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Completeness of the inner kth Reiffen pseudometric

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Abstract. We give an example of a Zalcman-type domain in $\mathbb{C}$ which is complete with respect to the integrated form of the $(k+1)$st Reiffen pseudometric, but not complete with respect to the kth one.

0. Introduction. In 1989 M. Klimek introduced for any domain in $\mathbb{C}^n$ an extremal plurisubharmonic function that generalized Green's function of one complex variable. Using that function he defined a biholomorphically invariant pseudodistance and studied its basic properties. For details we refer the readers to [Kl]. Later, S. Kobayashi started to study the similar inner pseudodistance $\int A_D$, which is the integrated form of the Azukawa pseudometric A_D (see [Ko] for details). It turns out that in some cases the Azukawa pseudometric may be approximated by the kth Reiffen pseudometrics $\gamma_D^{(k)}$. In 1995 S. Nivoche (see [Ni]) showed that $\lim_{k\to\infty} \gamma_D^{(k)} = A_D$ outside some pluripolar set for strictly hyperconvex domains D in $\mathbb{C}^n$. It seems that the study of properties of the integrated form $\int A_D$ of the Azukawa pseudodistance is more complicated than dealing with $\int\gamma_D^{(k)}$; one of the reasons is that in the definition of A_D we use a subfamily of plurisubharmonic functions while when defining $\gamma_D^{(k)}$ we may use only a subfamily of holomorphic functions. Therefore, in view of S. Nivoche's result it is convenient to begin with the study of $\int\gamma_D^{(k)}$. For the definitions and basic properties of all pseudometrics and pseudodistances mentioned above, we refer the readers to [Ja-Pf].

The aim of the paper is to show that there exists a bounded domain D in $\mathbb{C}$ which distinguishes completeness of the integrated forms of the kth and $(k+1)$st Reiffen pseudometric, i.e. D is $\int\gamma_D^{(k+1)}$-complete but not $\int\gamma_D^{(k)}$-complete. Our example is a Zalcman-type domain (see definitions below).

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We also show that for any Zalcman-type domain that distinguishes completeness as above, the relevant pseudometric $\gamma_D^{(k)}$ has to satisfy a special growth condition.

1. Definitions and main results. Let E denote the unit disc in $\mathbb{C}$ and let $D \subset \mathbb{C}$ be an arbitrary domain. For $k \in \mathbb{N}$ we define

$$\gamma_D^{(k)}(z;X) = \sup\left\{\left|\frac{1}{k!}\, f^{(k)}(z)X\right|^{1/k} : f \in \mathcal{O}(D,E),\ \operatorname{ord}_z f \geq k\right\},$$

where $\operatorname{ord}_z f$ denotes the order of the zero of f at z. We call $\gamma_D^{(k)}$ the *kth Reiffen pseudometric*.

For a piecewise $\mathcal{C}^1$-curve $\alpha : [0,1] \to D$ (we write $\alpha \in \mathcal{C}^1_{\mathrm{p}}([0,1],D)$) put

$$L_{\gamma_D^{(k)}}(\alpha) := \int_0^1 \gamma_D^{(k)}(\alpha(t);\alpha'(t))\,dt.$$

Define

$$\textstyle\int\gamma_D^{(k)}(z,w) := \inf\{L_{\gamma_D^{(k)}}(\alpha) : \alpha \in \mathcal{C}^1_{\mathrm{p}}([0,1],D),\ \alpha(0)=z,\ \alpha(1)=w\}, \qquad z,w \in D.$$

We call $\int\gamma_D^{(k)}$ the *integrated form* of $\gamma_D^{(k)}$.

Now we recall some properties of pseudodistances we will use in what follows (see [Ja-Pf] for details). Observe that

$$c_D \leq c_D^i = \textstyle\int\gamma_D^{(1)} \leq \int\gamma_D^{(k)}, \quad k \in \mathbb{N}, \tag{1}$$

where c_D (resp. c_D^i) denotes the Carathéodory (resp. inner Carathéodory) pseudodistance on D. We will also use the fact that if $E_* := E \setminus \{0\}$, then

$$c_{E_*} = c_E|_{E_* \times E_*}. \tag{2}$$

It should be pointed out that all pseudodistances we are interested in are contractive, i.e.

$$d_D(w,z) \geq d_G(f(w),f(z)), \qquad w,z \in D,\ f \in \mathcal{O}(D,G),$$

where $d = c$ or $d = \int\gamma^{(k)}$.

