 (2011)

Normality criteria and multiple values II

by YAN XU (Nanjing) and JIANMING CHANG (Changshu)

Abstract. Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D, let ψ ($\not\equiv 0, \infty$) be a meromorphic function in D, and k be a positive integer. If, for every $f \in \mathcal{F}$ and $z \in D$, (1) $f \neq 0$, $f^{(k)} \neq 0$; (2) all zeros of $f^{(k)} - \psi$ have multiplicities at least $(k+2)/k$; (3) all poles of ψ have multiplicities at most k, then $\mathcal{F}$ is normal in D.

1. Introduction. Let D be a domain in $\mathbb{C}$, and $\mathcal{F}$ be a family of meromorphic functions defined in D. Then $\mathcal{F}$ is said to be *normal* on D, in the sense of Montel, if for any sequence $\{f_n\} \subset \mathcal{F}$ there exists a subsequence $\{f_{n_j}\}$ such that $\{f_{n_j}\}$ converges spherically locally uniformly on D to a meromorphic function or ∞ (see [5, 8, 15]).

Yang [14] and Schwick [10] proved

THEOREM A. *Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D, let k be a positive integer, and let $\varphi(z)$ ($\not\equiv 0$) be an analytic function in D. If, for each $f \in \mathcal{F}$, $f \neq 0$ and $f^{(k)}(z) \neq \varphi(z)$ in D, then $\mathcal{F}$ is normal.*

Bergweiler and Langley [2] (cf. [1]) proved

THEOREM B. *Let $\mathcal{F}$ be a family of meromorphic functions defined in D, and let $k \geq 2$ be a positive integer. If, for each $f \in \mathcal{F}$, $f \neq 0$ and $f^{(k)} \neq 0$ in D, then $\{f'/f : f \in \mathcal{F}\}$ is normal.*

The holomorphic case was proved by Schwick [9]. Theorem B can be considered as the normal families analogue arising according to Bloch's principle from the well-known Picard type theorem: *Let f be meromorphic in $\mathbb{C}$ and let $k \geq 2$. If f and $f^{(k)}$ have no zeros, then f has the form $f(z) = e^{az+b}$ or $f(z) = (az+b)^{-n}$, where $a, b \in \mathbb{C}$, $a \neq 0$, and $n \in \mathbb{N}$*, which was conjectured by Hayman [6] and proved by Frank [4] for $k \geq 3$ and by Langley [7] for $k = 2$.

2010 *Mathematics Subject Classification*: Primary 30D05; Secondary 30D45.
Key words and phrases: meromorphic function, normal family, exceptional functions, multiplicity.

Let $D = \{z : |z| < 1\}$ and $\mathcal{F} = \{f_n(z) = nz\}$. Noting that $f_n'/f_n = 1/z$ for each n, $\{f_n'/f_n\}$ is obviously normal in D, but $\mathcal{F}$ is not normal in D. So the normality of $\{f'/f : f \in \mathcal{F}\}$ does not imply that of $\mathcal{F}$.

In [12] (cf. [3, 11]), we showed the following result

THEOREM C. *Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D, let ψ ($\not\equiv 0$) be a holomorphic function in D, and k be a positive integer. If, for each $f \in \mathcal{F}$,*

(a) *$f \neq 0$ and $f^{(k)} \neq 0$ in D;*
(b) *all zeros of $f^{(k)}(z) - \psi(z)$ have multiplicities at least $(k+2)/k$ in D,*

then $\mathcal{F}$ is normal.

REMARK 1. In fact, $(k+2)/k = 3$ for the case $k = 1$, and $1 < (k+2)/k < 2$ for the case $k \geq 2$. The number $(k+2)/k$ in Theorem C is sharp, which can be seen from two examples in [3].

It is natural to ask: *does Theorem C hold if we only assume that $\psi(z)$ is meromorphic?* In this paper, we prove the following result.

THEOREM 1. *Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain $D \subset \mathbb{C}$, let k be a positive integer, and let ψ ($\not\equiv \infty$) be a nonvanishing meromorphic function in D. If, for each $f \in \mathcal{F}$,*

(1) *$f \neq 0$ and $f^{(k)} \neq 0$ in D,*
(2) *all zeros of $f^{(k)}(z) - \psi(z)$ have multiplicities at least $(k+2)/k$ in D,*
(3) *all poles of ψ have multiplicities at most k in D,*

then $\mathcal{F}$ is normal.

