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Implicit difference schemes for mixed problems related to parabolic functional differential equations

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Abstract. Solutions of initial boundary value problems for parabolic functional differential equations are approximated by solutions of implicit difference schemes. The existence and uniqueness of approximate solutions is proved. The proof of the stability is based on a comparison technique with nonlinear estimates of the Perron type for given operators. It is shown that the new methods are considerably better than the explicit difference schemes. Numerical examples are presented.

1. Introduction. For any metric spaces X and Y we denote by $C(X,Y)$ the class of all continuous functions defined on X and taking values in Y. We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components. Write

$$E_0 = [-b_0, 0] \times [-b, b], \qquad E = [0, a] \times [-b, b],$$

where $a > 0$, $b_0 \in \mathbb{R}_+$, $\mathbb{R}_+ = (0, \infty)$ and $b = (b_1, \ldots, b_n)$, $b_i > 0$ for $i = 1, \ldots, n$. Suppose that $\phi_0 : E \to \mathbb{R}$ and $\phi = (\phi_1, \ldots, \phi_n) : E \to \mathbb{R}^n$ are given functions. Write $\varphi(t,x) = (\phi_0(t,x), \phi(t,x))$ for $(t,x) \in E$. We assume that $0 \le \phi_0(t,x) \le t$ for $(t,x) \in E$ and $-b \le \phi(t,x) \le b$ for $(t,x) \in E$. For $(t,x) \in [0,a] \times [-b,b]$ we define

$$D[t,x] = \{(\tau, y) \in \mathbb{R}^{1+n} : \tau \le 0,\ (t+\tau, x+y) \in E_0 \cup E\}.$$

It is clear that $D[t,x] = [-b_0 - t, 0] \times [-b-x, b-x]$. For a function $z : E_0 \cup E \to \mathbb{R}$ and a point $(t,x) \in [0,a] \times [-b,b]$ we define a function $z_{(t,x)} : D[t,x] \to \mathbb{R}$ as follows:

$$z_{(t,x)}(\tau, y) = z(t+\tau, x+y), \qquad (\tau, y) \in D[t,x].$$

Then $z_{(t,x)}$ is the restriction of z to the set $(E_0 \cup E) \cap ([-b_0, t] \times \mathbb{R}^n)$ and this restriction is shifted to the set $D[t,x]$. Write $I = [-b_0 - a, 0]$ and $B = I \times [-2b, 2b]$. Then $D[t,x] \subset B$ for $(t,x) \in [0,a] \times [-b,b]$. Let $M_{n\times n}$ be the

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class of all $n \times n$ symmetric matrices with real entries. Put $\Xi = E \times \mathbb{R} \times C(B,\mathbb{R}) \times \mathbb{R}^n \times M_{n\times n}$ and suppose that $F : \Xi \to \mathbb{R}$ is a given function. We consider the functional differential equation

$$\partial_t z(t,x) = F(t,x,z(t,x),z_{\varphi(t,x)},\partial_x z(t,x),\partial_{xx} z(t,x)) \tag{1.1}$$

where $\partial_x z = (\partial_{x_1} z,\ldots,\partial_{x_n} z)$, $\partial_{xx} z = [\partial_{x_i x_j} z]_{i,j=1,\ldots,n}$. Now we formulate initial boundary conditions for (1.1). Write

$$S_i = \{x \in [-b,b] : x_i = b_i\}, \quad S_{n+i} = \{x \in [-b,b] : x_i = -b_i\}, \quad i = 1,\ldots,n,$$

and

$$Q_1^+ = S_1, \qquad Q_i^+ = S_i \setminus \bigcup_{j=1}^{i-1} S_j, \qquad Q_i^- = S_{n+i} \setminus \bigcup_{j=1}^{n+i-1} S_j, \qquad i = 1,\ldots,n.$$

Set

$$\partial_0 E_i^+ = [0,a] \times Q_i^+, \qquad \partial_0 E_i^- = [0,a] \times Q_i^-, \qquad i = 1,\ldots,n,$$

and

$$\partial_0 E = \bigcup_{i=1}^{n} (\partial_0 E_i^+ \cup \partial_0 E_i^-).$$

Suppose that $\beta,\gamma,\Psi : \partial_0 E \to \mathbb{R}$, $\psi : E_0 \to \mathbb{R}$ are given functions. The following initial boundary conditions are associated with (1.1):

$$z(t,x) = \psi(t,x) \quad \text{on } E_0, \tag{1.2}$$

$$\beta(t,x)z(t,x) + \gamma(t,x)\partial_{x_i} z(t,x) = \Psi(t,x) \quad \text{on } \partial_0 E_i^+,\ i = 1,\ldots,n, \tag{1.3}$$

$$\beta(t,x)z(t,x) - \gamma(t,x)\partial_{x_i} z(t,x) = \Psi(t,x) \quad \text{on } \partial_0 E_i^-,\ i = 1,\ldots,n. \tag{1.4}$$

We are interested in establishing a method of numerical approximation of solutions to (1.1)–(1.4) by means of solutions of an associated implicit difference schemes and in estimating the difference between the exact and approximate solutions.

