ANNALES
POLONICI MATHEMATICI
LXXIX.3 (2002)

Existence of positive solutions for second order m-point boundary value problems

by Ruyun Ma (Lanzhou)

Abstract. Let $\alpha,\beta,\gamma,\delta \ge 0$ and $\varrho := \gamma\beta+\alpha\gamma+\alpha\delta > 0$. Let $\psi(t) = \beta+\alpha t$, $\phi(t)=\gamma+\delta-\gamma t$, $t\in[0,1]$. We study the existence of positive solutions for the m-point boundary value problem

$$\begin{cases} u''+h(t)f(u)=0, \quad 0<t<1,\\ \alpha u(0)-\beta u'(0)=\sum_{i=1}^{m-2} a_i u(\xi_i),\\ \gamma u(1)+\delta u'(1)=\sum_{i=1}^{m-2} b_i u(\xi_i),\end{cases}$$

where $\xi_i\in(0,1)$, $a_i,b_i\in(0,\infty)$ (for $i\in\{1,\dots,m-2\}$) are given constants satisfying $\varrho-\sum_{i=1}^{m-2}a_i\phi(\xi_i)>0$, $\varrho-\sum_{i=1}^{m-2}b_i\psi(\xi_i)>0$ and

$$\Delta := \begin{vmatrix} -\sum_{i=1}^{m-2} a_i\psi(\xi_i) & \varrho-\sum_{i=1}^{m-2}a_i\phi(\xi_i)\\ \varrho-\sum_{i=1}^{m-2}b_i\psi(\xi_i) & -\sum_{i=1}^{m-2}b_i\phi(\xi_i)\end{vmatrix} < 0.$$

We show the existence of positive solutions if f is either superlinear or sublinear by a simple application of a fixed point theorem in cones. Our result extends a result established by Erbe and Wang for two-point BVPs and a result established by the author for three-point BVPs.

1. Introduction. The study of multi-point boundary value problems for linear second order ordinary differential equations was initiated by Il'in and Moiseev [6]. Motivated by [6], Gupta [4] studied certain three-point boundary value problems for nonlinear ordinary differential equations. Since then, more general nonlinear multi-point boundary value problems have been studied by several authors. We refer the reader to [4–6, 8–10] for some relevant references.

2000 *Mathematics Subject Classification*: 34B10, 34B18.

Key words and phrases: multi-point boundary value problems, positive solutions, fixed point theorem, cones.

Supported by the NSFC and GG-110-10736-1003 and NWNU-KJCXGC-212.

In this paper, we are interested in the existence of positive solutions of the second order m-point boundary value problem

$$(1.1)\qquad \begin{cases} u'' + h(t)f(u) = 0, \quad 0 < t < 1, \\ \alpha u(0) - \beta u'(0) = \displaystyle\sum_{i=1}^{m-2} a_i u(\xi_i), \\ \gamma u(1) + \delta u'(1) = \displaystyle\sum_{i=1}^{m-2} b_i u(\xi_i), \end{cases}$$

where $\xi_i \in (0,1)$, $a_i, b_i \in (0,\infty)$ (for $i \in \{1,\dots,m-2\}$) are given constants.

If $a_i = b_i = 0$ for $i = 1,\dots,m-2$, then the m-point BVP (1.1) reduces to the two-point BVP

$$(1.2)\qquad \begin{cases} u'' + h(t)f(u) = 0, \quad 0 < t < 1, \\ \alpha u(0) - \beta u'(0) = 0, \\ \gamma u(1) + \delta u'(1) = 0. \end{cases}$$

In 1994, Erbe and Wang [3] obtained the following excellent result for (1.2).

Theorem A ([3, Theorem 1]). *Suppose that*

(A1) $f \in C([0,\infty),[0,\infty))$;
(A2) $h \in C([0,1],[0,\infty))$ *and* $h(t) \equiv 0$ *on no subinterval of* $(0,1)$;
(A3) $\alpha, \beta, \gamma, \delta \ge 0$, *and* $\varrho := \gamma\beta + \alpha\gamma + \alpha\delta > 0$.

