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Asymptotic properties of third order functional dynamic equations on time scales

by I. Kubiaczyk (Poznań) and S. H. Saker (Riyadh and Mansoura)

Abstract. The purpose of this paper is to study the asymptotic properties of nonoscillatory solutions of the third order nonlinear functional dynamic equation

$$[p(t)[(r(t)x^{\Delta}(t))^{\Delta}]^{\gamma}]^{\Delta} + q(t)f(x(\tau(t))) = 0, \quad t \geq t_0,$$

on a time scale $\mathbb{T}$, where $\gamma > 0$ is a quotient of odd positive integers, and p, q, r and τ are positive right-dense continuous functions defined on $\mathbb{T}$. We classify the nonoscillatory solutions into certain classes C_i, $i = 0, 1, 2, 3$, according to the sign of the Δ-quasi-derivatives and obtain sufficient conditions in order that $C_i = \emptyset$. Also, we establish some sufficient conditions which ensure the property A of the solutions. Our results are new for third order dynamic equations and involve and improve some results previously obtained for differential and difference equations. Some examples are worked out to demonstrate the main results.

1. Introduction. The study of dynamic equations on time scales, which goes back to its founder Stefan Hilger [22], is an area of mathematics that has recently received a lot of attention. It has been created in order to unify the study of differential and difference equations. Many results concerning differential equations carry over quite easily to corresponding results for difference equations, while other results seem to be completely different from their continuous counterparts. The study of dynamic equations on time scales reveals such discrepancies, and helps avoid proving results twice, once for differential equations and once again for difference equations. The general idea is to prove a result for a dynamic equation where the domain of the unknown function is a so-called time scale $\mathbb{T}$, which may be an arbitrary closed subset of the reals. This way results not only related to the set of real numbers or set of integers but also pertaining to more general time scales are obtained.

The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus, i.e., when $\mathbb{T} = \mathbb{R}$,

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$\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$ where $q > 1$. Dynamic equations on time scales have an enormous potential for applications, e.g. in population dynamics, where they may be used to model the growth of insect populations that grow continuously while in season, die out in winter, while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population (see [6]). The two books [6] and [7] on the subject of time scales, i.e., measure chains, by Bohner and Peterson summarize and organize much of time scales calculus.

For completeness, we recall the following concepts related to the notion of time scales. A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on $\mathbb{R}$. The *forward jump operator* and the *backward jump operator* are defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \qquad \rho(t) := \sup\{s \in \mathbb{T} : s < t\},$$

where $\sup \emptyset = \inf \mathbb{T}$. A point $t \in \mathbb{T}$ is said to be *left-dense* if $\rho(t) = t$ and $t > \inf \mathbb{T}$, *right-dense* if $\sigma(t) = t$, *left-scattered* if $\rho(t) < t$, and *right-scattered* if $\sigma(t) > t$. A function $g : \mathbb{T} \to \mathbb{R}$ is said to be *right-dense continuous* (rd-continuous) provided g is continuous at right-dense points, while at left-dense points, left hand limits exist and are finite. The set of all rd-continuous functions is denoted by $C_{\mathrm{rd}}(\mathbb{T})$. The *graininess function* μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^\sigma(t)$ stands for $f(\sigma(t))$.

Definition 1.1. Fix $t \in \mathbb{T}$ and let $x : \mathbb{T} \to \mathbb{R}$. Define $x^\Delta(t)$ to be the number (if it exists) with the property that given any $\epsilon > 0$ there is a neighborhood U of t with

$$|[x(\sigma(t)) - x(s)] - x^\Delta(t)[\sigma(t) - s]| \le \epsilon |\sigma(t) - s| \quad \text{for all } s \in U.$$

In this case, we say $x^\Delta(t)$ is the (*delta*) *derivative* of x at t and that x is (*delta*) *differentiable* at t.

We will frequently use the results in the following theorem which is due to Hilger [22].

Theorem 1.2. *Assume that $g : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}$.*

(i) *If g is differentiable at t, then g is continuous at t.*
(ii) *If g is continuous at t and t is right-scattered, then g is differentiable at t with*

$$g^\Delta(t) = \frac{g(\sigma(t)) - g(t)}{\mu(t)}.$$

(iii) *If g is differentiable and t is right-dense, then*

$$g^\Delta(t) = \lim_{s \to t} \frac{g(t) - g(s)}{t - s}.$$

(iv) *If g is differentiable at t, then $g(\sigma(t)) = g(t) + \mu(t)g^{\Delta}(t)$.*

In this paper, we will refer to the (delta) integral which we can define as follows:

DEFINITION 1.3. If $G^{\Delta}(t) = g(t)$, then the *Cauchy* (*delta*) *integral* of g is defined by

$$\int_a^t g(s)\Delta s := G(t) - G(a).$$

It can be shown (see [6]) that if $g \in C_{\mathrm{rd}}(\mathbb{T})$, then the Cauchy integral $G(t) := \int_{t_0}^t g(s)\Delta s$ exists for $t_0 \in \mathbb{T}$, and satisfies $G^{\Delta}(t) = g(t)$ for all $t \in \mathbb{T}$. For a more general definition of the delta integral see [6], [7].

In the last few years, there has been increasing interest in obtaining sufficient conditions for the qualitative properties of solutions of different classes of dynamic equations on time scales (see [1–3, 8, 9, 29–35] and the references cited therein). To the best of our knowledge, there are few papers which have dealt with various types of third order dynamic equations on time scales (see [14, 16–18, 26, 37]). In this paper, we are concerned with the asymptotic properties of nonoscillatory solutions and with the property A (defined below) of the third order nonlinear functional dynamic equation

$$(p(t)[(r(t)(x(t))^{\Delta})^{\Delta}]^{\gamma})^{\Delta} + q(t)f(x(\tau(t))) = 0, \tag{1.1}$$

on an arbitrary time scale $\mathbb{T}$, where p, q and r are positive real-valued rd-continuous functions defined on $\mathbb{T}$, $\tau : \mathbb{T} \to \mathbb{T}$, $\lim_{t\to\infty} \tau(t) = \infty$ and $f \in C(\mathbb{R}, \mathbb{R})$, $uf(u) > 0$ for $u \neq 0$. Throughout this paper, we assume that

$$\int_{t_0}^{\infty} \frac{1}{p^{1/\gamma}(t)}\Delta t = \infty, \quad \int_{t_0}^{\infty} \frac{1}{r(t)}\Delta t = \infty, \tag{1.2}$$

or

$$\int_{t_0}^{\infty} \frac{1}{p^{1/\gamma}(t)}\Delta t < \infty, \quad \int_{t_0}^{\infty} \frac{1}{r(t)}\Delta t < \infty. \tag{1.3}$$

Equation (1.1) is called a *delay dynamic equation* if $\tau(t) < t$, and an *advanced dynamic equation* if $\tau(t) > t$. For (1.1), we define the Δ-*quasi-derivatives* of a solution $x(t)$ by

$$x^{[0]} = x, \quad x^{[1]} = r(x^{[0]})^{\Delta}, \quad x^{[2]} = p((x^{[1]})^{\Delta})^{\gamma}, \quad x^{[3]} = (x^{[2]})^{\Delta}.$$