It is well known that every domain $D \subset \mathbb{C}$ biholomorphic to a bounded domain is c-hyperbolic (i.e. c_D is a distance). Hence, using (1), we find that $\int\gamma_D^{(k)}$ is also a distance for any bounded domain D.

We use a special notion of completeness. Let us recall the relevant definitions. Let D be a bounded domain in $\mathbb{C}$.

We say that D is $\int\gamma_D^{(k)}$-*complete* if any $\int\gamma_D^{(k)}$-Cauchy sequence $(z_n)_{n\in\mathbb{N}} \subset D$ converges to a point $z_0 \in D$ with respect to the natural topology in D, i.e. $|z_n - z_0| \to 0$ as $n \to \infty$.

We call a domain D $\int\gamma_D^{(k)}$-*finitely compact* if all $\int\gamma_D^{(k)}$-balls in D are relatively compact (with respect to the natural topology in D).

Since $\int\gamma_D^{(k)}$ is an inner distance (cf. [Ja-Pf, Proposition 4.3.2]) for any $k \in \mathbb{N}$, it turns out that these different notions are equivalent (see [Ja-Pf, Theorem 7.3.2]).

Before we present our main result we need the following definition. Let $B(a,r) := \{z \in \mathbb{C} : |z-a| < r\}$. For all sequences $(a_n), (r_n) \subset \mathbb{R}_{>0}$ such that $a_n \to 0,\ 2r_n < a_n,\ \overline{B}(a_n, r_n) \subset E_*$ for $n \in \mathbb{N}$ and $\overline{B}(a_n, r_n) \cap \overline{B}(a_m, r_m) = \emptyset$ whenever $n \neq m$, we define

$$D := E_* \setminus \bigcup_{n=1}^{\infty} \overline{B}(a_n, r_n). \tag{3}$$

We call such a domain a *Zalcman-type domain.*

Our main result is the following

THEOREM 1. *For any $k \in \mathbb{N}$ there exists a Zalcman-type domain D_k which is $\int\gamma_{D_k}^{(k+1)}$-complete, but not $\int\gamma_{D_k}^{(k)}$-complete.*

As we will see in the proof of Theorem 1, $\gamma_D^{(k)}$ satisfies some special growth conditions. The following result shows how carefully we must choose the appropriate domain.

THEOREM 2. *Let $D \subset \mathbb{C}$ be a bounded domain, $d_D(z) := \inf_{w \in \partial D} |z-w|$, $k, l \in \mathbb{N}$, and let $0 \leq \alpha < 1$ be such that $\gamma_D^{(k)}(z;1) \leq c(d_D(z))^{-\alpha},\ z \in D$, for some positive constant c. Then*

$$\gamma_D^{(k+l)}(z;1) \leq c'(d_D(z))^{-\alpha'}, \quad z \in D,$$

for some positive constants c' and $\alpha' < 1$.

Since such growth gives us noncompleteness of the Zalcman-type domain D, we get

COROLLARY 3. *If for a Zalcman-type domain D there exist positive constants c and $\alpha < 1$ such that*

$$\limsup_{0 > z \to 0} \gamma_D^{(k)}(z;1) < c|z|^{-\alpha},$$

then D is $\int\gamma_D^{(l)}$-noncomplete for any $l \geq k$.

Therefore, to get Theorem 1 it is natural that we are interested in the domains for which $\int\gamma_D^{(k)}$ has growth tempered as follows:

$$\gamma_D^{(k)}(z;1) \leq \frac{c}{d_D(z)(-\log d_D(z))^{\alpha}}, \quad \alpha > 1,$$

for $z \in D$ such that $d_D(z) < 1$.

2. Proof of Theorem 1. In the proof of Theorem 1 we will use the following lemmas. We present their proofs at the end of this section.