Then (1.2) *has at least one positive solution if either*

(i) $f_0 = 0$ *and* $f_\infty = \infty$, *or*
(ii) $f_0 = \infty$ *and* $f_\infty = 0$,

where

$$f_0 := \lim_{u\to 0^+} \frac{f(u)}{u}, \qquad f_\infty := \lim_{u\to\infty} \frac{f(u)}{u}.$$

This result has been extended and developed by many authors (see Erbe, Hu and Wang [2] and Lian, Wong and Yeh [7] for some references).

If $\alpha = \gamma = 1$, $\beta = \delta = 0$, $a_i = 0$ for $i = 1,\dots,m-2$, and $b_j = 0$ for $j = 2,\dots,m-2$, then (1.1) reduces to the three-point BVP

$$(1.3)\qquad \begin{cases} u'' + h(t)f(u) = 0, \quad 0 < t < 1, \\ u(0) = 0, \quad u(1) = bu(\xi). \end{cases}$$

In 1998, Ma [8] obtained the following result for (1.3).

Theorem B ([8, Theorem 1]). *Suppose that*

(H1) $0 < b\xi < 1$;
(H2) $f \in C([0,\infty),[0,\infty))$;
(H3) $h \in C([0,1],[0,\infty))$ *and there exists* $t_0 \in [\xi,1]$ *such that* $h(t_0) > 0$.

Then (1.3) *has at least one positive solution if either*

(i) $f_0 = 0$ *and* $f_\infty = \infty$, *or*
(ii) $f_0 = \infty$ *and* $f_\infty = 0$.

Theorem B has been extended by Webb [10]. We remark that in the proof of Theorem B, we rewrite (1.3) as the equivalent integral equation

$$(1.4)\qquad u(t) = -\int_0^t (t-s)h(s)f(u(s))\,ds - \frac{bt}{1-b\xi}\int_0^\xi (\xi-s)h(s)f(u(s))\,ds + \frac{t}{1-b\xi}\int_0^1 (1-s)h(s)f(u(s))\,ds$$

which contains one positive term and two negative terms and is not convenient for studying the existence of positive solutions.

In this paper, we consider the more general m-point BVP (1.1). To deal with (1.1), we give a new integral equation which is equivalent to (1.1) and only contains two positive terms. Our main result (see Theorem 3.1 below) extends and unifies the main results of [2, 3, 7, 8].

By a positive solution of (1.1) we understand a function $u(t)$ which is positive on $(0,1)$ and satisfies the differential equation and the boundary conditions in (1.1).

The main tool of this paper is the following well-known Guo–Krasnosel'skiĭ fixed point theorem.

Theorem C (see [3]). *Let E be a Banach space, and let $K \subset E$ be a cone. Assume Ω_1, Ω_2 are open bounded subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$$

be a completely continuous operator such that either

(i) $\|Au\| \le \|u\|,\ u \in K \cap \partial\Omega_1$, *and* $\|Au\| \ge \|u\|,\ u \in K \cap \partial\Omega_2$; *or*
(ii) $\|Au\| \ge \|u\|,\ u \in K \cap \partial\Omega_1$, *and* $\|Au\| \le \|u\|,\ u \in K \cap \partial\Omega_2$.

Then A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

2. The preliminary lemmas. Set

$$(2.1)\qquad \psi(t) := \beta + \alpha t, \qquad \phi(t) := \gamma + \delta - \gamma t, \qquad t \in [0,1],$$

and

$$\Delta := \begin{vmatrix} -\sum_{i=1}^{m-2} a_i\psi(\xi_i) & \varrho - \sum_{i=1}^{m-2} a_i\phi(\xi_i) \\ \varrho - \sum_{i=1}^{m-2} b_i\psi(\xi_i) & -\sum_{i=1}^{m-2} b_i\phi(\xi_i) \end{vmatrix}.$$

LEMMA 2.1. *Let* (A3) *hold. Assume*

$$\text{(H4)} \qquad \Delta \neq 0.$$

Then for $y \in C[0,1]$, *the problem*

$$(2.2) \qquad \begin{cases} u'' + y(t) = 0, & 0 < t < 1, \\ \alpha u(0) - \beta u'(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \\ \gamma u(1) + \delta u'(1) = \sum_{i=1}^{m-2} b_i u(\xi_i) \end{cases}$$

has a unique solution

$$(2.3) \qquad u(t) = \int_0^1 G(t,s) y(s) ds + A(y)\psi(t) + B(y)\phi(t)$$

where

$$(2.4) \qquad G(t,s) := \frac{1}{\varrho} \begin{cases} \phi(t)\psi(s), & 0 < s < t < 1, \\ \phi(s)\psi(t), & 0 < t < s < 1, \end{cases}$$

