ANNALES
POLONICI MATHEMATICI
100.3 (2011)

Doubly warped product submanifolds of (κ,μ)-contact metric manifolds

by Sibel Sular and Cihan Özgür (Balikesir)

Abstract. We establish sharp inequalities for C-totally real doubly warped product submanifolds in (κ,μ)-contact space forms and in non-Sasakian (κ,μ)-contact metric manifolds.

1. Introduction. Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and f_1, f_2 differentiable, positive-valued functions on M_1 and M_2, respectively. The *doubly warped product* $M = {}_{f_2}M_1 \times {}_{f_1}M_2$ is the product manifold $M_1 \times M_2$ equipped with the metric

$$g = f_2^2 g_1 + f_1^2 g_2.$$

More explicitly, if $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$ are canonical projections, then the metric g is given by

$$g = (f_2 \circ \pi_2)^2 \pi_1^* g_1 + (f_1 \circ \pi_1)^2 \pi_2^* g_2.$$

The functions f_1 and f_2 are called *warping functions*. If either $f_1 \equiv 1$ or $f_2 \equiv 1$, but not both, then we get a warped product. If both $f_1 \equiv 1$ and $f_2 \equiv 1$, then we obtain a Riemannian product manifold. If neither f_1 nor f_2 is constant, then we have a *non-trivial* doubly warped product [Ün].

For a doubly warped product ${}_{f_2}M_1 \times {}_{f_1}M_2$, let D_1 and D_2 denote the distributions obtained from the vectors on M_1 and M_2, respectively.

Assume that

$$x : {}_{f_2}M_1 \times {}_{f_1}M_2 \to \widetilde{M}$$

is an isometric immersion of a doubly warped product ${}_{f_2}M_1 \times {}_{f_1}M_2$ into a Riemannian manifold $\widetilde{M}$. We denote by σ the second fundamental form of x and by $H_i = (1/n_i)\operatorname{trace}\sigma_i$ the partial mean curvatures, where $\operatorname{trace}\sigma_i$ is the trace of σ restricted to M_i and $n_i = \dim M_i$ $(i = 1, 2)$. The immersion

2010 *Mathematics Subject Classification*: Primary 53C40; Secondary 53C25.
Key words and phrases: doubly warped product manifold, (κ,μ)-contact space form, non-Sasakian (κ,μ)-contact metric manifold, C-totally real submanifold.

DOI: 10.4064/ap100-3-2

x is called *mixed totally geodesic* if $\sigma(X,Z)=0$ for any vector fields X and Z tangent to D_1 and D_2, respectively.

If ${}_{f_2}M_1 \times {}_{f_1}M_2$ is a doubly warped product, we have

$$\nabla_X Y = \nabla^1_X Y - \frac{f_2^2}{f_1^2} g_1(X,Y)\nabla^2(\ln f_2)$$

and

$$\nabla_X Z = Z(\ln f_2)X + X(\ln f_1)Z,$$

for any vector fields X, Y tangent to M_1, and Z tangent to M_2, where ∇^1 and ∇^2 are the Levi-Civita connections of the Riemannian metrics g_1 and g_2, respectively. Here, $\nabla^2(\ln f_2)$ denotes the gradient of $\ln f_2$ with respect to the metric g_2.

If X and Z are unit vector fields, it follows that the sectional curvature $K(X\wedge Z)$ of the plane section spanned by X and Z is given by

$$K(X\wedge Z) = \frac{1}{f_1}\{(\nabla^1_X X)f_1 - X^2 f_1\} + \frac{1}{f_2}\{(\nabla^2_Z Z)f_2 - Z^2 f_2\}.$$

Consequently, we obtain

$$n_2\frac{\Delta f_1}{f_1} + n_1\frac{\Delta f_2}{f_2} = \sum_{1\le j\le n_1<s\le n} K(e_j\wedge e_s), \tag{1.1}$$

for a local orthonormal frame $\{e_1,\dots,e_{n_1},e_{n_1+1},\dots,e_n\}$ such that $e_1,\dots,e_{n_1}$ are tangent to M_1 and $e_{n_1+1},\dots,e_n$ are tangent to M_2.

In [Ch-2002], B. Y. Chen proved the following result for a warped product submanifold of a Riemannian manifold of constant sectional curvature:

Theorem 1.1. *Let $x : M_1\times_f M_2 \to \widetilde{M}(c)$ be an isometric immersion of an n-dimensional warped product $M_1\times_f M_2$ into an m-dimensional Riemannian manifold $\widetilde{M}(c)$ of constant sectional curvature c. Then*

$$\frac{\Delta f}{f} \le \frac{n^2}{4n_2}\|H\|^2 + n_1 c, \tag{1.2}$$

where $n_i=\dim M_i$, $n=n_1+n_2$, and Δ is the Laplacian operator of M. Equality holds in (1.2) identically if and only if x is a mixed totally geodesic immersion and $n_1H_1=n_2H_2$, where H_i, $i=1,2$, are the partial mean curvature vectors.

In [MM], K. Matsumoto and I. Mihai studied warped product submanifolds in Sasakian space forms. In [Mi-2004] and [Mi-2005], A. Mihai considered warped product submanifolds in complex space forms and quaternion space forms, respectively. Recently, in [MAEM], C. Murathan, K. Arslan, R. Ezentaş and I. Mihai studied warped product submanifolds in Kenmotsu space forms. Later, B. Y. Chen and F. Dillen extended inequality (1.2) to multiply warped product submanifolds in arbitrary Riemannian manifolds

[ChDi]. Recently, in [Tri], M. M. Tripathi established basic inequalities for C-totally real warped product submanifolds of (κ,μ)-contact space forms and non-Sasakian (κ,μ)-contact metric manifolds.