REMARK 2. The following example shows that condition (3) in Theorem 1 cannot be omitted.

EXAMPLE 1. Let $k \in \mathbb{N}$, $D = \{z : |z| < 1\}$, $\psi(z) = 1/z^{k+1}$, and

$$\mathcal{F} = \left\{ f_n(z) = \frac{1}{nz} : z \in D \right\}.$$

Clearly, $f_n(z) \neq 0$ and $f_n^{(k)}(z) = (-1)^k k!/(nz^{k+1}) \neq 0$. We also have

$$f_n^{(k)}(z) - \psi(z) = \left(\frac{(-1)^k k!}{n} - 1 \right) \frac{1}{z^{k+1}} \neq 0.$$

Thus conditions (1) and (2) in Theorem 1 are satisfied. But $\mathcal{F}$ is not normal in D.

Since normality is a local property, combining Theorems C and 1 we obtain the following theorem.

THEOREM 2. *Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain $D \subset \mathbb{C}$, let k be a positive integer, and let ψ ($\not\equiv 0, \infty$) be a meromorphic function in D. If, for each $f \in \mathcal{F}$,*

(1) *$f \neq 0$ and $f^{(k)} \neq 0$ in D,*
(2) *all zeros of $f^{(k)}(z) - \psi(z)$ have multiplicities at least $(k+2)/k$ in D,*
(3) *all poles of ψ have multiplicities at most k in D,*

then $\mathcal{F}$ is normal.

2. Lemmas. The following is a local version of Zalcman's lemma due to Xue and Pang [13] (cf. [16]).

LEMMA 1. *Let $\mathcal{F}$ be a family of functions meromorphic in a domain D such that $f \neq 0$ for each $f \in \mathcal{F}$. If $\mathcal{F}$ is not normal at $z_0 \in D$, then, for each $\alpha \geq 0$, there exist a sequence of points $z_n \in D$, $z_n \to z_0$, a sequence of positive numbers $\rho_n \to 0$, and a sequence of functions $f_n \in \mathcal{F}$ such that*

$$g_n(\zeta) = \frac{f_n(z_n + \rho_n\zeta)}{\rho_n^\alpha} \to g(\zeta)$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on $\mathbb{C}$.

LEMMA 2. *Let k, l be two integers with $k \geq l \geq 0$. Then there does not exist any rational function f such that $f \neq 0$, $f^{(k)} \neq 0$, and all zeros of $f^{(k)}(z) - 1/(z-\alpha)^l$ have multiplicity at least $(k+2)/k$ in $\mathbb{C}$, where α is a complex number.*

Proof. Suppose that such a rational function f exists. Since $f \neq 0$ and $f^{(k)} \neq 0$, we see that f is a nonpolynomial rational function and has the form

$$f(z) = \frac{A}{(z-z_1)^{m_1} \cdots (z-z_t)^{m_t}},$$

where $A \neq 0$ is a constant, and $m_1, \ldots, m_t$ are positive integers. Using the results of Frank [4] for $k \geq 3$ and Langley [7] for $k = 2$, we know that f has the form

$$f(z) = \frac{A}{(z-z_1)^m} \tag{1}$$

for $k \geq 2$. Set $m = m_1 + \cdots + m_t$. Then

$$f'(z) = \frac{-A(mz^{t-1} + b_{t-2}z^{t-2} + \cdots + b_0)}{(z-z_1)^{m_1+1} \cdots (z-z_t)^{m_t+1}},$$

where $b_{t-2}, \ldots, b_0$ are constants. In view of $f' \neq 0$, we get $t = 1$. It follows that f also has the form (1) for $k = 1$. Thus

$$f^{(k)}(z) = \frac{B}{(z-z_1)^{m+k}}, \tag{2}$$

where m is a positive integer.

For $l = 0$, we know that

$$f^{(k)}(z) - \frac{1}{(z-\alpha)^l} = f^{(k)}(z) - 1 = \frac{B - (z-z_1)^{m+k}}{(z-z_1)^{m+k}}$$

has only simple zeros, a contradiction.

Next we consider the case $l \geq 1$. If $\alpha = z_1$, then

$$f^{(k)}(z) - \frac{1}{(z-\alpha)^l} = \frac{B - (z-z_1)^{m+k-l}}{(z-z_1)^{m+k}},$$

and thus $f^{(k)}(z) - 1/(z-\alpha)^l$ has only simple zeros, a contradiction. Thus $\alpha \neq z_1$.