$$(2.5) \qquad A(y) := \frac{1}{\Delta} \begin{vmatrix} \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i, s) y(s)\, ds & \varrho - \sum_{i=1}^{m-2} a_i \phi(\xi_i) \\ \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i, s) y(s)\, ds & -\sum_{i=1}^{m-2} b_i \phi(\xi_i) \end{vmatrix}$$

$$(2.6) \qquad B(y) := \frac{1}{\Delta} \begin{vmatrix} -\sum_{i=1}^{m-2} a_i \psi(\xi_i) & \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i, s) y(s)\, ds \\ \varrho - \sum_{i=1}^{m-2} b_i \psi(\xi_i) & \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i, s) y(s)\, ds \end{vmatrix}.$$

Proof. Since ψ and ϕ are two linearly independent solutions of the equation $u'' = 0$, we know that any solution of $u''(t) = y(t)$ can be represented as

$$(2.7) \qquad u(t) = \int_0^1 G(t,s) y(s)\, ds + A\psi(t) + B\phi(t)$$

where G is as in (2.4).

It is easy to check that the function defined by (2.7) is a solution of (2.2) if A and B are defined by (2.5) and (2.6), respectively.

Now we show that the function defined by (2.7) is a solution of (2.2) only if A and B are as in (2.5) and (2.6), respectively.

Let u as in (2.7) be a solution of (2.2). Then

$$u(t) = \int_0^t \frac{1}{\varrho}\,\psi(s)\phi(t)y(s)\,ds + \int_t^1 \frac{1}{\varrho}\,\phi(s)\psi(t)y(s)\,ds + A\psi(t) + B\phi(t),$$

$$u'(t) = \phi'(t)\int_0^t \frac{1}{\varrho}\,\psi(s)y(s)\,ds + \psi'(t)\int_t^1 \frac{1}{\varrho}\,\phi(s)y(s)\,ds + A\psi'(t) + B\phi'(t),$$

$$\begin{aligned} u''(t) = {}& \phi''(t)\int_0^t \frac{1}{\varrho}\,\psi(s)y(s)\,ds + \phi'(t)\frac{1}{\varrho}\,\psi(t)y(t) \\ &+ \psi''(t)\int_t^1 \frac{1}{\varrho}\,\phi(s)y(s)\,ds - \psi'(t)\frac{1}{\varrho}\,\phi(t)y(t) + A\psi''(t) + B\phi''(t), \end{aligned}$$

so that

$$u''(t) = \frac{1}{\varrho}\,[\psi(t)\phi'(t) - \phi(t)\psi'(t)]y(t) = -y(t). \tag{2.8}$$

Since

$$u(0) = \beta\int_0^1 \frac{1}{\varrho}\,\phi(s)y(s)\,ds + A\beta + B(\gamma+\delta),$$

$$u'(0) = \alpha\int_0^1 \frac{1}{\varrho}\,\phi(s)y(s)\,ds + A\alpha + B(-\gamma),$$

we have

$$B(\gamma\alpha + \delta\alpha + \gamma\beta) = \sum_{i=1}^{m-2} a_i\Big[\int_0^1 G(\xi_i, s)y(s)\,ds + A\psi(\xi_i) + B\phi(\xi_i)\Big]. \tag{2.9}$$

Since

$$u(1) = \delta\int_0^1 \frac{1}{\varrho}\,\psi(s)y(s)\,ds + A(\alpha+\beta) + B\delta,$$

$$u'(1) = -\gamma\int_0^1 \frac{1}{\varrho}\,\psi(s)y(s)\,ds + A\alpha + B(-\gamma),$$

we have

$$A(\gamma\alpha + \delta\alpha + \gamma\beta) = \sum_{i=1}^{m-2} b_i\Big[\int_0^1 G(\xi_i, s)y(s)\,ds + A\psi(\xi_i) + B\phi(\xi_i)\Big]. \tag{2.10}$$

From (2.9) and (2.10), we get

$$\begin{cases} \Big[-\sum_{i=1}^{m-2} a_i\psi(\xi_i)\Big]A + \Big[\varrho - \sum_{i=1}^{m-2} a_i\phi(\xi_i)\Big]B = \sum_{i=1}^{m-2} a_i\int_0^1 G(\xi_i, s)y(s)\,ds, \\ \Big[\varrho - \sum_{i=1}^{m-2} b_i\psi(\xi_i)\Big]A - \Big[\sum_{i=1}^{m-2} b_i\phi(\xi_i)\Big]B = \sum_{i=1}^{m-2} b_i\int_0^1 G(\xi_i, s)y(s)\,ds, \end{cases}$$

which implies A and B satisfy (2.5) and (2.6), respectively. This completes the proof of the lemma.