LEMMA 4. *If D is a Zalcman-type domain, then for every $k \in \mathbb{N}$ there exists a positive constant $c_1 = c_1(k)$ such that for every $f \in \mathcal{O}(D,E)$ we have*

$$|f^{(k)}(z)| \le c_1 + c_1 \sum_{n=1}^{\infty} \frac{r_n}{(a_n - z)^{k+1}}, \quad z \in [-1/2, 0).$$

LEMMA 5. *For every $k \in \mathbb{N}$ there exists a Zalcman-type domain D_k such that*

(a) $$\limsup_{0>z\to 0} \smallint\gamma_{D_k}^{(k)}(-1/2, z) < \infty,$$

(b) $$\lim_{z\to 0} \smallint\gamma_{D_k}^{(k+1)}(w,z) = \infty, \quad w \in D_k.$$

Proof of Theorem 1. We fix $k \in \mathbb{N}$ and take D_k as in Lemma 5. The $\smallint\gamma_{D_k}^{(k)}$-noncompleteness of D_k is a direct consequence of (a).

Now we will prove that D_k is $\smallint\gamma_{D_k}^{(k+1)}$-complete. To do this we show that

$$\lim_{z\to z_0} \smallint\gamma_{D_k}^{(k+1)}(w,z) = \infty, \quad z_0 \in \partial D_k,\ w \in D_k.$$

We fix $w \in D_k$. There are three possibilities.

1° If $z_0 = 0$ then, using (b), we are done.

2° If $|z_0| = 1$ then, using (1) and the contractivity of $\smallint\gamma^{(k+1)}$, we get

$$\lim_{z\to z_0} \smallint\gamma_{D_k}^{(k+1)}(w,z) \ge \lim_{z\to z_0} \smallint\gamma_{E}^{(k+1)}(w,z) = \lim_{z\to z_0} c_E(w,z) = \infty.$$

3° If $z_0 \in \partial B(a_n, r_n)$ then, using (1), contractivity of the Carathéodory pseudodistance and (2), we get

$$\begin{aligned}\lim_{z\to z_0} \smallint\gamma_{D_k}^{(k+1)}(w,z) &\ge \lim_{z\to z_0} c_{D_k}(w,z) \ge \lim_{z\to z_0} c_{\mathbb{C}\setminus \overline{B}(a_n,r_n)}(w,z)\\ &= \lim_{z\to z_0} c_{E_*}\left(\frac{r_n}{w-a_n}, \frac{r_n}{z-a_n}\right)\\ &= \lim_{z\to z_0} c_{E}\left(\frac{r_n}{w-a_n}, \frac{r_n}{z-a_n}\right) = \infty,\end{aligned}$$

since $|r_n/(z-a_n)| \to 1$ as $z \to z_0$. ∎

We are left with the proofs of Lemmas 4 and 5.

Proof of Lemma 4. Let D be as in (3). We define

$$D_{(s)} := E \setminus \Big(\overline{B}(a_{(s)}, r_{(s)}) \cup \bigcup_{n=1}^{s} \overline{B}(a_n, r_n)\Big), \quad s \in \mathbb{N},$$

where the numbers $a_{(s)}, r_{(s)} > 0$ are chosen so that $0 \in B(a_{(s)}, r_{(s)})$, $\overline{B}(a_n, r_n) \subset B(a_{(s)}, r_{(s)})$ for $n > s$ and $\overline{B}(a_{(s)}, r_{(s)}) \cap \overline{B}(a_s, r_s) = \emptyset$. Obviously, $D_{(s)}$ is an $(s+2)$-connected domain and $D_{(s)} \subset D$. Observe that for $\varepsilon > 0$ small enough

$$D_{(s)}^{(\varepsilon)} := (1-\varepsilon)E \setminus \Big(\overline{B}(a_{(s)}, r_{(s)} + \varepsilon) \cup \bigcup_{n=1}^{s} \overline{B}(a_n, r_n + \varepsilon)\Big) \subset\subset D_{(s)}$$

is also an $(s+2)$-connected domain. Then the Cauchy integral formula for $D_{(s)}^{(\varepsilon)}$ gives us