We say that equation (1.1) has the *property A* if any proper solution $x(t)$ is either oscillatory or satisfies $|x^{[i]}(t)| \downarrow 0$ as $t \to \infty$ for $i = 0, 1, 2$. The solutions of (1.1) are called of *Kneser type* if the Δ-quasi-derivatives satisfy $x^{[i]}(t)x^{[i+1]}(t) < 0$ for $i = 0, 1, 2$ for all sufficiently large t. Originally, the property A was introduced for differential systems and equations; for more details we refer the reader to the book [24]. As a special case of (1.1), Erbe, Peterson and Saker [18] considered the third order linear dynamic equation

$$x^{\Delta\Delta\Delta}(t) + q(t)x(t) = 0, \tag{1.4}$$

on an arbitrary time scale $\mathbb{T}$, where $q(t)$ is a positive real-valued rd-continuous function on $\mathbb{T}$, and proved that if $x(t)$ is a solution of (1.4) and

$$I(q) = \int_{t_0}^{\infty} q(s)\Delta s = \infty,$$

then (1.4) has the property A, i.e., every solution $x(t)$ of (1.4) is oscillatory or satisfies

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x^{\Delta}(t) = \lim_{t\to\infty} x^{\Delta\Delta}(t) = 0.$$

One of our aims in this paper is to extend this result and establish some sufficient conditions for (1.1) to have the property A.

Since we are interested in the oscillatory and asymptotic behavior of solutions near infinity, we assume that $\sup \mathbb{T} = \infty$, and define the time scale interval $[t_0, \infty)_{\mathbb{T}}$ by $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$. By a solution of (1.1), we mean a function $x^{[0]} \in C_{\mathrm{rd}}([t_0, \infty), \mathbb{R})$ such that $x^{[1]}, x^{[2]} \in C_{\mathrm{rd}}([t_0, \infty), \mathbb{R})$ satisfying (1.1) for all $t \geq t_0$, where C_{rd} is the space of rd-continuous functions. A solution of (1.1) is said to be *proper* if it is defined on the interval $[t_0, \infty)$ and is nontrivial in any neighborhood of infinity. So the solutions vanishing in some neighborhood of infinity will be excluded from our considerations and we are interested only in the asymptotic behavior of proper solutions. A proper solution $x(t)$ of (1.1) is said to be *oscillatory* if it is neither eventually positive nor eventually negative; otherwise it is *nonoscillatory*. Equation (1.1) is said to be oscillatory if there exists at least one oscillatory solution, and nonoscillatory if all solutions are nonoscillatory.

In the following, we recall some results on third order dynamic equations on time scales that motivate our study. Morelli and Peterson [26] considered the third order dynamic equation

$$x^{\Delta\Delta\Delta}(t) + p(t)(x^{\sigma})^{\Delta} + q(t)x^{\sigma} = 0, \tag{1.5}$$

studied the asymptotic behavior of solutions, and established some sufficient conditions for existence of nonoscillatory solutions. Erbe, Peterson and Saker [16] initiated the study of oscillation of third order nonlinear dynamic equations and considered the equation

$$(c(t)((a(t)x^{\Delta}(t))^{\Delta}))^{\Delta} + q(t)f(x(t)) = 0, \quad t \geq t_0, \tag{1.6}$$

where $a(t)$, $c(t)$ and $q(t)$ are positive real-valued rd-continuous functions which satisfy

$$\int_{t_0}^{\infty} \frac{1}{c(t)} \Delta t = \infty, \qquad \int_{t_0}^{\infty} \frac{1}{a(t)} \Delta t = \infty, \tag{1.7}$$

and $f \in C(\mathbb{R}, \mathbb{R})$ satisfies $uf(u) > 0$ and $f(u)/u \geq K > 0$, for $u \neq 0$. Erbe, Peterson and Saker [17] extended the results in [16] and studied the oscillation of third order nonlinear dynamic equations of the form

$$[c(t)[(a(t)x^{\Delta}(t))^{\Delta}]^{\gamma}]^{\Delta} + f(t, x(t)) = 0, \quad t \geq t_0, \tag{1.8}$$

on a time scale $\mathbb{T}$, where $a(t) > 0$, $c(t) > 0$ are rd-continuous on $\mathbb{T}$, $\gamma \geq 1$ is a quotient of odd integers, and there exists a positive rd-continuous function q such that $|f(t, u)| \geq q(t)|u^{\gamma}|$ and $uf(t, u) > 0$, $u \neq 0$, and

$$\int_{t_0}^{\infty} \left(\frac{1}{c(t)}\right)^{1/\gamma} \Delta t = \infty, \qquad \int_{t_0}^{\infty} \frac{1}{a(t)} \Delta t = \infty. \tag{1.9}$$

They applied the Riccati transformation technique and established some sufficient conditions which guarantee that each solution $x(t)$ is oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$. Yu and Wang in [37] studied the oscillation of a third order dynamic equation of the form

$$\left(\frac{1}{a_2(t)}\left(\left(\frac{1}{a_1(t)}(x^{\Delta}(t))^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}\right)^{\Delta} + q(t)f(x(t)) = 0, \tag{1.10}$$

where $f \in C(\mathbb{R}, \mathbb{R})$, $uf(u) > 0$ and $f(u)/u \geq K > 0$, for $u \neq 0$, or $f'(u) > C > 0$, α_i is a quotient of odd positive integers, $i = 1, 2$, and $a_1(t)$, $a_2(t)$ and $q(t)$ are positive rd-continuous functions satisfying

$$\int_{t_0}^{\infty} q(t) \Delta t = \infty, \qquad \int_{t_0}^{\infty} (a_i(t))^{\alpha_i} \Delta t = \infty \quad \text{for } i = 1, 2. \tag{1.11}$$

They applied the Riccati technique of [16] and established some sufficient conditions which guarantee that each solution $x(t)$ of (1.10) is oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$. Erbe, Hassan and Peterson [14] studied the oscillation of the nonlinear functional dynamic equation

$$[c(t)[(a(t)x^{\Delta}(t))^{\Delta}]^{\gamma}]^{\Delta} + f(t, x(\tau(t))) = 0, \quad t \geq t_0, \tag{1.12}$$

and established several sufficient conditions which ensure that each solution $x(t)$ is oscillatory or $\lim_{t\to\infty} x(t) = 0$. They assumed that γ is a quotient of odd positive integers, $f \in C(\mathbb{T} \times \mathbb{R}, \mathbb{R})$ and satisfies $uf(t, u) > 0$ and $f(u)/u^{\gamma} \geq p(t)$ for $u \neq 0$, τ is defined on $\mathbb{T}$ and satisfies $\lim_{t\to\infty} \tau(t) = \infty$, and a and c are positive rd-continuous functions on $\mathbb{T}$ and satisfy (1.2) or (1.3).