In the following, we will make the following assumption:

$$\text{(H5)} \qquad \Delta < 0, \quad \varrho - \sum_{i=1}^{m-2} a_i\phi(\xi_i) > 0, \quad \varrho - \sum_{i=1}^{m-2} b_i\psi(\xi_i) > 0.$$

It is easy to see that if $\alpha = \gamma = 1$, $\beta = \delta = 0$, $a_i = 0$ for $i = 1, \ldots, m-2$, $b_1 > 0$ and $b_j = 0$ for $j = 2, \ldots, m-2$, then (H5) reduces to

$$0 < b_1\xi_1 < 1,$$

which is a key condition in [8, Theorem 1].

Lemma 2.2. *Let* (A3) *and* (H5) *hold. Then for* $y \in C[0,1]$ *with* $y \geq 0$, *the unique solution* u *of the problem* (2.2) *satisfies*

$$u(t) \geq 0, \qquad t \in [0,1].$$

Proof. This is an immediate consequence of the facts that $G \geq 0$ on $[0,1] \times [0,1]$ and $A(y) \geq 0$, $B(y) \geq 0$.

We note that if (H5) does not hold, then $y \in C[0,1]$ with $y \geq 0$ does not imply that the unique solution u of (2.2) is positive. We can see this from the following result:

Lemma 2.3 ([8, Lemma 3]). *Let* $b\xi > 1$. *If* $y \in C[0,1]$ *and* $y \geq 0$, *then* (1.3) *has no positive solution.*

Lemma 2.4. *Let* (A3) *and* (H5) *hold. Let* $\sigma \in (0, 1/2)$ *be a constant. Then for* $y \in C[0,1]$ *with* $y \geq 0$, *the unique solution* u *of the problem* (2.2) *satisfies*

$$\min\{u(t) \mid t \in [\sigma, 1-\sigma]\} \geq \Gamma\|u\|$$

where $\|u\| = \max\{u(t) \mid t \in [0,1]\}$ *and*

$$\Gamma := \min\{\phi(1-\sigma)/\phi(0), \psi(\sigma)/\psi(1)\}. \tag{2.11}$$

Proof. We see from (2.4) and (2.3) that

$$0 \leq G(t,s) \leq G(s,s), \qquad t \in [0,1],$$

which implies

$$u(t) \leq \int_0^1 G(s,s)y(s)\,ds + A(y)\psi(t) + B(y)\phi(t), \qquad t \in [0,1]. \tag{2.12}$$

Applying (2.4), we find that for $t \in [\sigma, 1-\sigma]$,

$$\frac{G(t,s)}{G(s,s)} = \begin{cases} \phi(t)/\phi(s), & 0 \le s \le t \le 1, \\ \psi(t)/\psi(s), & 0 \le t \le s \le 1, \end{cases} \tag{2.13}$$
$$\ge \begin{cases} \phi(1-\sigma)/\phi(0), & 0 \le s \le t \le 1-\sigma, \\ \psi(\sigma)/\psi(1), & \sigma \le t \le s \le 1, \end{cases}$$
$$\ge \Gamma,$$

where Γ is an in (2.11). Thus for $t \in [\sigma, 1-\sigma]$,

$$\begin{aligned} u(t) &= \int_0^1 \frac{G(t,s)}{G(s,s)} G(s,s)y(s)\,ds + A(y)\psi(t) + B(y)\phi(t) \\ &\ge \Gamma \int_0^1 G(s,s)y(s)\,ds + A(y)\psi(t) + B(y)\phi(t) \\ &\ge \Gamma \Big[\int_0^1 G(s,s)y(s)\,ds + A(y)\psi(t) + B(y)\phi(t)\Big] \ge \Gamma\|u\|. \end{aligned}$$

3. The main result. The main result of the paper is the following

THEOREM 3.1. *Let* (H2), (A3) *and* (H5) *hold. Assume that*

(H6) $h \in C([0,1],[0,\infty))$ *and there exists* $t_0 \in [0,1]$ *such that* $h(t_0) > 0$.