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int\limits_{|\zeta|=1-\varepsilon} \frac{f(\zeta)}{(\zeta - z)^{k+1}}\, d\zeta - \frac{k!}{2\pi i} \int\limits_{|\zeta - a_{(s)}| = r_{(s)}+\varepsilon} \frac{f(\zeta)}{(\zeta - z)^{k+1}}\, d\zeta$$
$$- \sum_{n=1}^{s} \frac{k!}{2\pi i} \int\limits_{|\zeta - a_n| = r_n + \varepsilon} \frac{f(\zeta)}{(\zeta - z)^{k+1}}\, d\zeta, \quad z \in D_{(s)}^{(\varepsilon)},$$

for every $f \in \mathcal{O}(D, E)$. Then for $z \in [-1/2, a_{(s)} - r_{(s)} - \varepsilon - \sqrt[2(k+1)]{r_{(s)} + \varepsilon})$ we have

$$\begin{aligned}
|f^{(k)}(z)| \le{}& \frac{k!}{2\pi} \int_0^{2\pi} \frac{1-\varepsilon}{|(1-\varepsilon)e^{it} - z|^{k+1}}\, dt \\
&+ \frac{k!}{2\pi} \int_0^{2\pi} \frac{r_{(s)} + \varepsilon}{|(r_{(s)} + \varepsilon)e^{it} + a_{(s)} - z|^{k+1}}\, dt \\
&+ \sum_{n=1}^{s} \frac{k!}{2\pi} \int_0^{2\pi} \frac{r_n + \varepsilon}{|(r_n + \varepsilon)e^{it} + a_n - z|^{k+1}}\, dt \\
\le{}& k!\, \frac{1-\varepsilon}{(1/2 - \varepsilon)^{k+1}} + k!\, \frac{r_{(s)} + \varepsilon}{(\sqrt[2(k+1)]{r_{(s)} + \varepsilon})^{k+1}} \\
&+ k! \sum_{n=1}^{s} \frac{r_n + \varepsilon}{\big(\frac{1}{2}(a_n - z - \varepsilon)\big)^{k+1}},
\end{aligned}$$

since $a_n - z - \varepsilon > a_n > 2r_n$. Letting $\varepsilon \to 0$ we get

$$|f^{(k)}(z)| \le k! 2^{k+1} + k! \sqrt{r_{(s)}} + k! 2^{k+1} \sum_{n=1}^{s} \frac{r_n}{(a_n - z)^{k+1}}$$

for $z \in [-1/2, a_{(s)} - r_{(s)} - \sqrt[2(k+1)]{r_{(s)}})$.

Now, since s is an arbitrary natural number, we may let $s \to \infty$ to get

$$|f^{(k)}(z)| \le k! 2^{k+1} + k! 2^{k+1} \sum_{n=1}^{\infty} \frac{r_n}{(a_n - z)^{k+1}}, \quad z \in [-1/2, 0),$$

since $a_{(s)}$ and $r_{(s)}$ tend to 0 as $s \to \infty$. ∎

Proof of Lemma 5. We fix $k \in \mathbb{N}$ and define

$$D_k := E_* \setminus \bigcup_{n=4}^{\infty} \overline{B}(a_n, r_{k,n}), \tag{4}$$

where $a_n := 2^{-n}$, $r_{k,n} := 2^{-n}n^{-k-1}$. It is easy to check that (4) is a Zalcman-type domain, since $a_{n+1} + r_{k,n+1} < a_n - r_{k,n}$ and $2r_{k,n} < a_n$ for all $k \in \mathbb{N}$, $n \geq 4$.

(a) It is sufficient to show that

$$\gamma_{D_k}^{(k)}(z;1) \leq \frac{c_2}{-z(-\log(-z))^{(k+1)/k}}, \quad z \in [-1/2, 0), \tag{5}$$

for some absolute constant $c_2 = c_2(k) > 0$. Indeed, using (5), it is easy to get

$$\limsup_{0>z\to 0} \int\gamma_{D_k}^{(k)}(-1/2, z) \leq \int_{-1/2}^{0} \gamma_{D_k}^{(k)}(z;1)\,dz < \infty,$$

and the proof of (a) is finished.

We will get (5) if we prove that

$$|f^{(k)}(z)| \leq \frac{c_3}{(-z)^k(-\log(-z))^{k+1}}, \qquad z \in [-1/2, 0),\ f \in \mathcal{O}(D_k, E), \tag{6}$$

where $c_3 = c_3(k)$ is a positive constant.