The dynamic equation (1.1) in its general form includes linear and nonlinear third order differential and difference equations, and depends on the time scale $\mathbb{T}$. When $\mathbb{T}=\mathbb{R}$, we have $\sigma(t)=t$, $\mu(t)=0$, $x^{\Delta}(t)=x'(t)$ and (1.1) becomes the third order nonlinear functional differential equation

$$(p(t)((r(t)x'(t))')^{\gamma})'+q(t)f(x(\tau(t)))=0,\quad t\in[t_0,\infty). \tag{1.13}$$

Oscillation and asymptotic properties of some special case of (1.13), when $\gamma=1$ and $\tau(t)=t$, have been investigated by many authors (see [4, 5, 10–13, 21, 24, 25, 28] and the references therein). Most of the results in the above mentioned papers concern the third order differential equation (1.13) or some special cases under the assumption

$$\int_{t_0}^{\infty} q(s)\,ds=\infty, \tag{1.14}$$

which does not hold when $q(t)=\alpha/t^{\gamma}$ for $\gamma>1$. In this paper, we give a new condition (see Theorem 2.8, Examples 2.13, 2.14) different from (1.14), which can be applied when $q(t)=\alpha/t^3$. So the results in this paper when $\mathbb{T}=\mathbb{R}$ partially improve known results for third order differential equations.

When $\mathbb{T}=\mathbb{N}$, we have $\sigma(t)=t+1$, $\mu(t)=1$, $x^{\Delta}(t)=\Delta x(t)=x(t+1)-x(t)$. In this case, (1.1) becomes the third order difference equation

$$\Delta(p(n)\,(\Delta(r(n)\Delta x(n)))^{\gamma})+q(n)f(x(\tau(n)))=0,\quad n\in[n_0,\infty)_{\mathbb{N}}. \tag{1.15}$$

Oscillation and asymptotic properties of some special case of (1.15), when $\gamma=1$ and $\tau(n)=n$, have been investigated by several authors (see [27, 36, 19, 20] and the references therein).

If $\mathbb{T}=h\mathbb{N}_0$, $h>0$, then $\sigma(t)=t+h$, $\mu(t)=h$, $x^{\Delta}(t)=\Delta_h x(t):=(x(t+h)-x(t))/h$. In this case, (1.1) becomes the third order difference equation with step size h,

$$\Delta_h(p(t)\,(\Delta_h(r(t)\Delta_h x(t)))^{\gamma})+q(t)f(x(\tau(t)))=0,\quad t\in[0,\infty)_{h\mathbb{N}_0}, \tag{1.16}$$

If $\mathbb{T}=q^{\mathbb{N}_0}=\{t: t=q^k,\ k\in\mathbb{N}_0,\ q>1\}$, then $\sigma(t)=pt$, $\mu(t)=(q-1)t$, $x^{\Delta}(t)=D_q x(t):=(x(q\,t)-x(t))/(p-1)t$ (where D_q is the so-called quantum derivative which has important applications in quantum mechanics [23]). In this case (1.1) becomes the third order q-difference equation

$$D_q(p(t)\,(D_q(r(t)D_q x(t)))^{\gamma})+q(t)f(x(\tau(t)))=0,\quad t\in[t_0,\infty)_{\mathbb{T}}. \tag{1.17}$$

Also, many other time scales can be considered, for example, $\mathbb{T}=\mathbb{N}_0^2=\{t=n^2: n\in\mathbb{N}_0\}$, where $\sigma(t)=(\sqrt{t}+1)^2$ and $\mu(t)=1+2\sqrt{t}$, $x^{\Delta}(t)=(x((\sqrt{t}+1)^2)-x(t))/(1+2\sqrt{t})$, and $\mathbb{T}=\mathbb{T}_n=\{H_n: n\in\mathbb{N}_0\}$ where $\{H_n\}$ is the so-called sequence of harmonic numbers defined by

$$H_0=0,\qquad H_n=\sum_{k=1}^{n}\frac{1}{k},\qquad n\in\mathbb{N}_0.$$

Here $\sigma(H_n)=H_{n+1}$, $\mu(H_n)=1/(n+1)$, $x^{\Delta}(H_n)=(n+1)\Delta_n x(H_n)$.

Our aim in this paper is to study the asymptotic properties of nonoscillatory solutions of (1.1). We classify the nonoscillatory solutions into classes C_i for $i = 0, 1, 2, 3$ according to the sign of the Δ-quasi-derivatives and establish some sufficient conditions which ensure that $C_i = \emptyset$. We also establish some sufficient conditions which ensure that the equation (1.1) has the property A, which means that every solution $x(t)$ of (1.1) is oscillatory or satisfies $\lim_{t\to\infty} x^{[0]}(t) = \lim_{t\to\infty} x^{[1]}(t) = \lim_{t\to\infty} x^{[2]}(t) = 0$. Our results are new for third order dynamic equations and include and improve some results previously obtained for differential and difference equations (1.13) and (1.15). For (1.16) and (1.17) the results are essentially new. Some examples are worked out to demonstrate the main results.

2. Main results. In this section, first we classify the nonoscillatory solutions into classes C_i for $i = 0, 1, 2, 3$. Second, we study the asymptotic properties of nonoscillatory solutions of (1.1) and establish some sufficient condition for (1.1) to have the property A. We note that if $x(t)$ is a solution of (1.1) then $z = -x$ is also a solution of (1.1), since $uf(u) > 0$ for $u \neq 0$. Thus, when considering nonoscillatory solutions of (1.1) we can restrict our attention to positive ones. We start with the following lemma which gives the sign of the Δ-quasi-derivatives of $x(t)$ of (1.1).

LEMMA 2.1. *Assume that:*

(h_1) *p, q, r and τ are positive rd-continuous functions defined on $\mathbb{T}$, $\tau : \mathbb{T} \to \mathbb{T}$, $\lim_{t\to\infty} \tau(t) = \infty$,*
(h_2) *$f \in C(\mathbb{R}, \mathbb{R})$ and $uf(u) > 0$.*

If $x(t)$ is a nonoscillatory solution of (1.1), *then there exists $T > t_0$ such that $x^{[i]}(t) \neq 0$ for $i = 0, 1, 2$ and $t \geq T$.*

Proof. Without loss of generality, we may assume that $x(t) > 0$ and pick $t_1 > t_0$ so that $x(\tau(t)) > 0$ for $t > t_1$. Then from (1.1), we see that $x^{[3]}(t) = -q(t)f(x(\tau(t))) < 0$ for $t \geq t_1$ and there exists $t_2 > t_1$ such that $x^{[2]}(t)$ is decreasing and either positive or negative for $t \geq t_2$. Thus, $x^{[1]}$ is either increasing or decreasing for $t \geq t_2$, and so there exists $T > t_2$ such that $x^{[1]}(t)$ is either positive or negative for $t \geq T$. ∎

In view of Lemma 2.1, all nonoscillatory solutions of (1.1) belong to the following classes:

$$C_0 = \{x : \exists T \text{ such that } x(t)x^{[1]}(t) < 0,\ x(t)x^{[2]}(t) > 0 \text{ for } t \geq T\},$$
$$C_1 = \{x : \exists T \text{ such that } x(t)x^{[1]}(t) > 0,\ x(t)x^{[2]}(t) < 0 \text{ for } t \geq T\},$$
$$C_2 = \{x : \exists T \text{ such that } x(t)x^{[1]}(t) > 0,\ x(t)x^{[2]}(t) > 0 \text{ for } t \geq T\},$$
$$C_3 = \{x : \exists T \text{ such that } x(t)x^{[1]}(t) < 0,\ x(t)x^{[2]}(t) < 0 \text{ for } t \geq T\}.$$

In the following theorems, we classify the nonoscillatory solutions of (1.1) into C_i for $i = 0, 1, 2, 3$ according to the sign of their Δ-quasi-derivatives, and some sufficient conditions ensuring that $C_i = \emptyset$ are given.