Then (1.1) *has at least one positive solution if either*

(i) $f_0 = 0$ *and* $f_\infty = \infty$, *or*
(ii) $f_0 = \infty$ *and* $f_0 = 0$.

REMARK 3.2. Condition (H6) is weaker than (H3).

REMARK 3.3. Theorem 3.1 extends [3, Theorem 1] and [8, Theorem 1].

Proof of Theorem 3.1. Since $h \in C[0,1]$, we may assume that $t_0 \in (0,1)$ in (H6). Take $\sigma \in (0,1/2) > 0$ such that $t_0 \in (\sigma, 1-\sigma)$ and let Γ be defined by (2.11).

Superlinear case. Suppose then that $f_0 = 0$ and $f_\infty = \infty$. We wish to show the existence of a positive solution of (1.1). Now (1.1) has a solution $u = u(t)$ if and only if u solves the operator equation

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)h(s)f(u(s))\,ds + A(h(\cdot)f(u(\cdot)))\psi(t) \\ &\quad + B(h(\cdot)f(u(\cdot)))\phi(t) \\ &:= (Tu)(t) \end{aligned} \tag{3.1}$$

where ϕ and ψ, G, A and B are defined by (2.1), (2.4), (2.5) and (2.6), respectively. Clearly

$$
\begin{aligned}
(3.2)\quad &|A(h(\cdot)f(u(\cdot)))| \\
&\le \frac{1}{-\Delta}\begin{vmatrix} \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)h(s)\,ds & \varrho - \sum_{i=1}^{m-2} a_i\phi(\xi_i) \\ \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,s)h(s)\,ds & -\sum_{i=1}^{m-2} b_i\phi(\xi_i) \end{vmatrix} \|f(u)\| \\
&:= \widetilde{A}\|f(u)\|
\end{aligned}
$$

and

$$
\begin{aligned}
(3.3)\quad &|B(h(\cdot)f(u(\cdot)))| \\
&\le \frac{1}{-\Delta}\begin{vmatrix} -\sum_{i=1}^{m-2} a_i\psi(\xi_i) & \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)h(s)\,ds \\ \varrho - \sum_{i=1}^{m-2} b_i\psi(\xi_i) & \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,s)h(s)\,ds \end{vmatrix} \|f(u)\| \\
&:= \widetilde{B}\|f(u)\|.
\end{aligned}
$$

Define

$$(3.4)\qquad K = \{u \in C[0,1] \mid u \ge 0,\ \min\{u(t) \mid t \in [\sigma, 1-\sigma]\} \ge \Gamma\|u\|\}.$$

It is obvious that K is a cone in $C[0,1]$. Moreover, by Lemmas 2.2 and 2.4, $TK \subset K$. It is also easy to check that $T : K \to K$ is completely continuous.

Now since $f_0 = 0$, we may choose $H_1 > 0$ so that $f(u) \le \varepsilon u$ for $0 < u < H_1$, where $\varepsilon > 0$ satisfies

$$(3.5)\qquad \varepsilon\Big(\int_0^1 G(s,s)h(s)\,ds + \widetilde{A}\|\psi\| + \widetilde{B}\|\phi\|\Big) \le 1.$$

Thus, if $u \in K$ and $|u|_0 = H_1$, then from (3.1)–(3.5) and the fact that $G(t,s) \le G(s,s)$ and $0 \le \psi(t) \le \psi(1)$, we have

$$
\begin{aligned}
(3.6)\qquad Tu(t) &= \int_0^1 G(t,s)h(s)f(u(s))\,ds \\
&\quad + A(h(\cdot)f(u(\cdot)))\psi(t) + B(h(\cdot)f(u(\cdot)))\phi(t) \\
&\le \Big(\int_0^1 G(s,s)h(s)\,ds + \widetilde{A}\|\psi\| + \widetilde{B}\|\phi\|\Big)\|f(u)\| \\
&\le \varepsilon\Big(\int_0^1 G(s,s)h(s)\,ds + \widetilde{A}\|\psi\| + \widetilde{B}\|\phi\|\Big)\|u\| \le \|u\|.
\end{aligned}
$$

Now if we let

$$\Omega_1 = \{u \in C[0,1] \mid \|u\| < H_1\}, \tag{3.7}$$

then (3.6) shows that $\|Tu\| \le \|u\|$ for $u \in K \cap \partial\Omega_1$.