Now we will prove (6). Let $z \in [-1/2, 0)$. Then there exist unique $m \in \mathbb{N}$ and $b \in (1, 2]$ such that $z = -b/2^m$. Observe that

$$\sum_{n=4}^{\infty} \frac{r_{k,n}}{(a_n - z)^{k+1}} \leq \sum_{n=4}^{m} \frac{r_{k,n}}{a_n^{k+1}} + \sum_{n=m}^{\infty} \frac{r_{k,n}}{(-z)^{k+1}}.$$

Now we estimate both series. For the first one we have

$$\sum_{n=4}^{m} \frac{r_{k,n}}{a_n^{k+1}} = \sum_{n=4}^{m} \frac{2^{nk}}{n^{k+1}} \leq \frac{2^{mk}}{m^{k+1}} \sum_{n=0}^{\infty} \left(\frac{8}{9}\right)^n = 9\,\frac{2^{mk}}{m^{k+1}}, \tag{7}$$

while the second series is estimated as follows:

$$\sum_{n=m}^{\infty} \frac{r_{k,n}}{(-z)^{k+1}} = \sum_{n=m}^{\infty} \frac{2^{m(k+1)}}{2^n n^{k+1} b^{k+1}} \leq \frac{2^{m(k+1)}}{2^m m^{k+1}} \sum_{n=0}^{\infty} \frac{1}{2^n} \leq 2\,\frac{2^{mk}}{m^{k+1}}. \tag{8}$$

Using the estimates (7) and (8) we get

$$\sum_{n=4}^{\infty} \frac{r_{k,n}}{(a_n - z)^{k+1}} \leq \frac{11 \cdot 2^k (\log 2)^{k+1} 2^{mk}}{b^k (\log(2^m/b))^{k+1}} = \frac{c_4}{(-z)^k(-\log(-z))^{k+1}},$$

where $c_4 = c_4(k) := 11 \cdot 2^k (\log 2)^{k+1}$. Using Lemma 4, we obtain

$$|f^{(k)}(z)| \le c_1\left(1+\frac{c_4}{(-z)^k(-\log(-z))^{k+1}}\right) \le \frac{2c_1c_4}{(-z)^k(-\log(-z))^{k+1}},$$

which gives us (6) with the constant $c_3 := 2c_1c_4$.

(b) We show that

$$\gamma_{D_k}^{(k+1)}(z;1) \ge \frac{c_5}{|z|\log(1/|z|)}, \qquad |z| < 1/4, \tag{9}$$

for some absolute constant $c_5 = c_5(k) > 0$.

Assume for a while that (9) holds. We fix $w \in D_k$. Then for any $|z| < 1/4$ and for any curve $\alpha \in \mathcal{C}_p^1([0,1], D_k)$ such that $\alpha(0) = z,\ \alpha(1) = w$ we have

$$\int_0^1 \gamma_{D_k}^{(k+1)}(\alpha(t);\alpha'(t))\,dt \ge \int_0^{t_\alpha} \frac{c_5|\alpha'(t)|\,dt}{|\alpha(t)|\log(1/|\alpha(t)|)}, \tag{10}$$

where $t_\alpha := \min\{t \in [0,1] : |\alpha(t)| = 1/4\}$; if $|\alpha(t)| < 1/4$ for $t \in [0,1]$, then $t_\alpha := 1$. Observe that, since

$$\frac{\partial}{\partial t}|\alpha(t)| = \frac{\alpha'(t)\overline{\alpha(t)} + \overline{\alpha'(t)}\alpha(t)}{2|\alpha(t)|} = \frac{\operatorname{Re}(\alpha'(t)\overline{\alpha(t)})}{|\alpha(t)|} \le |\alpha'(t)|,$$

the following estimate holds (if $t_\alpha = 1$ then, in what follows, instead of $1/4$ we write $|w|$):

$$\int_0^{t_\alpha} \frac{|\alpha'(t)|\,dt}{|\alpha(t)|\log(1/|\alpha(t)|)} \ge \int_{|z|}^{1/4} \frac{dx}{x\log(1/x)} = \log\log\frac{1}{|z|} - \log\log 4.$$

Taking the infimum over all such curves α, using (10) and the estimate above, we obtain

$$\int\gamma_{D_k}^{(k+1)}(w,z) \ge c_5\left(\log\log\frac{1}{|z|} - \log\log 4\right).$$

Hence, if we let $z \to 0$, we obtain (b).