Difference schemes for parabolic functional differential equations are obtained by replacing partial derivatives with difference operators. Moreover, because solutions of difference equations are defined on the mesh and equation (1.1) contains the functional variable $z_{\varphi(t,x)}$ which is an element of the space $C(D[\varphi(t,x)],\mathbb{R})$, some interpolating operators are needed. This leads to difference functional problems which satisfy consistency conditions on classical solutions of original problems. The main question in these considerations is to find sufficient conditions for the stability of difference functional schemes.

There are the following motivations for the construction of implicit difference schemes related to (1.1)–(1.4). Two types of assumptions are needed in theorems on the stability of explicit difference schemes generated by (1.1)–(1.4). The first type conditions concern regularity of F. It is assumed that

(i) the function F of the variables (t, x, p, w, q, s), $q = (q_1, \ldots, q_n)$, $s = [s_{ij}]_{i,j=1}^n$, is of class C^1 with respect to (q, s) and the functions

$$\partial_q F = (\partial_{q_1} F, \ldots, \partial_{q_n} F), \qquad \partial_s F = [\partial_{s_{ij}} F]_{i,j=1,\ldots,n},$$

are bounded.

(ii) F satisfies global estimates of the Perron type with respect to (p, w).

The second type conditions concern the mesh. It is required that difference schemes corresponding to (1.1) satisfy the condition

$$(1.5) \qquad 1 - 2h_0 \sum_{i=1}^{n} \frac{1}{h_i^2} \partial_{s_{ii}} F(P) + h_0 \sum_{\substack{i,j=1 \\ j \neq i}}^{n} \frac{1}{h_i h_j} |\partial_{s_{ij}} F(P)| \geq 0, \qquad P \in \Xi,$$

where h_0 and $(h_1, \ldots, h_n)$ are steps of the mesh with respect to t and $(x_1, \ldots, x_n)$, respectively. Note that if equation (1.1) has the form

$$\partial_t z(t,x) = \sum_{i=1}^{n} \partial_{x_i x_i} z(t,x) + f(t, x, z(t,x), z_{\varphi(t,x)}, \partial_x z(t,x))$$

where $f : E \times \mathbb{R} \times C(B, \mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}$ then condition (1.5) has the form

$$1 - 2h_0 \sum_{i=1}^{n} \frac{1}{h_i^2} > 0.$$

It is clear that strong assumptions on relations between h_0 and $(h_1, \ldots, h_n)$ are required in (1.5).

The aim of the paper is to show that there are difference schemes for (1.1)–(1.4) such that assumption (1.5) can be omitted in the theorem on the convergence of the difference method.

In recent years, a number of papers concerning numerical methods for parabolic equations have been published. Difference approximations of nonlinear equations with initial boundary conditions of the Dirichlet type were considered in [2], [18]. The results and methods presented in the above papers were extended in [4], [9], [16], [17] to functional differential problems. Numerical treatment of functional differential equations with initial boundary conditions of the Neumann type can be found in [7], [10]. In those papers explicit difference schemes were considered. A comparison technique is used in the investigations of the stability of nonlinear difference functional problems. Various monotone iterative methods and finite difference schemes for computing of numerical solutions of reaction diffusion equations with time delay were studied in [13], [14], [19].

Implicit difference schemes for parabolic functional differential equations were first treated in [6], [8] (see also [11]). Nonlinear equations with initial boundary conditions of the Dirichlet type and mixed problems of the Neumann type were considered.

In this paper we start the investigation of implicit difference methods for the first mixed problem related to nonlinear parabolic functional differential equations. Our results are new also in the case of differential equations without functional dependence. The paper is a continuation of [4], [6], [8], [12], [16] and it generalizes some results of those papers.

There are the following differences between known theorems and our results.

1. Difference schemes for parabolic equations with initial boundary conditions of the Neumann type have the following property. Besides mesh points $(t^{(r)}, x^{(m)}) \in \mathbb{R}^{1+n}$ satisfying the condition $(t^{(r)}, x^{(m)}) \in E$ some additional mesh points are needed. More precisely, an approximate solution must be defined at some points $(t^{(r)}, x^{(m)})$ such that $(t^{(r)}, x^{(m)}) \notin E$ (see [7], [8]). In this approach, the error of an implicit difference scheme is estimated by $C\sqrt{\|h\|}$, where $C > 0$. We show that under natural assumptions on given functions and on the mesh the error of implicit difference schemes is estimated by $C\|h\|$.

2. It is usually assumed that the function F of the variables (t, x, p, w, q, s) satisfies nonlinear estimates of the Perron type with respect to (p, w) and the assumption is global. Our conditions are more general. We prove that solutions of difference schemes are bounded uniformly with respect to h and we find estimates of classical solutions to (1.1)–(1.4). Then we assume the local Perron type condition for F with respect to (p, w). It is clear that there are differential equations with deviated variables and differential integral equations such that local estimates with respect to (p, w) hold and global conditions required in [4], [7]–[9], [16] are not satisfied.