In [Ol], A. Olteanu established the following general inequality for arbitrary isometric immersions of doubly warped product manifolds in arbitrary Riemannian manifolds:

Theorem 1.2. *Let x be an isometric immersion of an n-dimensional doubly warped product $M = {}_{f_2}M_1 \times {}_{f_1}M_2$ into an arbitrary m-dimensional Riemannian manifold $\widetilde{M}$. Then*

$$n_2 \frac{\Delta_1 f_1}{f_1} + n_1 \frac{\Delta_2 f_2}{f_2} \leq \frac{n^2}{4} \|H\|^2 + n_1 n_2 \max \widetilde{K}, \tag{1.3}$$

where $n_i = \dim M_i$, $n = n_1 + n_2$, Δ_i is the Laplacian operator of M_i, $i = 1,2$, and $\max \widetilde{K}(p)$ denotes the maximum of the sectional curvature function of $\widetilde{M}$ restricted to 2-plane sections of the tangent space T_pM of M at each point p in M. Moreover, equality holds in (1.3) identically if and only if the following two statements hold:

(1) *x is a mixed totally geodesic immersion satisfying $n_1 H_1 = n_2 H_2$, where H_i, $i = 1,2$, are the partial mean curvature vectors of M_i,*

(2) *at each point $p = (p_1,p_2) \in M$, the sectional curvature function $\widetilde{K}$ of $\widetilde{M}$ satisfies $\widetilde{K}(u,v) = \max \widetilde{K}(p)$ for each unit vector $u \in T_{p_1}M_1$ and each unit vector $v \in T_{p_2}M_2$.*

Motivated by the studies of the above authors, we prove similar inequalities for C-totally real doubly warped product submanifolds of (κ,μ)-contact space forms and non-Sasakian (κ,μ)-contact metric manifolds.

The paper is organized as follows: In Section 2, we give a brief introduction to submanifolds, (κ,μ)-contact metric manifolds, (κ,μ)-contact space forms and non-Sasakian (κ,μ)-contact metric manifolds. In Section 3, we prove basic inequalities for (κ,μ)-contact space forms and non-Sasakian (κ,μ)-contact metric manifolds. In Section 4, as applications we prove that if the functions f_1 and f_2 are harmonic then $M = {}_{f_2}M_1 \times {}_{f_1}M_2$ does not admit minimal immersions under certain conditions.

2. Preliminaries. Let M be an m-dimensional Riemannian manifold and $p \in M$. Denote by $K(\pi)$ or $K(u,v)$ the sectional curvature of M associated with a plane section $\pi \subset T_pM$, where $\{u,v\}$ is an orthonormal basis of π. For any n-dimensional subspace $L \subseteq T_pM$, $2 \leq n \leq m$, its scalar curvature $\tau(L)$ is given by

$$\tau(L) = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j),$$

where $\{e_1, \ldots, e_n\}$ is any orthonormal basis of L [Ch-2000]. If $L = T_pM$, then $\tau(L)$ is just the scalar curvature $\tau(p)$ of M at p.

For an n-dimensional submanifold M in a Riemannian m-manifold $\widetilde{M}$, we denote by ∇ and $\widetilde{\nabla}$ the Levi-Civita connections of M and $\widetilde{M}$, respectively. The Gauss and Weingarten formulas are

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y) \quad \text{and} \quad \widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X Y,$$

respectively, for vector fields X, Y tangent to M, and N normal to M, where σ denotes the second fundamental form, $\nabla^{\perp}$ the normal connection and A the shape operator of M [Ch-1973].

Denote by R and $\widetilde{R}$ the Riemannian curvature tensors of M and $\widetilde{M}$, respectively. Then the equation of Gauss is given by

$$\begin{aligned} R(X,Y,Z,W) &= \widetilde{R}(X,Y,Z,W) \\ &\quad + g(\sigma(Y,Z),\sigma(X,W)) - g(\sigma(X,Z),\sigma(Y,W)), \end{aligned}$$

for all vector fields X, Y, Z, W tangent to M [Ch-1973].

For any orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM, the mean curvature vector is given by

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} \sigma(e_i, e_i),$$

where $n = \dim M$. The submanifold M is *totally geodesic* in $\widetilde{M}$ if $\sigma = 0$, and *minimal* if $H = 0$.

We write

$$\sigma_{ij}^r = g(\sigma(e_i, e_j), e_r), \qquad i, j \in \{1, \ldots, n\},\ r \in \{n+1, \ldots, m\},$$

for the coefficients of the second fundamental form σ with respect to $e_1, \ldots, e_n$, $e_{n+1}, \ldots, e_m$, and set

$$\|\sigma\|^2 = \sum_{i,j=1}^{n} g(\sigma(e_i, e_j), \sigma(e_i, e_j)).$$

Let M be a local n-dimensional Riemannian manifold and $\{e_1, \ldots, e_n\}$ be a local orthonormal frame on M. For a differentiable function f on M, the Laplacian Δf of f is given by

$$\Delta f = \sum_{j=1}^{n} \{(\nabla_{e_j} e_j) f - e_j e_j f\}.$$

We will need the following Chen's Lemma:

Lemma 2.1 ([Ch-1993]). *Let $n \geq 2$ and $a_1, \ldots, a_n, b$ be real numbers such that*

$$\Big(\sum_{i=1}^{n} a_i\Big)^2 = (n-1)\Big(\sum_{i=1}^{n} a_i^2 + b\Big).$$

Then $2a_1a_2 \geq b$*, with equality holding if and only if*

$$a_1 + a_2 = a_3 = \cdots = a_n.$$

A $(2m+1)$-dimensional Riemannian manifold $\widetilde{M}$ is said to be an *almost contact metric manifold* [Bl-2002] if there exist on $\widetilde{M}$ a $(1,1)$-tensor field φ, a vector field ξ, a 1-form η and a Riemannian metric g satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0,$$
$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X,\xi),$$

for any vector fields X, Y on $\widetilde{M}$. An almost contact metric manifold is a *contact metric manifold* if

$$g(X, \varphi Y) = d\eta(X,Y),$$

for all X, Y on $\widetilde{M}$.

A contact metric manifold is a *Sasakian manifold* if the Riemannian curvature tensor $\widetilde{R}$ of $\widetilde{M}$ satisfies

$$\widetilde{R}(X,Y)\xi = \eta(Y)X - \eta(X)Y,$$

for all vector fields X, Y on $\widetilde{M}$.

In a contact metric manifold $\widetilde{M}$, a $(1,1)$-tensor field h is given by

$$h = \frac{1}{2} L_\xi \varphi,$$

where L_ξ is the Lie derivative in the characteristic direction ξ. Moreover h is symmetric and satisfies

$$h\xi = 0, \quad h\varphi + \varphi h = 0,$$
$$\widetilde{\nabla}\xi = -\varphi - \varphi h, \quad \operatorname{trace}(h) = \operatorname{trace}(\varphi h) = 0,$$

where $\widetilde{\nabla}$ is the Levi-Civita connection.

The tangent sphere bundle of a flat Riemannian manifold admits a contact metric structure satisfying $R(X,Y)\xi = 0$ [Bl-2002]. The (κ,μ)-nullity condition on a contact metric manifold is considered as a generalization of both $R(X,Y)\xi = 0$ and the Sasakian case. The (κ,μ)-*nullity distribution* $N(\kappa,\mu)$ [BKP] of a contact metric manifold $\widetilde{M}$ is defined by

$$N(\kappa,\mu) : p \mapsto N_p(\kappa,\mu) = \{Z \in T_pM \mid R(X,Y)Z = (\kappa I + \mu h)(g(Y,Z)X - g(X,Z)Y)\},$$

for all $X, Y \in TM$ where $(\kappa,\mu) \in \mathbb{R}^2$ and I is the identity map. If ξ belongs to the (κ,μ)-nullity distribution $N(\kappa,\mu)$ then the contact metric manifold $\widetilde{M}$ is called a (κ,μ)-*contact metric manifold*. In particular the condition

$$R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY)$$

holds on a (κ,μ)-contact metric manifold.

On a (κ,μ)-contact metric manifold we have

$$h^2 = (\kappa-1)\varphi^2 \quad \text{and} \quad \kappa \leq 1.$$

For a (κ,μ)-contact metric manifold, the conditions to be a Sasakian manifold, $\kappa = 1$, and $h = 0$ are all equivalent. When $\kappa < 1$, the non-zero eigenvalues of h are $\lambda = \mp\sqrt{1-\kappa}$ each with multiplicity m. The eigenspace relative to the eigenvalue 0 is span$\{\xi\}$. Also, for $\kappa \neq 1$, the subbundle $D = \ker(\eta)$ can be decomposed into the eigenspace distributions D_+ and D_- relative to the eigenvalues λ and $-\lambda$, respectively. These distributions are orthogonal to each other and have dimension m [Bl-2002].

For a unit vector field X orthogonal to ξ in an almost contact metric manifold, the sectional curvature $\widetilde{K}(X,\varphi X)$ is called a *φ-sectional curvature*. On a $(2m+1)$-dimensional $(m > 3)$, (κ,μ)-contact metric manifold $\widetilde{M}$, if the φ-sectional curvature at $p \in \widetilde{M}$ is independent of the φ-section at p, then it is constant [Kou]. If the (κ,μ)-contact metric manifold $\widetilde{M}$ has constant φ-sectional curvature c, then it is said to be a *(κ,μ)-contact space form* and denoted by $\widetilde{M}(c)$. The Riemannian curvature tensor of a (κ,μ)-contact space form $\widetilde{M}(c)$ is given by

$$\begin{aligned}(2.1)\qquad \widetilde{R}(X,Y,Z,W) &= \frac{c+3}{4}\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} \\ &+ \frac{c-1}{4}\{2g(X,\varphi Y)g(\varphi Z,W) + g(X,\varphi Z)g(\varphi Y,W) - g(Y,\varphi Z)g(\varphi X,W)\} \\ &+ \frac{c+3-4\kappa}{4}\{\eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W) \\ &+ g(X,Z)\eta(Y)\eta(W) - g(Y,Z)\eta(X)\eta(W)\} \\ &+ \frac{1}{2}\{g(hY,Z)g(hX,W) - g(hX,Z)g(hY,W) \\ &+ g(\varphi hX,Z)g(\varphi hY,W) - g(\varphi hY,Z)g(\varphi hX,W)\} \\ &+ g(\varphi Y,\varphi Z)g(hX,W) - g(\varphi X,\varphi Z)g(hY,W) \\ &+ g(hX,Z)g(\varphi^2 Y,W) - g(hY,Z)g(\varphi^2 X,W) \\ &+ \mu\{\eta(Y)\eta(Z)g(hX,W) - \eta(X)\eta(Z)g(hY,W) \\ &+ g(hY,Z)\eta(X)\eta(W) - g(hX,Z)\eta(Y)\eta(W)\},\end{aligned}$$

for all vector fields X,Y,Z,W on $\widetilde{M}(c)$ [Kou]. If $\kappa = 1$ then a (κ,μ)-contact space form $\widetilde{M}(c)$ becomes a *Sasakian space form* and the equation (2.1) reduces to