It remains to prove (9). According to the definition of $\gamma_{D_k}^{(k+1)}$ we will get (9) if we prove that for every $|z| < 1/4$ there exists a function $f_z \in \mathcal{O}(D_k, E)$ such that $f_z(z) = f_z'(z) = \ldots = f_z^{(k)}(z) = 0$ and

$$|f_z^{(k+1)}(z)| \ge \frac{c_6}{(|z|\log(1/|z|))^{k+1}}, \tag{11}$$

with some absolute constant $c_6 = c_6(k) > 0$.

Now we will construct such a function. Let $|z| < 1/4$. Then there exist unique $m \in \mathbb{N},\ b \in (1,2]$ and $\theta \in [0, 2\pi)$ such that $z = be^{i\theta}/2^m$. Observe that $m \ge 3$. We define

$$\widetilde{f}_z(\lambda) := \sum_{j=0}^{k} \alpha_{b,\theta}^j (2^{-m-j-1} - \lambda)^{-1} + 2^{m+1}\beta_{b,\theta}, \quad \lambda \in D_k,$$

where $\alpha^0_{b,\theta} := 1$ and $\alpha^1_{b,\theta},\ldots,\alpha^k_{b,\theta},\beta_{b,\theta} \in \mathbb{C}$ are constants, depending only on b and θ (and not on m), taken to satisfy the condition $\widetilde{f}_z(z) = \ldots = \widetilde{f}_z^{(k)}(z) = 0$. Clearly, $\widetilde{f}_z$ is a holomorphic function in D_k.

In Lemma 6 below we show that the definition of $\widetilde{f}_z$ is correct (i.e. that the numbers $\alpha^1_{b,\theta},\ldots,\alpha^k_{b,\theta},\beta_{b,\theta}$ exist), but first observe that

$$[(2^{-m-j-1}-\lambda)^{-1}]^{(l)} = l!(2^{-m-j-1}-\lambda)^{-l-1}. \tag{12}$$

LEMMA 6. *For every $z = be^{i\theta}/2^m$, where $b \in [1,2]$, $\theta \in [0,2\pi]$ and $m \ge 3$, the numbers $\alpha^1_{b,\theta},\ldots,\alpha^k_{b,\theta},\beta_{b,\theta}$ as above exist and their moduli can be estimated from above by a positive constant α which is independent of b and θ. Moreover,*

$$B_{k,b,\theta} := \sum_{j=0}^{k}\left(\frac{2^j}{1-2^{j+1}be^{i\theta}}\right)^{k+2}\alpha^j_{b,\theta} \neq 0.$$

In particular, $B_k := \min\{|B_{k,b,\theta}| : b \in [1,2],\ \theta \in [0,2\pi]\} > 0$.

Later we will give the proof of Lemma 6. Using (12) we obtain

$$\begin{aligned}
|\widetilde{f}_z^{(k+1)}(z)| &= \left|\sum_{j=0}^{k}\alpha^j_{b,\theta}(k+1)!\left(\frac{1}{2^{m+j+1}}-\frac{be^{i\theta}}{2^m}\right)^{-(k+2)}\right| \\
&\ge \left|\sum_{j=0}^{k}\alpha^j_{b,\theta}\left(\frac{2^{m+j+1}}{1-2^{j+1}be^{i\theta}}\right)^{k+2}\right| \\
&= 2^{(m+1)(k+2)}|B_{k,b,\theta}| \ge c_7 2^{m(k+2)},
\end{aligned}$$

where c_7 is a positive constant depending only on k.

Now, with the help of Lemma 6, we estimate the supremum of the function $\widetilde{f}_z$ on D_k:

$$\begin{aligned}
\|\widetilde{f}_z\|_{D_k} &\le \sum_{j=0}^{k}\frac{|\alpha^j_{b,\theta}|}{r_{k,m+j+1}} + 2^{m+1}|\beta_{b,\theta}| \\
&\le \frac{\alpha(k+2)}{r_{k,m+k+1}} = \alpha(k+2)2^{m+k+1}(m+k+1)^{k+1} \\
&\le c_8 2^m(m-1)^{k+1},
\end{aligned}$$

where $c_8 > 0$ depends only on k.