Further, since $f_\infty = \infty$, there exists $\widehat{H}_2 > 0$ such that $f(u) \ge \varrho_0 u$ for $u \ge \widehat{H}_2$, where $\varrho_0 > 0$ is chosen so that

$$\varrho_0 \gamma \int_{\sigma}^{1-\sigma} G(t_0,s)h(s)\,ds \ge 1. \tag{3.8}$$

Let $H_2 = \max\{2H_1, \widehat{H}_2/\Gamma\}$ and $\Omega_2 = \{u \in C[0,1] \mid \|u\| < H_2\}$. Then $u \in K$ and $\|u\| = H_2$ implies

$$\min_{\sigma \le t \le 1-\sigma} u(t) \ge \Gamma\|u\| \ge \widehat{H}_2,$$

and so

$$\begin{aligned} Tu(t_0) &= \int_0^1 G(t_0,s)h(s)f(u(s))\,ds \\ &\quad + A(h(\cdot)f(u(\cdot)))\psi(t) + B(h(\cdot)f(u(\cdot)))\phi(t) \\ &\ge \int_0^1 G(t_0,s)h(s)f(u(s))\,ds \ge \int_{\sigma}^{1-\sigma} G(t_0,s)h(s)\varrho_0 u(s)\,ds \\ &\ge \varrho_0 \Gamma \int_{\sigma}^{1-\sigma} G(t_0,s)h(s)\,ds\,\|u\|. \end{aligned} \tag{3.9}$$

Hence, $\|Tu\| \ge \|u\|$ for $u \in K \cap \partial\Omega_2$. Therefore, by the first part of Theorem C, T has a fixed point u in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ such that $H_1 \le \|u\| \le H_2$. This completes the superlinear part of the theorem.

Sublinear case. Suppose next that $f_0 = \infty$ and $f_\infty = 0$. We first choose $H_3 > 0$ such that $f(y) \ge My$ for $0 < y < H_3$, where

$$M\Gamma \int_{\sigma}^{1-\sigma} G(t_0,s)h(s)\,ds \ge 1. \tag{3.10}$$

By using the method to get (3.9), we obtain

$$\begin{aligned} Tu(t_0) &\ge \int_0^1 G(t_0,s)h(s)f(u(s))\,ds \\ &\ge \int_{\sigma}^{1-\sigma} G(t_0,s)h(s)Mu(s)\,ds \ge M\Gamma \int_{\sigma}^{1-\sigma} G(t_0,s)h(s)\,ds\,\|u\|. \end{aligned} \tag{3.11}$$

Thus, if we let $\Omega_3 = \{u \in C[0,1] \mid \|u\| < H_3\}$ then

$$\|Tu\| \ge \|u\|, \qquad u \in K \cap \partial\Omega_3.$$

Now, since $f_\infty = 0$, there exists $\widehat{H}_4 > 0$ so that $f(u) \le \lambda u$ for $u \ge \widehat{H}_4$, where $\lambda > 0$ satisfies

$$\lambda\Big(\int_0^1 G(s,s)h(s)\,ds + \widetilde{A}\|\psi\| + \widetilde{B}\|\phi\|\Big) \le 1. \tag{3.12}$$

We consider two cases:

Case 1. Suppose f is bounded, say $f(y) \le N$ for all $y \in [0,\infty)$. In this case choose

$$H_4 = \max\Big\{2H_3, N\Big(\int_0^1 G(s,s)h(s)\,ds + \widetilde{A}\|\psi\| + \widetilde{B}\|\phi\|\Big)\Big\}$$

so that for $u \in K$ with $\|u\| = H_4$ we have

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s)h(s)f(u(s))\,ds + A(h(\cdot)f(u(\cdot)))\psi(t) + B(h(\cdot)f(u(\cdot)))\phi(t) \\ &\le \Big(\int_0^1 G(s,s)h(s)\,ds + \widetilde{A}\|\psi\| + \widetilde{B}\|\phi\|\Big)N \le H_4 \end{aligned}$$

and therefore $\|Tu\| \le \|u\|$.