In this section, the following notation will be used:

$$I(u) := \int_{t_0}^{\infty} u(s)\Delta s,$$

$$I(u, v) := \int_{t_0}^{\infty} u(t) \int_{t_0}^{t} v(s)\Delta s\Delta t,$$

where u and v are positive rd-continuous functions. Now, we begin to classify the nonoscillatory solutions and start by the following theorem which gives a condition ensuring that $C_3 = \emptyset$.

Theorem 2.2. *Assume that* (h_1)–(h_2) *hold. If*

$$I(1/r, 1/p^{1/\gamma}) = \infty, \tag{2.1}$$

then the class C_3 *is empty.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we assume that the solution $x(t)$ is positive and pick $t_1 > t_0$ so that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t > t_1$. To prove that C_3 is empty, we prove that the case $x(t)x^{[1]}(t) < 0$, $x(t)x^{[2]}(t) < 0$ for $t \geq T$ is impossible. Assume for the sake of contradiction that there exists $T > t_1$ such that $x^{[2]}(t) < 0$ and $x^{[1]}(t) < 0$ for $t \geq T$. Denote $a_0 = x^{[2]}(T) < 0$. Since $x^{[2]}$ is decreasing, we have $p(t)((x^{[1]}(t))^{\Delta})^{\gamma} < a_0$ for $t \geq T$. This implies that $(x^{[1]}(t))^{\Delta} < a_0^{1/\gamma}(p(s))^{-1/\gamma}$. Integrating from T to t, we get

$$x^{[1]}(t) < x^{[1]}(T) + a_0^{1/\gamma} \int_{T}^{t} \frac{1}{p^{1/\gamma}(s)}\Delta s.$$

Now, since $x^{[1]}(T) < 0$, we see after integrating again from T to t that

$$x(t) < x(T) + a_0^{1/\gamma} \int_{T}^{t} \frac{1}{r(s)} \int_{T}^{s} \frac{1}{p^{1/\gamma}(u)}\Delta u\Delta s.$$

Letting $t \to \infty$, by (2.1) we get $\lim_{t\to\infty} x(t) = -\infty$, which contradicts the fact that $x(t) > 0$. ∎

The following theorem gives conditions ensuring that $C_1 = \emptyset$.

Theorem 2.3. *Assume that* (h_1)–(h_2) *hold. If*

$$I(1/r) < \infty, \qquad I(1/r, 1/p^{1/\gamma}) = \infty, \tag{2.2}$$

then the class C_1 *is empty.*

Proof. Assume that C_1 is nonempty and let $x \in C_1$ be a nonoscillatory solution of (1.1). Without loss of generality we may assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$, $x^{[1]}(t) > 0$, $x^{[2]}(t) < 0$, and $x^{[3]}(t) < 0$ for $t \geq t_1$. Since $x^{[3]} < 0$, $x^{[2]}$ is decreasing and $x^{[2]}(t) < x^{[2]}(t_1) = c_1 < 0$ for $t \geq t_1$. This leads to $(x^{[1]}(t))^{\Delta} < c_1^{1/\gamma} p^{1/\gamma}(t)$. Integrating from t_1 to t yields

$$x^{[1]}(t) < c_2 + c_1^{1/\gamma} \int_{t_1}^{t} \frac{1}{p^{1/\gamma}(s)} \Delta s,$$

where $c_2 = x^{[1]}(t_1) > 0$. Therefore,

$$(x^{[0]}(t))^{\Delta} < c_2 r^{-1}(t) + c_1^{1/\gamma} \frac{1}{r(t)} \int_{t_1}^{t} \frac{1}{p^{1/\gamma}(s)} \Delta s.$$

This leads to

$$x^{[0]}(t) < x^{[0]}(t_1) + c_2 \int_{t_1}^{t} \frac{1}{r(s)} \Delta s + c_1^{1/\gamma} \int_{t_1}^{t} \frac{1}{r(s)} \int_{t_1}^{s} \frac{1}{p^{1/\gamma}(\theta)} \Delta\theta \Delta s.$$

Letting $t \to \infty$, we deduce by (2.2) that $x^{[0]}(t) \to -\infty$, contrary to $x(t) > 0$ for large t. ∎

The following theorem gives conditions ensuring that $C_2 = \emptyset$.

THEOREM 2.4. *Assume that* (h_1)–(h_2) *hold. Then* C_2 *is empty if any one of the following conditions holds:*

$$I(q) = \infty, \qquad \liminf_{|u| \to \infty} |f(u)| > 0, \tag{2.3}$$

$$f'(u) \geq 0 \text{ for } u \in \mathbb{R} \text{ and } \int_{t_0}^{\infty} q(t) f\Big(c_4 \int_{T}^{\tau(t)} \frac{1}{r(s)} \Delta s\Big) \Delta t = \infty, \tag{2.4}$$

for all $c_4 > 0$ *and* $T \geq t_0$.

Proof. Assume that C_2 is nonempty and let $x \in C_2$ be a nonoscillatory solution of (1.1). We may assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$ and $x(\tau(t)) > 0$, $x^{[1]}(t) > 0$, $x^{[2]}(t) > 0$, and $x^{[3]}(t) < 0$ for $t \geq t_1$. First, we consider the case when (2.3) holds. Since x is increasing for $t \geq t_0$, we have $\lim_{t\to\infty} x(t) = c \in \mathbb{R} \cup \{\infty\}$, and since $\lim_{t\to\infty} \tau(t) = \infty$, we can assume that $\lim_{t\to\infty} x(\tau(t)) = c \in \mathbb{R} \cup \{\infty\}$. In view of this, (2.3) and the conditions imposed on f, there exists a positive constant $K > 0$ such that $f(x(\tau(t))) \geq K$ for $t \geq t_1$. By integrating (1.1) from t_1 to t, we have

$$x^{[2]}(t) = x^{[2]}(t_0) - \int_{t_0}^{t} q(s) f(x(s)) \Delta s \leq x^{[2]}(t_0) - K x^{\gamma}(t_0) \int_{t_0}^{t} q(s) \Delta s.$$

This together with $x^{[2]}(t) > 0$ for $t \geq t_0$ leads to

$$\int_{t_0}^{\infty} q(s)\Delta s \leq \frac{x^{[2]}(t_0)}{K},$$

which contradicts $I(q) = \infty$.