Observe that for $f_z := \widetilde{f}_z/\|\widetilde{f}_z\|_{D_k} \in \mathcal{O}(D_k,E)$ we obtain (11), because

$$|f_z^{(k+1)}(z)| \ge \frac{c_7 2^{m(k+1)}}{c_8(m-1)^{k+1}} \ge \frac{c_7(\log 2)^{k+1}2^{m(k+1)}}{c_8 b^{k+1}(\log(2^m/b))^{k+1}} = \frac{c_6}{(|z|\log(1/|z|))^{k+1}}.$$

Thus the proof of Lemma 5 is complete. ∎

We are left with the proof of Lemma 6.

Proof of Lemma 6. First we will construct $\alpha^1_{b,\theta},\dots,\alpha^k_{b,\theta}$. We show that the system of k equations

$$\widetilde{f}'_z(z) = \dots = \widetilde{f}^{(k)}_z(z) = 0 \tag{13}$$

in k unknowns $\alpha^1_{b,\theta},\dots,\alpha^k_{b,\theta}$ always has a solution, i.e. its determinant is not zero.

Observe that, by (12), the system (13) is equivalent to

$$\sum_{j=0}^{k} l!\left(\frac{2^{m+j+1}}{1-2^{j+1}be^{i\theta}}\right)^{l+1}\alpha^j_{b,\theta} = 0, \quad l=1,\dots,k, \tag{14}$$

which is equivalent to

$$\sum_{j=1}^{k}\left(\frac{2^{j}}{1-2^{j+1}be^{i\theta}}\right)^{l+1}\alpha^j_{b,\theta} = -\left(\frac{1}{1-2be^{i\theta}}\right)^{l+1}, \quad l=1,\dots,k. \tag{15}$$

To simplify notation put

$$A^j_{b,\theta} := \frac{2^j}{1-2^{j+1}be^{i\theta}}, \quad j=0,\dots,k,$$

and observe that all $|A^j_{b,\theta}|\in[1/8,1]$ and that $A^\mu_{b,\theta}\neq A^\nu_{b,\theta}$ whenever $\mu\neq\nu$. Now (15) is equivalent to

$$\sum_{j=1}^{k}(A^j_{b,\theta})^{l+1}\alpha^j_{b,\theta} = -(A^0_{b,\theta})^{l+1}, \quad l=1,\dots,k, \tag{16}$$

and it is easy to see that

$$|\det[(A^j_{b,\theta})^{l+1}]^k_{j,l=1}| = |A^1_{b,\theta}\dots A^k_{b,\theta}|^2\prod_{k\ge\mu>\nu\ge1}|A^\mu_{b,\theta}-A^\nu_{b,\theta}|\ge\varepsilon>0, \tag{17}$$

where ε is a constant independent of b and θ. Hence the choice of the numbers $\alpha^1_{b,\theta},\dots,\alpha^k_{b,\theta}$ is always possible; now we may take

$$\beta_{b,\theta} := -\sum_{j=0}^{k}A^j_{b,\theta}\alpha^j_{b,\theta}.$$

Now we prove the existence of the constant α. Since

$$|\beta_{b,\theta}| \le \sum_{j=0}^{k}|A^j_{b,\theta}\alpha^j_{b,\theta}| \le (k+1)\max\{|\alpha^j_{b,\theta}| : 0\le j\le k\},$$

it is enough to deal with the numbers $\alpha^j_{b,\theta}$.

Observe that $|A^j_{b,\theta}|^{l+1} \in [2^{-3(k+1)}, 1]$ for $j = 0, \ldots, k$, $l = 1, \ldots, k$, $b \in [1,2]$ and $\theta \in [0, 2\pi]$. Since det is a continuous function, it is bounded on compact sets. This observation and (17) give us global upper bounds $\alpha^1, \ldots, \alpha^k$ of $|\alpha^1_{b,\theta}|, \ldots, |\alpha^k_{b,\theta}|$, independent of b and θ. Hence one may take

$$\alpha := (k+1)\max\{|\alpha^j| : 0 \le j \le k\}.$$

It remains to prove that $B_{k,b,\theta} \neq 0$. Observe that this is equivalent to

$$\sum_{j=1}^{k} (A^j_{b,\theta})^{k+2} \alpha^j_{b,\theta} \neq -(A^0_{b,\theta})^{k+2}. \tag{18}$$