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) &= \frac{c+3}{4}\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} \\ &+ \frac{c-1}{4}\{2g(X,\varphi Y)g(\varphi Z,W)\end{aligned}$$

$$+ g(X,\varphi Z)g(\varphi Y,W) - g(Y,\varphi Z)g(\varphi X,W) + \eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W) + g(X,Z)\eta(Y)\eta(W) - g(Y,Z)\eta(X)\eta(W)\}.$$

The Riemannian curvature tensor $\widetilde{R}$ of a non-Sasakian (κ,μ)-contact metric manifold $\widetilde{M}$ is given by

$$\begin{aligned}(2.2)\qquad \widetilde{R}(X,Y,Z,W) &= \left(1-\frac{\mu}{2}\right)\{g(Y,Z)g(X,W)-g(X,Z)g(Y,W)\}\\ &-\frac{\mu}{2}\{2g(X,\varphi Y)g(\varphi Z,W)+g(X,\varphi Z)g(\varphi Y,W)-g(Y,\varphi Z)g(\varphi X,W)\}\\ &+g(Y,Z)g(hX,W)-g(X,Z)g(hY,W)\\ &-g(Y,W)g(hX,Z)+g(X,W)g(hY,Z)\\ &+\frac{1-\mu/2}{1-\kappa}\{g(hY,Z)g(hX,W)-g(hX,Z)g(hY,W)\}\\ &+\frac{\kappa-\mu/2}{1-\kappa}\{g(\varphi hY,Z)g(\varphi hX,W)-g(\varphi hX,Z)g(\varphi hY,W)\}\\ &+\eta(X)\eta(W)\{(\kappa-1+\mu/2)g(Y,Z)-(\mu-1)g(hY,Z)\}\\ &-\eta(X)\eta(Z)\{(\kappa-1+\mu/2)g(Y,W)-(\mu-1)g(hY,W)\}\\ &+\eta(Y)\eta(Z)\{(\kappa-1+\mu/2)g(X,W)-(\mu-1)g(hX,W)\}\\ &-\eta(Y)\eta(W)\{(\kappa-1+\mu/2)g(X,Z)-(\mu-1)g(hX,Z)\},\end{aligned}$$

for all vector fields X,Y,Z,W on $\widetilde{M}$ ([Bo-1999], [Bo-2000]). A 3-dimensional non-Sasakian (κ,μ)-contact metric manifold has constant φ-sectional curvature, but this is not true for higher dimensions. A non-Sasakian (κ,μ)-contact metric manifold has constant φ-sectional curvature c if and only if $\mu=\kappa+1$ [Kou].

3. Main results. A submanifold M normal to ξ in a contact metric manifold $\widetilde{M}$ is said to be a *C-totally real submanifold* [YK]. It follows that φ maps any tangent space of M into the normal space, that is, $\varphi(T_pM)\subset T_p^{\perp}M$ for any $p\in M$.

For a C-totally real submanifold in a contact metric manifold, it is easy to see that

$$g(A_\xi X,Y) = -g(\widetilde{\nabla}_X\xi,Y) = g(\varphi X+\varphi hX,Y),$$

which means that $A_\xi=(\varphi h)^T$, the tangent component of φh.

In this section, we consider inequalities for C-totally real doubly warped product submanifolds of (κ,μ)-contact space forms and non-Sasakian (κ,μ)-contact metric manifolds.

Now, we begin with the following theorem:

THEOREM 3.1. *Let $M = {}_{f_2}M_1 \times_{f_1} M_2$ be an n-dimensional C-totally real doubly warped product submanifold of a $(2m+1)$-dimensional (κ,μ)-contact space form $\widetilde{M}(c)$. Then*

$$\begin{aligned}(3.1)\qquad n_2\frac{\Delta_1 f_1}{f_1} + n_1\frac{\Delta_2 f_2}{f_2} &\le \frac{n^2}{4}\|H\|^2 + \frac{n_1 n_2}{4}(c+3) + n_2\operatorname{trace}(h^T_{|M_1}) + n_1\operatorname{trace}(h^T_{|M_2})\\ &\quad + \frac{1}{4}\{(\operatorname{trace}(h^T))^2 - (\operatorname{trace}(h^T_{|M_1}))^2 - (\operatorname{trace}(h^T_{|M_2}))^2\\ &\quad - (\operatorname{trace}(A_\xi))^2 + (\operatorname{trace}(A_{\xi|M_1}))^2 + (\operatorname{trace}(A_{\xi|M_2}))^2\\ &\quad - \|h^T\|^2 + \|h^T_{|M_1}\|^2 + \|h^T_{|M_2}\|^2\\ &\quad + \|A_\xi\|^2 - \|A_{\xi|M_1}\|^2 - \|A_{\xi|M_2}\|^2\},\end{aligned}$$

where $n_i = \dim M_i$, $n = n_1 + n_2$ and Δ_i is the Laplacian of M_i, $i = 1,2$. Equality holds in (3.1) identically if and only if M is mixed totally geodesic and $n_1 H_1 = n_2 H_2$, where H_i, $i = 1,2$, are the partial mean curvature vectors.