Since

$$f^{(k)}(z) - \frac{1}{(z-\alpha)^l} = \frac{B(z-\alpha)^l - (z-z_1)^{m+k}}{(z-z_1)^{m+k}(z-\alpha)^l}, \tag{3}$$

there exists a point z_0 such that $f^{(k)}(z_0) - 1/(z_0-\alpha)^l = 0$. As all zeros of $f^{(k)}(z) - 1/(z-\alpha)^l$ have multiplicity at least $(k+2)/k$, we see from (3) that

$$B(z_0-\alpha)^l - (z_0-z_1)^{m+k} = 0, \tag{4}$$

$$lB(z_0-\alpha)^{l-1} - (m+k)(z_0-z_1)^{m+k-1} = 0. \tag{5}$$

Solving (4) and (5) for z_0, we obtain

$$z_0 = \frac{(m+k)\alpha - lz_1}{m+k-l},$$

which implies that $f^{(k)}(z) - 1/(z-\alpha)^l$ has only one zero z_0 as above. Thus

$$(z-z_1)^{m+k} - B(z-\alpha)^l = \left(z - \frac{(m+k)\alpha - lz_1}{m+k-l}\right)^{m+k}. \tag{6}$$

If $l < k$, then equating the coefficients of z^{m+k-1} in (6), we get

$$z_1 = \frac{(m+k)\alpha - lz_1}{m+k-l},$$

and so $\alpha = z_1$, a contradiction.

Therefore $l = k$, and (6) can be written as

$$(z-z_1)^{m+k} - B(z-\alpha)^k = \left(z - \frac{(m+k)\alpha - kz_1}{m}\right)^{m+k}. \tag{7}$$

For $m \geq 2$, equating the coefficients of z^{m+k-1} in (7), we also deduce that $\alpha = z_1$, a contradiction. For $m = 1$, by (7), we have

$$(z-z_1)^{k+1} - B(z-\alpha)^k = [z - (k+1)\alpha + kz_1]^{k+1}. \tag{8}$$

Equating the coefficients of z^k and z^{k-1} in (8), we get

$$(k+1)z_1 + B = (k+1)[(k+1)\alpha - kz_1], \tag{9}$$

$$\binom{k+1}{2} z_1^2 + k\alpha B = \binom{k+1}{2}[(k+1)\alpha - kz_1]^2. \tag{10}$$

From (9), we have $B = (k+1)^2(\alpha - z_1)$. Substituting this in (10) gives

$$(k^2-1)(\alpha - z_1)^2 = 0.$$

Noting $\alpha \neq z_1$, we obtain $k = l = 1$. Then we conclude from (3) and (6) that $f'(z) - 1/(z-\alpha)$ has one zero with multiplicity 2. But this contradicts the assumption that all zeros of $f'(z) - 1/(z-\alpha)$ have multiplicity at least $(k+2)/k = 3$ (here $k = l = 1$). Lemma 2 is proved. ∎

We shall use the standard notation of value distribution theory (see [5, 15]): $T(r,f), m(r,f), N(r,f), \bar{N}(r,f), \ldots$. We denote by $S(r,f)$ any function satisfying

$$S(r,f) = o\{T(r,f)\}$$

as $r \to \infty$, possibly outside a set of finite measure.

Lemma 3. *Let k be a positive integer, let $\mathcal{F} = \{f_n\}$ be a family of meromorphic functions defined in a domain D , and let $\varphi_n(z)$ be a sequence of holomorphic functions on D such that $\varphi_n \to \varphi$ locally uniformly on D, where $\varphi(z)$ $(\neq 0)$ is holomorphic on D. If $f_n \neq 0$, $f_n^{(k)} \neq 0$, and all zeros of $f_n^{(k)} - \varphi_n$ have multiplicity at least $(k+2)/k$, then $\mathcal{F}$ is normal in D.*

Proof. Suppose that $\mathcal{F}$ is not normal at $z_0 \in D$. By Lemma 1, there exist a sequence of functions $f_n \in \mathcal{F}$, a sequence of complex numbers $z_n \to z_0$ and a sequence of positive numbers $\rho_n \to 0$ such that

$$g_n(\zeta) = \frac{f_n(z_n + \rho_n\zeta)}{\rho_n^k} \to g(\zeta)$$

spherically uniformly on compact subsets of $\mathbb{C}$, where $g(\zeta)$ is a nonconstant meromorphic function on $\mathbb{C}$. Hurwitz's theorem implies that $g(\zeta) \neq 0$.