Next, we consider (2.4). Since $x^{[2]}(t) > 0$, we see that $(x^{[1]}(t))^{\Delta} > 0$, and so $x^{[1]}$ is increasing for $t \geq t_0$. So $x^{[1]}(t) > x^{[1]}(t_1) = c_4 > 0$. This gives

$$x^{\Delta}(t) \geq c_4 r^{-1}(t) \quad \text{ for } t \geq t_1.$$

Also since $\lim_{t\to\infty} \tau(t) = \infty$, we can choose $t > t_2$ sufficiently large such that $\tau(t) > t_2$ for $t \geq t_2$. Then by integrating $x^{\Delta}(t) \geq c_4 r^{-1}(t)$ from t_2 to $\tau(t)$, we get

$$x(\tau(t)) \geq c_4 \int_{t_1}^{\tau(t)} r^{-1}(s)\Delta s.$$

Since f is nondecreasing, we have

$$f(x(\tau(s))) \geq f\Big(c_4 \int_{t_1}^{\tau(s)} r^{-1}(u)\Delta u\Big).$$

Again integrating (1.1) from t_2 to t, we have

$$\begin{aligned} 0 &< x^{[2]}(t_2) - \int_{t_2}^{t} q(s) f(x(\tau(s)))\Delta s \\ &\leq x^{[2]}(t_2) - \int_{t_2}^{t} q(s) f\bigg(c_4 \int_{t_2}^{\tau(s)} \frac{1}{r(u)}\Delta u\bigg)\Delta s. \end{aligned}$$

This implies that

$$\int_{t_2}^{\infty} q(t) f\bigg(c_4 \int_{t_2}^{\tau(t)} \frac{1}{r(s)}\Delta s\bigg)\Delta t \leq x^{[2]}(t_2),$$

which is a contradiction. The proof is complete. ■

Theorem 2.5. *Assume that* (h_1)–(h_2) *hold and*

$$I(1/p^{1/\gamma}) = I(1/r) = \infty. \tag{2.5}$$

Then every nonoscillatory solution $x(t)$ *of* (1.1) *is in* $C_0 \cup C_2$.

Proof. We may assume that $x(t)$ is eventually positive and pick $t_1 \geq t_0$ such that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \geq t_1$. Then from Lemma 2.1, $x^{[0]}$, $x^{[1]}$ and $x^{[2]}$ are monotone and eventually of one sign. So to complete the proof, we show that only the following two cases are possible:

(I) $x^{[0]}(t) > 0$, $x^{[1]}(t) > 0$, $x^{[2]}(t) > 0$, for $t \geq t_1$ sufficiently large,
(II) $x^{[0]}(t) > 0$, $x^{[1]}(t) < 0$, $x^{[2]}(t) > 0$, for $t \geq t_1$ sufficiently large.

From (1.1) and (h_2) we see that $x^{[3]}(t) \leq 0$ for $t \geq t_1$. We claim that there is $t_2 \geq t_1$ such that $x^{[2]}(t) > 0$ for $t \geq t_2$. Suppose to the contrary that $x^{[2]}(t) \leq 0$ for $t \geq t_2$. Since $x^{[2]}$ is decreasing, there exists a negative constant C and $t_3 \geq t_2$ such that $x^{[2]}(t) \leq C$ for $t \geq t_3$. Consequently,

$$x^{[1]}(t) \leq x^{[1]}(t_3) + C^{1/\gamma} \int_{t_3}^{t} \frac{1}{p^{1/\gamma}(s)} \Delta s.$$

Letting $t \to \infty$, we obtain $x^{[1]}(t) \to -\infty$ by (2.5). Thus, there is $t_4 \geq t_3$ such that $r(t)(x^{[0]}(t))^{\Delta} \leq r(t_4)(x^{[0]}(t_4))^{\Delta} < 0$ for $t \geq t_4$. This implies after integrating from t_4 to t that

$$x^{[0]}(t) - x^{[0]}(t_4) \leq r(t_4)(x^{[0]}(t_4))^{\Delta} \int_{t_3}^{t} \frac{1}{r(s)} \Delta s,$$

which implies that $x^{[0]}(t) \to -\infty$ as $t \to \infty$ by (2.5), which contradicts the fact that $x^{[0]}(t) > 0$. Therefore $x^{[2]} > 0$. The proof is complete. ∎

For equation (1.4), the only admissible classes of nonoscillatory solutions are C_0 and C_2. For equation (1.6) when (1.7) holds, the only admissible classes of nonoscillatory solutions are also C_0 and C_2. Also for equation (1.8), if (1.9) holds then the only admissible classes of nonoscillatory solutions are C_0 and C_2.

In the following theorems, we study the asymptotic properties of nonoscillatory solutions and establish some sufficient conditions for equation (1.1) to have the property A.

Theorem 2.6. *Assume that* (h_1)–(h_2) *and* (2.5) *hold. Let* $x(t)$ *be a nonoscillatory solution of* (1.1) *belonging to* C_2. *Then:*

(i) $\lim_{t\to\infty} x^{[0]}(t) = \infty$.
(ii) *If* $\lim_{t\to\infty} x^{[2]}(t) \neq 0$, *then* $\lim_{t\to\infty} x^{[1]}(t) = \infty$.

Proof. Because $x(t)$ is a nonoscillatory solution of (1.1) and belongs to C_2, we can assume that there exists $T \geq t_0$ such that $x^{[0]}(t) > 0$, $x^{[0]}(\tau(t)) > 0$, $x^{[1]}(t) > 0$, $x^{[2]}(t) > 0$ for $t \geq T$. (The case when $x^{[0]}(t) < 0$, $x^{[1]}(t) < 0$, $x^{[2]}(t) < 0$ for $t \geq T^* > t_0$ may be proved by using similar arguments.)

(i) As $x^{[1]}$ is positive and increasing, we have $x^{[1]}(t) > x^{[1]}(T)$ for $t \geq T$. By integrating from T to t, we obtain

$$x^{[0]}(t) > x^{[0]}(T) + x^{[1]}(T) \int_{T}^{t} \frac{\Delta s}{r(s)};$$

as $t \to \infty$, assumption (2.5) implies the first assertion.

(ii) Since $x^{[2]}(t)=p(t)((x^{[1]}(t))^{\gamma})^{\Delta}$, integrating from T to t implies that

$$x^{[1]}(t)=x^{[1]}(T)+\int_T^t\left(\frac{1}{p(s)}x^{[2]}(s)\right)^{1/\gamma}\Delta s.$$

Taking into account that $x^{[2]}(t)$ is positive and decreasing, we get

$$x^{[1]}(t)\geq x^{[1]}(T)+(x^{[2]}(T))^{1/\gamma}\int_T^t\left(\frac{1}{p(s)}\right)^{1/\gamma}\Delta s;$$

as $t\to\infty$, assumption (2.5) implies the second assertion. ■

THEOREM 2.7. *Assume that* (h_1)–(h_2) *hold and there exists* $T>t_0$ *such that*

$$\int_T^\infty\frac{1}{r(s)}\int_T^s\left(\frac{1}{p(u)}\right)^{1/\gamma}\Delta u\Delta s=\infty. \tag{2.6}$$

Then every nonoscillatory solution $x(t)$ *of* (1.1) *which belongs to* C_2 *satisfies* $\lim_{t\to\infty}x^{[0]}(t)=\infty$.