Suppose (18) does not hold. Then we obtain the system of $k+1$ equations

$$\sum_{j=1}^{k} (A^j_{b,\theta})^{l+1} \alpha^j_{b,\theta} = -(A^0_{b,\theta})^{l+1}, \quad l = 1, \ldots, k+1, \tag{19}$$

which has the only solution $\alpha^j_{b,\theta}$, $j = 1, \ldots, k$. Now, if we remove from (19) the first equation, we obtain the system of k equations

$$\sum_{j=1}^{k} (A^j_{b,\theta})^{l+1} \alpha^j_{b,\theta} = -(A^0_{b,\theta})^{l+1}, \quad l = 2, \ldots, k+1, \tag{20}$$

which also has the same unique solution $\alpha^j_{b,\theta}$, $j = 1, \ldots, k$. But if we compare the solutions of (16) and (20) we get

$$A^0_{b,\theta}/A^j_{b,\theta} = 1, \quad j = 1, \ldots, k,$$

which is impossible and, consequently, (18) holds.

Now, since $|B_{k,b,\theta}|$ is, with respect to variables (b,θ), a positive and continuous function on the compact set $[1,2] \times [0,2\pi]$, we conclude that $B_k > 0$ and the proof of Lemma 6 is finished. ∎

3. Proof of Theorem 2. In the proof of Theorem 2 we will use the following lemma.

LEMMA 7. *If $D \subset \mathbb{C}$ is a bounded domain, then*

$$\frac{\gamma^{(k)}_D(z_0;1)}{(d_D(z_0))^{l/k}} \ge (\gamma^{(k+l)}_D(z_0;1))^{(k+l)/k}, \quad z_0 \in D,\ k,l \in \mathbb{N}.$$

Proof of Lemma 7. Fix $k, l \in \mathbb{N}$ and $z_0 \in D$. Then for $f \in \mathcal{O}(D,E)$ such that $f(z_0) = f'(z_0) = \ldots = f^{(k+l-1)}(z_0) = 0$ we define

$$g(z) := \begin{cases} f(z)/(z-z_0)^l, & z \neq z_0, \\ 0, & z = z_0. \end{cases}$$

Observe that g is a holomorphic function on D. Moreover, using the Taylor expansion of f at z_0 we obtain

$$g(z)=\sum_{j=k+l}^{\infty}\frac{f^{(j)}(z_0)}{j!}(z-z_0)^{j-l},\quad z\in D.$$

Then

$$g^{(m)}(z_0)=0,\quad m=0,1,\ldots,k-1, \tag{21}$$

and

$$g^{(k)}(z_0)=\frac{k!}{(k+l)!}f^{(k+l)}(z_0). \tag{22}$$

From the maximum principle we get $\|g\|_D\le 1/(d_D(z_0))^l$. Therefore, by (21), $h:=(d_D(z_0))^l g\in\mathcal{O}(D,E)$ satisfies the conditions in the definition of the kth Reiffen pseudometric. Hence, using (22), we obtain

$$\begin{aligned}\gamma_D^{(k)}(z_0;1)&\ge\sup_h\left(\frac{1}{k!}|h^{(k)}(z_0)|\right)^{1/k}=\left(\frac{(d_D(z_0))^l}{(k+l)!}\sup_f|f^{(k+l)}(z_0)|\right)^{1/k}\\&=((d_D(z_0))^l(\gamma_D^{(k+l)}(z_0;1))^{k+l})^{1/k}\\&=(d_D(z_0))^{l/k}(\gamma_D^{(k+l)}(z_0;1))^{(k+l)/k},\end{aligned}$$

and the proof of Lemma 7 is complete. ∎

Proof of Theorem 2. From Lemma 7 we get

$$\gamma_D^{(k+l)}(z;1)\le\frac{(\gamma_D^{(k)}(z;1))^{k/(k+l)}}{(d_D(z))^{l/(k+l)}},\quad z\in D,\ k,l\in\mathbb{N}.$$

Now, if $\gamma_D^{(k)}(z;1)\le c(d_D(z))^{-\alpha}$ then

$$\gamma_D^{(k+l)}(z;1)\le\frac{c^{k/(k+l)}}{(d_D(z))^{(\alpha k+l)/(k+l)}}=\frac{c'}{(d_D(z))^{\alpha'}},\quad z\in D,\ k,l\in\mathbb{N},$$

where $c':=c^{k/(k+l)}$ and $\alpha':=(\alpha k+l)/(k+l)<1$. ∎

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