Proof. We choose a local orthonormal frame $\{e_1,\dots,e_{n_1},e_{n_1+1},\dots,e_n\}$ such that $e_1,\dots,e_{n_1}$ are tangent to M_1, $e_{n_1+1},\dots,e_n$ are tangent to M_2 and e_{n+1} is parallel to the mean curvature vector H.

From the equation of Gauss, we have

$$2\tau(p) = n^2\|H\|^2(p) - \|\sigma\|^2(p) + 2\widetilde{\tau}(T_pM), \qquad p \in M,$$

where $\|\sigma\|^2$ is the squared norm of the second fundamental form σ of M in $\widetilde{M}$ and $\widetilde{\tau}(T_pM)$ is the scalar curvature of the subspace T_pM in $\widetilde{M}$.

We set

$$\delta = 2\tau - \frac{n^2}{2}\|H\|^2 - 2\widetilde{\tau}(T_pM). \tag{3.2}$$

The equation (3.2) can be written as follows:

$$n^2\|H\|^2 = 2(\delta + \|\sigma\|^2). \tag{3.3}$$

For the chosen local orthonormal frame, the relation (3.3) takes the form

$$\Big(\sum_{i=1}^{n}\sigma_{ii}^{n+1}\Big)^2 = 2\Big[\delta + \sum_{i=1}^{n}(\sigma_{ii}^{n+1})^2 + \sum_{i\neq j}(\sigma_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(\sigma_{ij}^r)^2\Big].$$

If we put $a_1 = \sigma_{11}^{n+1}$, $a_2 = \sum_{i=2}^{n_1}\sigma_{11}^{n+1}$ and $a_3 = \sum_{t=n_1+1}^{n}\sigma_{tt}^{n+1}$, then the above equation reduces to

$$\Big(\sum_{i=1}^{3}a_i\Big)^2=2\Big[\delta+\sum_{i=1}^{3}a_i^2+\sum_{1\le i\ne j\le n}(\sigma_{ij}^{n+1})^2+\sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(\sigma_{ij}^{r})^2 - \sum_{2\le j\ne k\le n_1}\sigma_{jj}^{n+1}\sigma_{kk}^{n+1}-\sum_{n_1+1\le s\ne t\le n}\sigma_{ss}^{n+1}\sigma_{tt}^{n+1}\Big].$$

Hence, a_1, a_2 and a_3 satisfy the assumption of Chen's Lemma (for $n=3$), which implies that

$$\Big(\sum_{i=1}^{3}a_i\Big)^2=2\Big(b+\sum_{i=1}^{3}a_i^2\Big)$$

with

$$b=\delta+\sum_{i=1}^{3}a_i^2+\sum_{1\le i\ne j\le n}(\sigma_{ij}^{n+1})^2+\sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(\sigma_{ij}^{r})^2 - \sum_{2\le j\ne k\le n_1}\sigma_{jj}^{n+1}\sigma_{kk}^{n+1}-\sum_{n_1+1\le s\ne t\le n}\sigma_{ss}^{n+1}\sigma_{tt}^{n+1}.$$

Then we get $2a_1a_2\ge b$, with equality holding $a_1+a_2=a_3$. Equivalently

$$\sum_{1\le j<k\le n_1}\sigma_{jj}^{n+1}\sigma_{kk}^{n+1}+\sum_{n_1+1\le s<t\le n}\sigma_{ss}^{n+1}\sigma_{tt}^{n+1} \ge\frac{\delta}{2}+\sum_{1\le\alpha<\beta\le n}(\sigma_{\alpha\beta}^{n+1})^2+\frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{\alpha,\beta=1}^{n}(\sigma_{\alpha\beta}^{r})^2. \tag{3.4}$$

Equality holds if and only if

$$\sum_{i=1}^{n_1}\sigma_{ii}^{n+1}=\sum_{t=n_1+1}^{n}\sigma_{tt}^{n+1}.$$

By making use of the Gauss equation again, we have

$$\begin{aligned} n_2\frac{\Delta_1 f_1}{f_1}+n_1\frac{\Delta_2 f_2}{f_2} &=\tau-\sum_{1\le j<k\le n_1}K(e_j\wedge e_k)-\sum_{n_1+1\le s<t\le n}K(e_s\wedge e_t)\\ &=\tau-\widetilde{\tau}(D_1)-\sum_{r=n+1}^{2m+1}\sum_{1\le j<k\le n_1}(\sigma_{jj}^r\sigma_{kk}^r-(\sigma_{jk}^r)^2)\\ &\quad-\widetilde{\tau}(D_2)-\sum_{r=n+1}^{2m+1}\sum_{n_1+1\le s<t\le n}(\sigma_{ss}^r\sigma_{tt}^r-(\sigma_{st}^r)^2).\end{aligned} \tag{3.5}$$