Proof. Again we can assume that there exists $T\geq t_0$ such that $x^{[0]}(t)>0$, $x^{[0]}(\tau(t))>0$, $x^{[1]}(t)>0$, $x^{[2]}(t)>0$ for $t\geq T$. From the definition of $x^{[1]}(t)$, we have

$$x^{[0]}(t)=x^{[0]}(T)+\int_T^t\frac{x^{[1]}(s)\Delta s}{r(s)}>\int_T^t\frac{x^{[1]}(s)\Delta s}{r(s)}. \tag{2.7}$$

Also, since $x^{[2]}(t)=p(t)((x^{[1]}(t))^{\Delta})^{\Delta\gamma}$, we get

$$x^{[1]}(t)=x^{[1]}(T)+\int_T^t\left(\frac{1}{p(s)}x^{[2]}(s)\right)^{1/\gamma}\Delta s>\int_T^t\left(\frac{1}{p(s)}x^{[2]}(s)\right)^{1/\gamma}\Delta s. \tag{2.8}$$

Taking into account that $x^{[2]}(t)$ is positive and decreasing, we deduce from (2.7) and (2.8) that

$$x^{[0]}(t)>\int_T^t\frac{1}{r(s)}\int_T^s\left(\frac{1}{p(u)}x^{[2]}(u)\right)^{1/\gamma}\Delta u\Delta s>x^{[2]}(T)\int_T^t\frac{1}{r(s)}\int_T^s\left(\frac{1}{p(u)}\right)^{1/\gamma}\Delta u\Delta s.$$

Assumption (2.6) implies that $\lim_{t\to\infty}x^{[0]}(t)=\infty$. ■

THEOREM 2.8. *Assume that* (h_1)–(h_2) *hold,* $f(u)\geq Ku^{\beta}$ *for* $\beta\geq 0$ *and*

(h_3) $\int_{t_0}^\infty r^{-1}(t)\int_t^\infty(p^{-1}(u)\int_u^\infty q(s)\Delta s)^{1/\gamma}\Delta u\Delta t=\infty$.

Then every nonoscillatory solution $x(t)$ *of* (1.1) *which belongs to* C_0 *satisfies* $\lim_{t\to\infty}x^{[0]}(t)=0$.

Proof. Because $x(t)$ is a nonoscillatory solution of (1.1) and belongs to C_0, we can assume that there exists $T\geq t_0$ such that $x^{[0]}(t)>0$,

$x^{[0]}(\tau(t)) > 0$, $x^{[1]}(t) < 0$, $x^{[2]}(t) > 0$ for $t \geq T$. (The case when $x^{[0]}(t) < 0$, $x^{[1]}(t) > 0$, and $x^{[2]}(t) < 0$ for $t \geq T^* > t_0$ is similar). We integrate (1.1) from t to ∞, noting that $x^{[2]}$ is positive and decreasing, and obtain

$$-x^{[2]}(t) + \int_t^\infty Kq(t)x^\beta(\tau(t))\Delta t \leq 0.$$

So

$$-p(t)((x^{[1]}(t))^\Delta)^\gamma + \int_t^\infty Kq(t)x^\beta(\tau(t))\Delta t \leq 0. \tag{2.9}$$

Integrating (2.9) from t to ∞ after dividing by $p(t)$ and using the fact that $x^{[1]}(t) < 0$, we get

$$r(t)(x^{[0]}(t))^\Delta + K^{1/\gamma}\int_t^\infty \left(\frac{1}{p(u)}\int_u^\infty q(s)x((\tau(s)))^\beta\Delta s\right)^{1/\gamma}\Delta t \leq 0. \tag{2.10}$$

Dividing (2.10) by $r(t)$ and integrating from T to ∞, we have

$$\int_T^\infty \frac{1}{r(t)}\int_t^\infty \left(\frac{1}{p(u)}\int_u^\infty q(s)(x(\tau(s)))^\beta\Delta s\right)^{1/\gamma}\Delta u\Delta t \leq \frac{x^{[0]}(T)}{K^{1/\gamma}}. \tag{2.11}$$

Now since $x^{[0]}(t)$ is positive and decreasing for $t \geq T$, it follows that

$$\lim_{t\to\infty} x^{[0]}(t) =: k \geq 0.$$

If we assume that $k > 0$, then we can assume that $x^\beta(\tau(t)) \geq (k/2)^\beta$ for $t_2 \geq T$. Thus from (2.11), we get

$$\int_T^\infty \frac{1}{r(t)}\int_t^\infty \left(\frac{1}{p(u)}\int_u^\infty q(s)\Delta s\right)^{1/\gamma}\Delta u\Delta t \leq \frac{2^{\beta/\gamma}x^{[0]}(t_1)}{K^{1/\gamma}k^{\beta/\gamma}},$$

which contradicts (h_3). Thus $\lim_{t\to\infty} x(t) = 0$. ■

THEOREM 2.9. *Assume that* (h_1)–(h_3) *and* (2.5) *hold and* $f(u) \geq Ku^\beta$ *for some* $\beta \geq 0$. *If there exists* $T > t_0$ *such that*

$$I(q) = \int_T^\infty q(s)\Delta s = \infty, \tag{2.12}$$

then (1.1) *has the property* A.

Proof. Assume that (1.1) does not have the property A and let $x(t)$ be a nonoscillatory solution of (1.1) which may be assumed to be positive for $t \geq T \geq t_0$. Now from Theorem 2.5, since (2.5) holds, there are two possibilities:

(i) $x^{[1]}(t) > 0$ for $t \geq T$,
(ii) $x^{[1]}(t) < 0$ for $t \geq T$.

Since $\lim_{t\to\infty}\tau(t)=\infty$, we can choose $T_1>T$ such that $\tau(t)>T$ for $t>T_1$. If (i) holds, then $x(t)$ is increasing and then from (h_2) we see that $f(x(\tau))\ge Kx^{\gamma}(T_1)$. Integrating (1.1) from T to t, we have

$$x^{[2]}(t)=x^{[2]}(T)-\int_T^t q(s)f(x(\tau(s)))\Delta s\le x^{[2]}(T)-Kx^{\beta}(T_1)\int_T^t q(s)\Delta s.$$

This implies by (2.12) that $\lim_{t\to\infty}x^{[2]}(t)=-\infty$, which is a contradiction.

If (ii) holds then $x(t)$ is decreasing and there are two cases:

Case 1: $\lim_{t\to\infty}x(t)=k>0$. Then $x(t)\ge k$ for $t\ge T$. Integrating (1.1) three times and following the proof of Theorem 2.8, we get a contradiction with (h_3).