We see that

$$g_n^{(k)}(\zeta) = f_n^{(k)}(z_n + \rho_n\zeta) \to g^{(k)}(\zeta) \tag{11}$$

spherically uniformly on every compact subset of $\mathbb{C}$ which contains no pole of $g(\zeta)$. From (11), we know that either $g^{(k)} \neq 0$ or $g^{(k)} \equiv 0$ for any $\zeta \in \mathbb{C}$ that is not a pole of $g(\zeta)$. Clearly, these also hold for all $\zeta \in \mathbb{C}$. If $g^{(k)} \equiv 0$, we deduce that g is a nonzero constant since $g \neq 0$, a contradiction. Therefore $g^{(k)} \neq 0$.

Since

$$g_n^{(k)}(\zeta) - \varphi_n(z_n + \rho_n\zeta) = f_n^{(k)}(z_n + \rho_n\zeta) - \varphi_n(z_n + \rho_n\zeta) \to g^{(k)}(\zeta) - \varphi(z_0),$$

Hurwitz's theorem implies that all zeros of $g^{(k)}(\zeta)-\varphi(z_0)$ have multiplicities at least $(k+2)/k$. It follows from Lemma 2 (for $l=0$) that g must be transcendental.

By Nevanlinna's first and second fundamental theorems, we have

$$\begin{aligned}T(r,g^{(k)}) &\le \bar N(r,g^{(k)})+\bar N\left(r,\frac{1}{g^{(k)}}\right)+\bar N\left(r,\frac{1}{g^{(k)}-\varphi(z_0)}\right)+S(r,g^{(k)})\\ &\le \frac{1}{k+1}N(r,g^{(k)})+\frac{k}{k+2}N\left(r,\frac{1}{g^{(k)}-\varphi(z_0)}\right)+S(r,g^{(k)})\\ &\le \frac{1}{k+1}T(r,g^{(k)})+\frac{k}{k+2}T\left(r,\frac{1}{g^{(k)}-\varphi(z_0)}\right)+S(r,g^{(k)})\\ &\le \frac{k^2+2k+2}{k^2+3k+2}T(r,g^{(k)})+S(r,g^{(k)}),\end{aligned}$$

a contradiction. Lemma 3 is proved. ■

3. Proof of Theorem 1. Without loss of generality, we may assume $D=\Delta=\{z:|z|<1\}$, and

$$\psi(z)=\frac{\varphi(z)}{z^l}\quad (z\in\Delta),$$

where l is a positive integer with $l\le k$, $\varphi(0)=1$, $\varphi(z)\ne 0,\infty$ on $\Delta'=\{z:0<|z|<1\}$. By Theorem C, it is enough to show that $\mathcal{F}$ is normal at $z=0$.

Suppose that $\mathcal{F}$ is not normal at $z=0$. By Lemma 1 (with $\alpha=k-l$), there exist a sequence of functions $f_n\in\mathcal{F}$, a sequence of complex numbers $z_n\to 0$ and a sequence of positive numbers $\rho_n\to 0$ such that

$$F_n(\zeta)=\frac{f_n(z_n+\rho_n\zeta)}{\rho_n^{k-l}}\to F(\zeta) \tag{12}$$

spherically uniformly on compact subsets of $\mathbb{C}$, where $F(\zeta)$ is a nonconstant meromorphic function on $\mathbb{C}$. By Hurwitz's theorem, $F(\zeta)\ne 0$.

Obviously, on every compact subsets of $\mathbb{C}$ which contains no poles of $F(\zeta)$,

$$F_n^{(k)}(\zeta)=\rho_n^l f_n^{(k)}(z_n+\rho_n\zeta)\to F^{(k)}(\zeta).$$

Since $f_n^{(k)}(z_n+\rho_n\zeta)\ne 0$, we see that either $F^{(k)}(\zeta)\ne 0$ or $F^{(k)}(\zeta)\equiv 0$ for any $\zeta\in\mathbb{C}$ that is not a pole of $F(\zeta)$. Obviously, these also hold for all $\zeta\in\mathbb{C}$. If $F^{(k)}(\zeta)\equiv 0$, then $F(\zeta)$ is a polynomial of degree at most $k-1$, but this contradicts the fact that $F(\zeta)\ne 0$ and $F(\zeta)$ is nonconstant. So $F^{(k)}(\zeta)\ne 0$.