In view of the equations (1.1), (3.4) and (3.5) we obtain

$$
\begin{aligned}
(3.6)\quad & n_2\frac{\Delta_1 f_1}{f_1}+n_1\frac{\Delta_2 f_2}{f_2} \\
&\leq \tau-\widetilde{\tau}(TM)+\sum_{1\leq s\leq n_1}\sum_{n_1+1\leq t\leq n}\widetilde{K}(e_s\wedge e_t) \\
&\quad -\frac{\delta}{2}-\sum_{1\leq j<t\leq n}(\sigma_{jt}^{n+1})^2-\frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{\alpha,\beta=1}^{n}(\sigma_{\alpha\beta}^{r})^2 \\
&\quad +\sum_{r=n+2}^{2m+1}\sum_{1\leq j<k\leq n_1}((\sigma_{jk}^{r})^2-\sigma_{jj}^{r}\sigma_{kk}^{r})+\sum_{r=n+2}^{2m+1}\sum_{n_1+1\leq s<t\leq n}((\sigma_{st}^{r})^2-\sigma_{ss}^{r}\sigma_{tt}^{r}) \\
&=\tau-\widetilde{\tau}(TM)+\sum_{1\leq s\leq n_1}\sum_{n_1+1\leq t\leq n}\widetilde{K}(e_s\Lambda e_t)-\frac{\delta}{2}-\sum_{r=n+1}^{2m+1}\sum_{j=1}^{n_1}\sum_{t=n_1+1}^{n}(\sigma_{jt}^{r})^2 \\
&\quad -\frac{1}{2}\sum_{r=n+2}^{2m+1}\Big(\sum_{j=1}^{n_1}\sigma_{jj}^{r}\Big)^2-\frac{1}{2}\sum_{r=n+2}^{2m+1}\Big(\sum_{t=n_1+1}^{n}\sigma_{tt}^{r}\Big)^2.
\end{aligned}
$$

Applying (3.2) in (3.6) we get

$$
\begin{aligned}
(3.7)\quad & n_2\frac{\Delta_1 f_1}{f_1}+n_1\frac{\Delta_2 f_2}{f_2} \\
&\leq \frac{n^2}{4}\|H\|^2-\widetilde{\tau}(TM)+\sum_{1\leq s\leq n_1}\sum_{n_1+1\leq t\leq n}\widetilde{K}(e_s\wedge e_t) \\
&\quad -\sum_{r=n+1}^{2m+1}\sum_{j=1}^{n_1}\sum_{t=n_1+1}^{n}(\sigma_{jt}^{r})^2-\frac{1}{2}\sum_{r=n+2}^{2m+1}\Big(\sum_{j=1}^{n_1}\sigma_{jj}^{r}\Big)^2-\frac{1}{2}\sum_{r=n+2}^{2m+1}\Big(\sum_{t=n_1+1}^{n}\sigma_{tt}^{r}\Big)^2.
\end{aligned}
$$

On the other hand, from (2.1) we can write the sectional curvature of $\widetilde{M}(c)$ as follows:

$$
\begin{aligned}
(3.8)\quad \widetilde{K}(e_i\wedge e_j)=\frac{c+3}{4}&+g(h^Te_i,e_i)+g(h^Te_j,e_j) \\
&+\frac{1}{2}\{g(h^Te_i,e_i)g(h^Te_j,e_j)-g(h^Te_i,e_j)^2 \\
&-g(A_\xi e_i,e_i)g(A_\xi e_j,e_j)+g(A_\xi e_i,e_j)^2\}
\end{aligned}
$$

(see equation (4.3) in [Tri]). Then, using (3.8) in (3.7), we obtain the inequality (3.1). ∎

Taking $h=0$ in (3.1), we obtain the following corollary:

COROLLARY 3.2 ([Ol]). *Let $M={}_{f_2}M_1\times{}_{f_1}M_2$ be an n-dimensional C-totally real doubly warped product submanifold of a Sasakian space form $\widetilde{M}(c)$. Then*

$$n_2\frac{\Delta_1 f_1}{f_1}+n_1\frac{\Delta_2 f_2}{f_2}\le\frac{n^2}{4}\|H\|^2+\frac{n_1n_2}{4}(c+3), \tag{3.9}$$

where $n_i=\dim M_i$, $n=n_1+n_2$ *and* Δ_i *is the Laplacian of* M_i, $i=1,2$. *Equality holds in* (3.9) *identically if and only if* M *is mixed totally geodesic and* $n_1H_1=n_2H_2$, *where* H_i, $i=1,2$, *are the partial mean curvature vectors.*

Similarly, we establish a sharp inequality for C-totally real doubly warped product submanifolds of non-Sasakian (κ,μ)-contact metric manifolds in the following theorem:

THEOREM 3.3. *Let* $M={}_{f_2}M_1\times_{f_1}M_2$ *be an* n*-dimensional* C*-totally real doubly warped product submanifold of a* $(2m+1)$*-dimensional non-Sasakian* (κ,μ)*-contact metric manifold* $\widetilde{M}$. *Then*

$$\begin{aligned}
n_2\frac{\Delta_1 f_1}{f_1}&+n_1\frac{\Delta_2 f_2}{f_2}\\
&\le\frac{n^2}{4}\|H\|^2+\frac{n_1n_2}{4}\left(1-\frac{\mu}{2}\right)\\
&+n_2\,\mathrm{trace}(h^T_{|M_1})+n_1\,\mathrm{trace}(h^T_{|M_2})\\
&+\frac{1}{2}\,\frac{1-\mu/2}{1-\kappa}\{(\mathrm{trace}(h^T))^2-(\mathrm{trace}(h^T_{|M_1}))^2-(\mathrm{trace}(h^T_{|M_2}))^2\}\\
&-\frac{1}{2}\,\frac{\kappa-\mu/2}{1-\kappa}\{(\mathrm{trace}(A_\xi))^2-(\mathrm{trace}(A_{\xi|M_1}))^2-(\mathrm{trace}(A_{\xi|M_2}))^2\}\\
&-\frac{1}{2}\,\frac{1-\mu/2}{1-\kappa}\{\|h^T\|^2-\|h^T_{|M_1}\|^2-\|h^T_{|M_2}\|^2\}\\
&+\frac{1}{2}\,\frac{\kappa-\mu/2}{1-\kappa}\{\|A_\xi\|^2-\|A_{\xi|M_1}\|^2-\|A_{\xi|M_2}\|^2\},
\end{aligned}\tag{3.10}$$