Case 2. If f is unbounded, then we know from (A1) that there exists $H_4 > \max\{2H_3, \widehat{H}_4/\Gamma\}$ such that

$$f(y) \le f(H_4) \qquad \text{for } 0 < y \le H_4.$$

Then for $u \in K$ and $\|u\| = H_4$, we have

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s)h(s)f(u(s))\,ds + A(h(\cdot)f(u(\cdot)))\psi(t) + B(h(\cdot)f(u(\cdot)))\phi(t) \\ &\le \Big(\int_0^1 G(s,s)h(s)\,ds + \widetilde{A}\|\psi\| + \widetilde{B}\|\psi\|\Big)\|f(u)\| \\ &\le \lambda\Big(\int_0^1 G(s,s)h(s)\,ds + \widetilde{A}\|\psi\| + \widetilde{B}\|\psi\|\Big)\|u\| \le H_4. \end{aligned}$$

Therefore, in either case we may put

$$\Omega_4 = \{u \in C[0,1] \mid \|u\| < H_4\},$$

and for $u \in K \cap \partial\Omega_4$ we have $\|Tu\| \le \|u\|$. By the second part of Theorem C, it follows that BVP (1.1) has a positive solution. Thus, we have completed the proof of Theorem 3.1.

REMARK 3.4. Erbe, Hu and Wang [2] and Lian, Wong and Yeh [7] studied the existence of multiple positive solutions of the two-point boundary value problem

$$\begin{cases} u'' + g(t,u) = 0, \quad 0 < t < 1, \\ \alpha u(0) - \beta u'(0) = 0, \\ \gamma u(1) + \delta u'(1) = 0. \end{cases}$$

It is easy to see from the proof of Theorem 3.1 that we can apply Lemmas 2.2 and 2.4 to establish the corresponding multiplicity results under condition (H5) for the m-point boundary value problem

$$\begin{cases} u'' + g(t,u) = 0, \quad 0 < t < 1, \\ \alpha u(0) - \beta u'(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \\ \gamma u(1) + \delta u'(1) = \sum_{i=1}^{m-2} b_i u(\xi_i), \end{cases}$$

and extend the multiplicity results of [2, 7] without any difficulties.

References

[1] H. Dang and K. Schmitt, *Existence of positive solutions for semilinear elliptic equations in annular domains*, Differential Integral Equations 7 (1994), 747–758.

[2] L. H. Erbe, S. C. Hu and H. Y. Wang, *Multiple positive solutions of some two-point boundary value problems*, J. Math. Anal. Appl. 203 (1994), 640–648.

[3] L. H. Erbe and H. Y. Wang, *On the existence of positive solutions of ordinary differential equations*, Proc. Amer. Math. Soc. 120 (1994), 743–748.

[4] C. P. Gupta, *Solvability of a three-point nonlinear boundary value problem for a second order ordinary differential equation*, J. Math. Anal. Appl. 168 (1992), 540–551.

[5] —, *A generalized multi-point boundary value problem for second order ordinary differential equations*, Appl. Math. Comput. 89 (1998), 133–146.

[6] V. A. Il'in and E. I. Moiseev, *Nonlocal boundary value problem of the first kind for a Sturm–Liouville operator in its differential and finite difference aspects*, Differ. Equ. 23 (1987), 803–810.

[7] W. C. Lian, F. H. Wong and C. C. Yeh, *On the existence of positive solutions of nonlinear second order differential equations*, Proc. Amer. Math. Soc. 124 (1996), 1117–1126.

[8] R. Y. Ma, *Positive solutions of nonlinear three-point boundary-value problems*, Electron. J. Differential Equations 34 (1999), 1–8.

[9] R. Y. Ma and N. Castaneda, *Existence of solutions of nonlinear m-point boundary-value problems*, J. Math. Anal. Appl. 256 (2001), 556–567.

[10] J. R. L. Webb, *Positive solutions of some three-point boundary value problems via fixed point theory*, Nonlinear Anal. 47 (2001), 4319–4332.

Department of Mathematics
Northwest Normal University
Lanzhou 730070, Gansu, People's Republic of China
E-mail: mary@nwnu.edu.cn

Reçu par la Rédaction le 28.2.2002
Révisé le 13.4.2002 (1321)