We distinguish the following two cases.

Case 1: $z_n/\rho_n\to\infty$. Set

$$g_n(\zeta)=z_n^{l-k}f_n(z_n(1+\zeta)).$$

Clearly, $g_n \neq 0$ and $g_n^{(k)} \neq 0$. Since

$$g_n^{(k)}(\zeta) - \frac{\varphi_n(z_n(1+\zeta))}{(1+\zeta)^l} = z_n^l \left[f_n^{(k)}(z_n(1+\zeta)) - \frac{\varphi(z_n(1+\zeta))}{(z_n(1+\zeta))^l} \right] = z_n^l [f_n^{(k)}(z_n(1+\zeta)) - \psi(z_n(1+\zeta))],$$

by the assumption of theorem, all zeros of $g_n^{(k)}(\zeta) - \varphi(z_n(1+\zeta))/(1+\zeta)^l$ have multiplicity at least $(k+2)/k$ in Δ. On the other hand, $\varphi(z_n(1+\zeta))/(1+\zeta)^l$ is holomorphic in Δ for each n, and

$$\frac{\varphi(z_n(1+\zeta))}{(1+\zeta)^l} \to \frac{1}{(1+\zeta)^l} \ (\neq 0)$$

for $\zeta \in \Delta$. Then, by Lemma 3, $\{g_n\}$ is normal in Δ.

Hence, we can find a subsequence $\{g_{n_j}\} \subset \{g_n\}$ and a function g such that

$$g_{n_j}(\zeta) = z_{n_j}^{l-k} f_{n_j}(z_{n_j}(1+\zeta)) \to g(\zeta) \tag{13}$$

spherically locally uniformly on Δ.

If $g(0) \neq \infty$, from (12) and (13), and noting $z_n/\rho_n \to \infty$, we have

$$F^{(k-l)}(\zeta) = \lim_{j\to\infty} f_{n_j}^{(k-l)}(z_{n_j} + \rho_{n_j}\zeta) = \lim_{j\to\infty} f_{n_j}^{(k-l)}\left(z_{n_j} + z_{n_j}\left(\frac{\rho_{n_j}}{z_{n_j}}\zeta \right)\right) = \lim_{j\to\infty} g_{n_j}^{(k-l)}\left(\frac{\rho_{n_j}}{z_{n_j}}\zeta \right) = g^{(k-l)}(0). \tag{14}$$

It follows from (14) that $F^{(k-l)}(\zeta)$ must be a finite constant, and then $F(\zeta)$ is a polynomial. But this is impossible since $F(\zeta)$ is nonconstant and $F(\zeta) \neq 0$.

If $g(0) = \infty$, then

$$g_{n_j}\left(\frac{\rho_{n_j}}{z_{n_j}}\zeta \right) = z_{n_j}^{l-k} f_{n_j}(z_{n_j} + \rho_{n_j}\zeta) \to g(0) = \infty,$$

and hence

$$F(\zeta) = \lim_{j\to\infty} \frac{f_{n_j}(z_{n_j} + \rho_{n_j}\zeta)}{\rho_{n_j}^{k-l}} = \lim_{j\to\infty} \left(\frac{z_{n_j}}{\rho_{n_j}} \right)^{k-l} z_{n_j}^{l-k} f_{n_j}(z_{n_j} + \rho_{n_j}\zeta) = \infty,$$

that is, $F(\zeta) \equiv \infty$, a contradiction.

Case 2: $z_n/\rho_n \nrightarrow \infty$. Taking a subsequence and renumbering, we may assume that $z_n/\rho_n \to \alpha$, a finite complex number.

We have

$$F_n^{(k)}(\zeta) - \frac{\rho_n^l \varphi(z_n + \rho_n\zeta)}{(z_n + \rho_n\zeta)^l} \to F^{(k)}(\zeta) - \frac{1}{(\alpha+\zeta)^l} \tag{15}$$

on $\mathbb{C} \setminus \{-\alpha\}$. Since

$$F_n^{(k)}(\zeta) - \frac{\rho_n^l \varphi(z_n + \rho_n\zeta)}{(z_n + \rho_n\zeta)^l} = \rho_n^l (f_n^{(k)}(z_n + \rho_n\zeta) - \psi(z_n + \rho_n\zeta)),$$

and $f_n^{(k)}(z_n + \rho_n\zeta) - \psi(z_n + \rho_n\zeta)$ has only zeros with multiplicity at least $(k+2)/k$, Hurwitz's theorem and (15) imply that all zeros of $F^{(k)}(\zeta) - 1/(\alpha+\zeta)^l$ have multiplicity at least $(k+2)/k$.