Case 2: $\lim_{t\to\infty}x(t)=0$. From the fact that $x^{[0]}(t)>0$ and $x^{[2]}(t)>0$ for $t\ge T$ it follows that $x^{[1]}(t)$ is increasing and $\lim_{t\to\infty}x^{[1]}(t)=\beta$ where $-\infty<\beta\le 0$. This implies that $x^{[1]}(t)\le\beta$ for all $t\ge T>t_0$, and hence

$$x^{[0]}(T)\ge x^{[0]}(t)-\beta\int_T^t r^{-1}(s)\Delta s,$$

which is impossible by (2.5) for $\beta<0$. Therefore $\lim_{t\to\infty}x^{[1]}(t)=0$. Now $x^{[1]}(t)<0$ and $x^{[3]}(t)<0$ for $t\ge T$ imply that $x^{[2]}$ is decreasing and $\lim_{t\to\infty}x^{[2]}(t)=\delta$ where $0\le\delta<\infty$. This implies that

$$x^{[1]}(T)\le x^{[1]}(t)-\delta^{1/\gamma}\int_T^t p^{-1/\gamma}(s)\Delta s,$$

which is again impossible by (2.5) for $\delta>0$, and hence $\delta=0$. ■

From the proof of Theorem 2.9, we see that if (2.5) holds then $x\in C_0\cup C_2$, while if $x^{[1]}(t)<0$ for $t\ge T$, then $x\in C_0$ is a solution of Kneser type, and we immediately get the following results.

Proposition 2.10. *Assume that* (h_1)–(h_3) *and* (2.5) *hold. Then every solution* $x(t)$ *of Kneser type satisfies* $\lim_{t\to\infty}x^{[i]}(t)=0$, $i=0,1,2$.

Theorem 2.11. *Assume that* (h_1)–(h_3) *and* (2.5) *hold, and* $f(u)\ge Ku^{\beta}$ *for some* $\beta\ge 0$. *If there exists* $T>t_0$ *such that*

$$\int_T^{\infty} q(s)\bigg(\int_T^{\tau(u)}\frac{\Delta u}{r(u)}\bigg)^{\beta}\Delta s=\infty, \tag{2.13}$$

then (1.1) *has the property* A.

Proof. Assume that (1.1) does not have the property A and let $x(t)$ be a nonoscillatory solution of (1.1) which may be assumed to be positive for $t\ge T\ge t_0$. Now, from Theorem 2.5, there are two possibilities.

(i) $x^{[1]}(t) > 0$ for $t \geq T$,
(ii) $x^{[1]}(t) < 0$ for $t \geq T$.

If $x^{[1]}(t) > 0$ for $t \geq T$, then $x(t)$ is increasing. Integrating (1.1) from T to t, we have

$$x^{[2]}(t) = x^{[2]}(T) - \int_T^t q(s) f(x(\tau(s)))\Delta s.$$

Since $f(x(\tau(t))) \geq Kx^{\beta}(\tau(t)))$, we have

$$x^{[2]}(t) \leq x^{[2]}(T) - K \int_T^t q(s) x^{\beta}(\tau(t))\Delta s.$$

Since $\lim_{t\to\infty} \tau(t) = \infty$, we can choose $T_1 > T$ such that $\tau(t) > T$ for $t \geq T_1$. So, since $x^{[1]}(t) \geq x^{[1]}(T)$, we get $x(\tau(t)) \geq x^{[1]}(T) \int_T^{\tau(t)} \frac{\Delta u}{r(s)}$. This implies that

$$x^{[2]}(t) \leq x^{[2]}(T) - K \left(x^{[1]}(T)\right)^{\beta} \int_T^t q(s) \left(\int_T^{\tau(u)} \frac{\Delta u}{r(u)} \right)^{\beta} \Delta s.$$

This leads by (2.13) to $\lim_{t\to\infty} x^{[2]}(t) = -\infty$, which is a contradiction. The proof for (ii) is similar to the proof of Theorem 2.9 and hence is omitted. The proof is complete. ∎

We now give some examples to illustrate the main results.

Example 2.12. Consider the equation

$$x^{\Delta\Delta\Delta}(t) + \frac{\alpha}{t^3} x(\sigma(t)) = 0, \quad t \in [1, \infty)_{\mathbb{T}}, \tag{2.14}$$

where $\mathbb{T}$ is a time scale. Here $\gamma = 1$, $K = 1$, $r(t) = p(t) = 1$, $q(t) = \alpha/t^3$ and $\tau(t) = \sigma(t) > t$. In this case it is clear that the conditions (h_1)–(h_3) and (2.5) hold. Then by Theorem 2.8 every solution $x(t)$ of (2.14) which belongs to C_0 satisfies $\lim_{t\to\infty} x(t) = 0$. In fact, if $\mathbb{T} = \mathbb{R}$, $\alpha = 6$ the equation becomes the third order differential equation

$$x^{'''}(t) + \frac{6}{t^3} x(t) = 0 \quad \text{for } t \in [1, \infty),$$

and one such solution is $x(t) = 1/t$.

Example 2.13. Consider the dynamic equation

$$x^{\Delta\Delta\Delta}(t) + \frac{\alpha}{t^3} x(2t) = 0 \quad \text{for } t \in [1, \infty)_{\mathbb{T}}. \tag{2.15}$$

Here $\gamma = 1$, $K = 1$, $r(t) = p(t) = 1$, $q(t) = \alpha/t^3$ and $\tau(t) = 2t > t$. In this case it is clear that the conditions (h_1)–(h_3) and (2.5) hold. Then by Theorem 2.8, every solution $x(t)$ of (2.15) which belongs to C_0 satisfies

$\lim_{t\to\infty} x(t) = 0$. In fact, $\mathbb{T} = \mathbb{R}$ and $\alpha = 3/8 > 1/4$, the equation becomes the third order advanced differential equation

$$x'''(t) + \frac{3}{8t^3}x(2t) = 0 \quad \text{for } t \in [1,\infty);$$

by the change of variable $\theta = 2t$ the equation is reduced to the third order differential equation

$$y'''(\theta) + \frac{24}{\theta^3}y(\theta) = 0 \quad \text{for } \theta \in [2,\infty),$$

and one solution is given by $y(\theta) = 1/\theta^2$.

EXAMPLE 2.14. Consider the third order dynamic equation

$$\left(\frac{1}{t^3}\left[\frac{1}{t^2}x^{\Delta}(t)\right]^{\Delta}\right)^{\Delta} + \alpha t^2 x^3(t^2) = 0 \quad \text{for } t \in [1,\infty)_{\mathbb{T}}. \tag{2.16}$$

Here $p(t) = 1/t^3$, $r(t) = 1/t^2$, $\gamma = 1$, $K = 1$, $\beta = 3$ and $q(t) = \alpha t^2$. It is clear that (h_1)–(h_2) hold. To apply Theorem 2.9 it remains to prove that (h_3) and (2.5) hold. In this case it is clear that

$$I(1/r) = \int_1^{\infty} t^2\Delta t = \infty, \qquad I(1/p) = \int_1^{\infty} t^3\Delta t = \infty,$$

$$\int_{t_0}^{\infty}\frac{1}{r(t)}\int_t^{\infty}\left(\frac{1}{p(u)}\int_u^{\infty} q(s)\Delta s\right)^{1/\gamma}\Delta u\Delta t = \int_1^{\infty} u^2\int_t^{\infty}\left(u^3\int_u^{\infty}\alpha s^2\Delta s\right)\Delta u\Delta t = \infty.$$

Then by Theorem 2.9 equation (2.16) has the property A. In fact if $\mathbb{T} = \mathbb{R}$ and $\alpha = 90$, then one solution is $x(t) = 1/t^2$, which satisfies $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x^{[1]}(t) = \lim_{t\to\infty} x^{[2]}(t) = 0$.