where $n_i=\dim M_i$, $n=n_1+n_2$ *and* Δ_i *is the Laplacian of* M_i, $i=1,2$. *Equality holds in* (3.10) *identically if and only if* $M={}_{f_2}M_1\times_{f_1}M_2$ *is mixed totally geodesic and* $n_1H_1=n_2H_2$, *where* H_i, $i=1,2$, *are the partial mean curvature vectors.*

Proof. We choose a local orthonormal frame $\{e_1,\dots,e_{n_1},e_{n_1+1},\dots,e_n\}$ such that $e_1,\dots,e_{n_1}$ are tangent to M_1, $e_{n_1+1},\dots,e_n$ are tangent to M_2 and e_{n+1} is parallel to the mean curvature vector H. Then from equation (2.2) we have

$$\begin{aligned}
\widetilde{K}(e_i\wedge e_j)=(1-\mu/2)&+g(h^Te_i,e_i)+g(h^Te_j,e_j)\\
&+\frac{1-\mu/2}{1-\kappa}\{g(h^Te_i,e_i)g(h^Te_j,e_j)-g(h^Te_i,e_j)^2\}\\
&+\frac{\kappa-\mu/2}{1-\kappa}\{g(A_\xi e_i,e_i)g(A_\xi e_j,e_j)-g(A_\xi e_i,e_j)^2\}
\end{aligned}\tag{3.11}$$

(see equation (4.9) in [Tri]). Similar to the proof of Theorem 3.1 we obtain (3.7). Then making use of (3.11) in (3.7) we obtain (3.10). ∎

4. Applications. As applications, we derive certain obstructions to the existence of minimal C-totally real doubly warped product submanifolds in (κ,μ)-contact space forms, in non-Sasakian (κ,μ)-contact metric manifolds and in Sasakian space forms.

COROLLARY 4.1. *Let $M = {}_{f_2}M_1 \times_{f_1} M_2$ be a C-totally real doubly warped product manifold. If the warping functions f_1 and f_2 are harmonic, then M admits no minimal immersion into a (κ,μ)-contact space form $\widetilde{M}(c)$ with*

$$\begin{aligned}(4.1)\quad 0 > \frac{n_1 n_2}{4}(c+3) &+ n_2 \operatorname{trace}(h^T_{|M_1}) + n_1 \operatorname{trace}(h^T_{|M_2})\\ &+ \frac{1}{4}\{(\operatorname{trace}(h^T))^2 - (\operatorname{trace}(h^T_{|M_1}))^2 - (\operatorname{trace}(h^T_{|M_2}))^2\\ &- (\operatorname{trace}(A_\xi))^2 + \operatorname{trace}((A_{\xi|M_1}))^2 + \operatorname{trace}((A_{\xi|M_2}))^2\\ &- \|h^T\|^2 + \|h^T_{|M_1}\|^2 + \|h^T_{|M_2}\|^2 + \|A_\xi\|^2 - \|A_{\xi|M_1}\|^2 - \|A_{\xi|M_2}\|^2\}.\end{aligned}$$

Proof. Suppose that f_1 and f_2 are harmonic, and M admits a minimal C-totally real immersion into a (κ,μ)-contact space form $\widetilde{M}(c)$. Then the inequality (3.1) turns into

$$\begin{aligned}0 \le \frac{n_1 n_2}{4}(c+3) &+ n_2 \operatorname{trace}(h^T_{|M_1}) + n_1 \operatorname{trace}(h^T_{|M_2})\\ &+ \frac{1}{4}\{(\operatorname{trace}(h^T))^2 - (\operatorname{trace}(h^T_{|M_1}))^2 - (\operatorname{trace}(h^T_{|M_2}))^2\\ &- (\operatorname{trace}(A_\xi))^2 + (\operatorname{trace}(A_{\xi|M_1}))^2 + (\operatorname{trace}(A_{\xi|M_2}))^2\\ &- \|h^T\|^2 + \|h^T_{|M_1}\|^2 + \|h^T_{|M_2}\|^2\\ &+ \|A_\xi\|^2 - \|A_{\xi|M_1}\|^2 - \|A_{\xi|M_2}\|^2\}.\end{aligned}$$

Thus we obtain the inequality (4.1). ∎

Similar to Corollary 4.1, we can give the following corollary:

COROLLARY 4.2. *Let $M = {}_{f_2}M_1 \times_{f_1} M_2$ be a C-totally real doubly warped product manifold. If the warping functions f_1 and f_2 are harmonic, then ${}_{f_2}M_1 \times {}_{f_1}M_2$ admits no minimal immersion into a (κ,μ)-contact metric manifold $\widetilde{M}$ with*

$$\begin{aligned}0 < \frac{n_1 n_2}{4}\left(1 - \frac{\mu}{2}\right) &+ n_2 \operatorname{trace}(h^T_{|M_1}) + n_1 \operatorname{trace}(h^T_{|M_2})\\ &+ \frac{1}{2}\,\frac{1-\mu/2}{1-\kappa}\{(\operatorname{trace}(h^T))^2 - (\operatorname{trace}(h^T_{|M_1}))^2 - (\operatorname{trace}(h^T_{|M_2}))^2\}\\ &- \frac{1}{2}\,\frac{\kappa-\mu/2}{1-\kappa}\{(\operatorname{trace}(A_\xi))^2 - (\operatorname{trace}(A_{\xi|M_1}))^2 - (\operatorname{trace}(A_{\xi|M_2}))^2\}\end{aligned}$$

$$-\frac{1}{2}\frac{1-\mu/2}{1-\kappa}\{\|h^T\|^2-\|h^T_{|M_1}\|^2-\|h^T_{|M_2}\|^2\}$$
$$+\frac{1}{2}\frac{\kappa-\mu/2}{1-\kappa}\{\|A_\xi\|^2-\|A_{\xi|_{M_1}}\|^2-\|A_{\xi|_{M_2}}\|^2\}.$$

If $h=0$ in Corollary 4.1, we have the following corollaries:

COROLLARY 4.3 ([Ol]). *If the warping functions f_1 and f_2 are harmonic, then ${}_{f_2}M_1\times{}_{f_1}M_2$ admits no minimal C-totally real immersion into a Sasakian space form $\widetilde{M}(c)$ with $c<-3$.*

COROLLARY 4.4 ([Ol]). *If the warping functions f_1 and f_2 are eigenfunctions of the Laplacian on M_1 and M_2, respectively, with positive eigenvalues, then ${}_{f_2}M_1\times{}_{f_1}M_2$ admits no minimal C-totally real immersion into a Sasakian space form $\widetilde{M}(c)$ with $c\le-3$.*

COROLLARY 4.5 ([Ol]). *If one of the warping functions f_1 and f_2 is harmonic and the other one is an eigenfunction of the Laplacian with a positive eigenvalue, then ${}_{f_2}M_1\times{}_{f_1}M_2$ admits no minimal C-totally real immersion into a Sasakian space form $\widetilde{M}(c)$ with $c\le-3$.*

Acknowledgements. The authors are grateful to the referee for his valuable comments towards the improvement of the paper.

References

[Bl-2002] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Progr. Math. 203, Birkhäuser Boston, Boston, MA, 2002.

[BKP] D. E. Blair, T. Koufogiorgos and B. J. Papantoniou, *Contact metric manifolds satisfying a nullity condition*, Israel J. Math. 91 (1995), 189–214.

[Bo-1999] E. Boeckx, *A class of locally φ-symmetric contact metric spaces*, Arch. Math. (Basel) 72 (1999), 466–472.

[Bo-2000] —, *A full classification of contact metric (κ,μ)-spaces*, Illinois J. Math. 44 (2000), 212–219.

[Ch-1973] B. Y. Chen, *Geometry of Submanifolds*, Pure Appl. Math. 22, Dekker, New York, 1973.

[Ch-1993] —, *Some pinching and classification theorems for minimal submanifolds*, Arch. Math. (Basel) 60 (1993), 568–578.

[Ch-2000] —, *Some new obstructions to minimal and Lagrangian, isometric immersions*, Japan J. Math. 26 (2000), 105–127.

[Ch-2002] —, *On isometric minimal immersions from warped products into real space forms*, Proc. Edinburgh Math. Soc. 45 (2002), 579–587.

[ChDi] B. Y. Chen and F. Dillen, *Optimal inequalities for multiply warped product submanifolds*, Int. Electron. J. Geom. 1 (2008), no. 1, 1–11.

[Kou] T. Koufogiorgos, *Contact Riemannian manifolds with constant φ-sectional curvature*, Tokyo J. Math. 20 (1997), 13–22.

[MM] K. Matsumoto and I. Mihai, *Warped product submanifolds in Sasakian space forms*, SUT J. Math. 38 (2002), 135–144.

[Mi-2004] A. Mihai, *Warped product submanifolds in complex space forms*, Acta Sci. Math. (Szeged) 70 (2004), 419–427.

[Mi-2005] —, *Warped product submanifolds in quaternion space forms*, Rev. Roumaine Math. Pures Appl. 50 (2005), 283–291.

[MAEM] C. Murathan, K. Arslan, R. Ezentas and I. Mihai, *Warped product submanifolds in Kenmotsu space forms*, Taiwanese J. Math. 10 (2006), 1431–1441.

[Ol] A. Olteanu, *A general inequality for doubly warped product submanifolds*, Math. J. Okayama Univ. 52 (2010), 133–142.

[Tri] M. M. Tripathi, *C-totally real warped product submanifolds*, arXiv:0806.0201.

[Ün] B. Ünal, *Doubly warped products*, PhD Thesis, Univ. of Missouri-Columbia, 2000.

[YK] K. Yano and M. Kon, *Anti-invariant Submanifolds*, Lecture Notes in Pure Appl. Math. 21. Dekker, New York, 1976.

Sibel Sular, Cihan Özgür
Department of Mathematics
Balikesir University
10145, Çağış, Balikesir, Turkey
E-mail: csibel@balikesir.edu.tr
cozgur@balikesir.edu.tr

Received 2.2.2010
and in final form 9.12.2010 (2159)