Using Nevanlinna's first and second fundamental theorems (for small functions), we have

$$\begin{aligned} T(r,F^{(k)}) &\le \bar{N}(r,F^{(k)}) + \bar{N}\left(r,\frac{1}{F^{(k)}}\right) + \bar{N}\left(r,\frac{1}{F^{(k)} - 1/(\alpha+\zeta)^l}\right) + S(r,F^{(k)}) \\ &\le \frac{1}{k+1} N(r,F^{(k)}) + \frac{k}{k+2} N\left(r,\frac{1}{F^{(k)} - 1/(\alpha+\zeta)^l}\right) + S(r,F^{(k)}) \\ &\le \frac{1}{k+1} T(r,F^{(k)}) + \frac{k}{k+2} T\left(r,\frac{1}{F^{(k)} - 1/(\alpha+\zeta)^l}\right) + S(r,F^{(k)}) \\ &\le \frac{k^2+2k+2}{k^2+3k+2} T(r,F^{(k)}) + S(r,F^{(k)}). \end{aligned}$$

This implies that F is a rational function. However, by Lemma 2, such an F does not exist, a contradiction. Theorem 1 is thus proved. ∎

Acknowledgements. Y. Xu was supported by NSFC (grant no. 10871094). J. M. Chang was supported by NSFC (grant no. 10871094), NSFU of Jiangsu, China (grant no. 08KJB110001) and the Qing Lan Project of Jiangsu, China.

References

[1] W. Bergweiler, *Normality and exceptional values of derivatives*, Proc. Amer. Math. Soc. 129 (2001), 121–129.
[2] W. Bergweiler and J. K. Langley, *Nonvanishing derivatives and normal families*, J. Anal. Math. 91 (2003), 353–367.
[3] M. L. Fang and J. M. Chang, *Normal families and multiple values*, Arch. Math. (Basel) 88 (2007), 560–568.
[4] G. Frank, *Eine Vermutung von Hayman über Nullstellen meromorpher Funktionen*, Math. Z. 149 (1976), 29–36.
[5] W. K. Hayman, *Meromorphic Functions*, Clarendon Press, Oxford, 1964.
[6] —, *Research Problems in Function Theory*, Athlone Press, London, 1967.
[7] J. K. Langley, *Proof of a conjecture of Hayman concerning f and f''*, J. London Math. Soc. (2) 48 (1993), 500–514.
[8] J. L. Schiff, *Normal Families*, Springer, New York, 1993.
[9] W. Schwick, *Normality criteria for families of meromorphic functions*, J. Anal. Math. 52 (1989), 241–289.

[10] W. Schwick, *Exceptional functions and normality*, Bull. London Math. Soc. 29 (1997), 425–432.

[11] J. Y. Xia and Y. Xu, *Normal families of meromorphic functions with multiple values*, J. Math. Anal. Appl. 354 (2009), 387–393.

[12] Y. Xu and J. M. Chang, *Normality criteria and multiple values*, Acta Math. Sinica (Chin. Ser.) 54 (2011), 265–270.

[13] G. F. Xue and X. C. Pang, *A criterion for normality of a family of meromorphic functions*, J. East China Norm. Univ. Natur. Sci. Ed. 1988, no. 2, 15–22.

[14] Le Yang, *Normality for families of meromorphic functions*, Sci. Sinica Ser. A 29 (1986), 1263–1274.

[15] Lo Yang, *Value Distribution Theory*, Springer, Berlin, and Sci. Press, Beijing, 1993.

[16] L. Zalcman, *Normal families: new perspectives*, Bull. Amer. Math. Soc. 35 (1998), 215–230.

Yan Xu
Institute of Mathematics
School of Mathematics
Nanjing Normal University
Nanjing 210046, P.R. China
E-mail: xuyan@njnu.edu.cn

Jianming Chang
Department of Mathematics
Changshu Institute of Technology
Changshu, Jiangsu, 215500, P.R. China
E-mail: jmchang@cslg.edu.cn

Received 5.12.2010
and in final form 18.12.2010 (2336)