EXAMPLE 2.15. Consider the dynamic equation

$$x^{\Delta\Delta\Delta}(t) + \frac{\alpha}{t^2}x(t^2) = 0 \quad \text{for } t \in [1,\infty)_{\mathbb{T}}. \tag{2.17}$$

Here $\gamma = 1$, $K = 1$, $\beta = 1$, $r(t) = p(t) = 1$, $q(t) = \alpha/t^2$ and $\tau(t) = t^2 > t$. In this case it is clear that the conditions (h_1)–(h_3) and (2.5) hold. To apply Theorem 2.11 it remains to prove that (2.13) holds. In this case it is clear:

$$\int_T^{\infty} q(s)\left(\int_T^{\tau(u)}\frac{\Delta u}{r(u)}\right)^{\beta}\Delta s = \int_1^{\infty}\frac{\alpha}{s^2}(s^2-1)\Delta s = \infty.$$

Then by Theorem 2.11 equation (2.17) has the property A. In fact if $\mathbb{T} = \mathbb{R}$ and $\alpha = 6$, then one solution is $x(t) = 1/t$, which satisfies $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x^{[1]}(t) = \lim_{t\to\infty} x^{[2]}(t) = 0$.

In the following, we consider the case when (1.3) holds and establish some sufficient conditions which guarantee that the solution $x(t)$ of (1.1) satisfies $\lim_{t\to\infty} x^{[0]}(t) = 0$ and $\lim_{t\to\infty} x^{[1]}(t) = 0$. Throughout the following we assume that (2.1) holds and so by Theorem 2.2, the class C_3 is empty. From the above results, if (1.3) holds and $x(t)$ is a solution of (1.1) then $x_0 \in C_0 \cup C_1 \cup C_2$. Throughout the following we define $B(t)$ by

$$B(t) := q(t)\Big(\int_T^{g(t)} \frac{1}{r(s)}\Delta s\Big)^{\gamma}.$$

where $T > t_0$ is such that $g(t) > T$.

THEOREM 2.16. *Assume that* (h_1)–(h_3), (1.3), (2.3) *and* (2.5) *hold, and* $uf(u) > 0$, $f(u) \geq Ku^{\gamma}$ *for some* $\gamma > 0$. *If*

$$\int_T^{\infty}\Big[\frac{1}{p(s)}\int_T^{s} B(u)\Delta u\Big]^{1/\gamma}\Delta s = \infty, \tag{2.18}$$

then every nonoscillatory solution $x(t)$ *of* (1.1) *satisfies* $\lim_{t\to\infty} x^{[0]}(t) = 0$ *and* $\lim_{n\to\infty} x^{[1]}(t) = 0$.

Proof. Without loss of generality, we may assume that $x(t) > 0$, $x(g(t)) > 0$ for $t \geq t_1$ where t_1 is large. The proof when $x(t) < 0$ is similar, since $uf(u) > 0$. Since (2.3) is satisfied we know that $x \in C_0 \cup C_1 \cup C_2$. If $x \in C_0$, then we are back in the proof of Theorem 2.8 and infer that $\lim_{t\to\infty} x(t) = 0$. If (2.5) holds then C_2 is empty. It remains to consider the case when $x \in C_1$. Thus there exists $T > t_1$ such that $x^{[1]}(t) > 0$ and $x^{[2]}(t) < 0$ for $t \geq T$. Now, from the definitions of the Δ-quasi-derivatives,

$$x^{\Delta}(t) = \frac{x^{[1]}(t)}{r(t)}.$$

Since $(x^{[1]}(t))^{\Delta} < 0$, $x^{[1]}$ is decreasing, and by integrating from T to t we have

$$x(t) - x(T) = \int_T^{t}\frac{x^{[1]}(s)}{r(s)}\Delta s \geq x^{[1]}(t)\int_T^{t}\frac{1}{r(s)}\Delta s. \tag{2.19}$$

Hence, there exists $T_1 > T$ such that

$$x(g(t)) \geq x^{[1]}(g(t))\int_T^{g(t)}\frac{1}{r(s)}\Delta s \quad \text{for } t \geq T_1.$$

Substituting in (1.1), we get

$$(p(t)((x^{[1]})^{\Delta})^{\gamma})^{\Delta} + Kq(t)\Big(\int_T^{g(t)}\frac{1}{r(s)}\Delta s\Big)^{\gamma}(x^{[1]}(g(t)))^{\gamma} \leq 0, \quad t \geq T_1. \tag{2.20}$$

Let $y(t) = x^{[1]}(t) > 0$. Then $y^{\Delta}(t) < 0$ and $y(t)$ satisfies the inequality

$$(p(t)(y^{\Delta}(t))^{\gamma})^{\Delta} + KB(t)y^{\gamma}(g(t)) \leq 0, \quad \text{for } t \geq T_1. \tag{2.21}$$

Now, since y is decreasing, the limit $\lim_{n\to\infty} y(t) = b \geq 0$ exists. We assert that $b = 0$. If not, then $y(\tau(t)) > y(t) > b > 0$ for $t \geq T_2$. Define

$$u(t) = p(t)(y^{\Delta}(t))^{\gamma}).$$

Then from (2.21) for $t \geq T_2$, we obtain

$$u^{\Delta}(t) = -KB(t)y(g(t)) \leq -b^{\gamma}KB(t).$$

Hence for $t \geq T_2$, we have

$$u(t) \leq u(T_2) - Kb^{\gamma}\int_{T_2}^{t} B(s)\Delta s < -Kb^{\gamma}\int_{T_2}^{t} B(s)\Delta s.$$

Since $u(T_2) = p(T_2)y^{\Delta}(T_2) < 0$. Integrating again from T_2 to t, we have

$$y(t) - y(T_2) \leq -K^{1/\gamma}b\int_{T_2}^{t}\left[\frac{1}{p(s)}\int_{T_2}^{s} B(u)\Delta u\right]^{1/\gamma}\Delta s.$$

By condition (2.18), we get $y(t) \to -\infty$ as $t \to \infty$, contrary to the fact that $y(t) > 0$ for $t \geq T_2$. Thus $b = 0$ and $\lim_{t\to\infty} x^{[1]}(t) = 0$. ∎

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I. Kubiaczyk
Faculty of Mathematics and Computer Science
Adam Mickiewicz University
Umultowska 87
61-614 Poznań, Poland
E-mail: kuba@amu.edu.pl

S. H. Saker
Department of Mathematics Skills, PY
King Saud University
Riyadh 11451, Saudi Arabia
E-mail: mathcoo@py.ksu.edu.sa
and
Department of Mathematics
Faculty of Science
Mansoura University
Mansoura, 35516, Egypt
E-mail: shsaker@mans.edu.eg

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