Differential equations with deviated variables and differential integral problems are particular cases of (1.1)–(1.4). Existence results for parabolic functional differential problems were discussed in [1], [5], [15]. Information on applications of functional differential equations can be found in [20].

2. Implicit difference schemes. A function $z : E_0 \cup E \to \mathbb{R}$ will be called *of class* C_* if z is continuous on $E_0 \cup E$, the partial derivatives $\partial_t z$, $\partial_x z = (\partial_{x_1} z, \ldots, \partial_{x_n} z)$, $\partial_{xx} z = [\partial_{x_i x_j} z]_{i,j=1,\ldots,n}$ exist on E and the functions $\partial_t z, \partial_x z, \partial_{xx} z$ are continuous on E. We consider solutions of (1.1)–(1.4) of class C_*. We will say that a function $F : \Xi \to \mathbb{R}$ satisfies *condition* (V) if for all $(t, x, p, w, q, s) \in \Xi$ and $\bar{w} \in C(B, \mathbb{R})$ such that $w(\tau, y) = \bar{w}(\tau, y)$ for $(\tau, y) \in D[\varphi(t, x)]$ we have $F(t, x, p, w, q, s) = F(t, x, p, \bar{w}, q, s)$. Condition (V) for F means that the value of F at $(t, x, p, w, q, s) \in \Xi$ depends on (t, x, p, q, s) and on the restriction of w to the set $D[\varphi(t, x)]$ only. Suppose that $z \in C(E_0 \cup E, \mathbb{R})$ and that F satisfies condition (V). Then the values $F(t, x, p, z_{\varphi(t,x)}, q, s)$ exist for $(t, x, p, q, s) \in E \times \mathbb{R} \times \mathbb{R}^n \times M_{n\times n}$.

For $x \in \mathbb{R}^n,\ U \in M_{n\times n}$ where $U = [u_{ij}]_{i,j=1}^n$ we write

$$\|x\| = \sum_{i=1}^n |x_i|, \qquad \|U\|_\infty = \max\Big\{\sum_{j=1}^n |u_{ij}| : 1 \le i \le n\Big\}.$$

For any sets X and Y we denote by $\mathcal{F}(X,Y)$ the class of all functions defined on X and taking values in Y. Let $\mathbb{N}$ and $\mathbb{Z}$ be the sets of natural numbers and integers, respectively. We define a mesh in $\mathbb{R}^{1+n}$ in the following way. Let $\mathbf{h} = (h_0, h),\ h = (h_1,\dots,h_n)$, stand for steps of the mesh. For $(r,m) \in \mathbb{Z}^{1+n},\ m = (m_1,\dots,m_n)$, we define nodal points as follows:

$$t^{(r)} = rh_0, \qquad x^{(m)} = (m_1h_1,\dots,m_nh_n) = (x_1^{(m_1)},\dots,x_n^{(m_n)}).$$

Let us denote by $\mathbb{H}$ the set of all $\mathbf{h}$ for which there exist $(M_1,\dots,M_n) = M \in \mathbb{Z}^n$ and $M_0 \in \mathbb{Z}$ such that $M_ih_i = b_i$ for $i = 1,\dots,n$, $M_0h_0 = b_0$. Let $K \in \mathbb{N}$ be defined by the relations $Kh_0 \le a < (K+1)h_0$. For $\mathbf{h} \in \mathbb{H}$ we put

$$\mathbb{R}^{1+n}_{\mathbf{h}} = \{(t^{(r)}, x^{(m)}) : (r,m) \in \mathbb{Z}^{1+n}\}$$

and

$$E_{0.\mathbf{h}} = E_0 \cap \mathbb{R}^{1+n}_{\mathbf{h}}, \qquad E_{\mathbf{h}} = E \cap \mathbb{R}^{1+n}_{\mathbf{h}}, \qquad B_{\mathbf{h}} = B \cap \mathbb{R}^{1+n}_{\mathbf{h}},$$
$$\partial_0 E^+_{\mathbf{h}.i} = \partial_0 E^+_i \cap \mathbb{R}^{1+n}_{\mathbf{h}}, \qquad \partial_0 E^-_{\mathbf{h}.i} = \partial_0 E^+_i \cap \mathbb{R}^{1+n}_{\mathbf{h}},$$
$$D_{\mathbf{h}}[r,m] = D[t^{(r)}, x^{(m)}] \cap \mathbb{R}^{1+n}_{\mathbf{h}},$$
$$\Xi_{\mathbf{h}} = E_{\mathbf{h}} \times \mathbb{R} \times C(B,\mathbb{R}) \times \mathbb{R}^n \times M_{n\times n}.$$

Difference operators are defined in the following way. Let $e_i \in \mathbb{R}^n$ be the canonical unit vectors. Write $J = \{(i,j) :\ i,j = 1,\dots,n,\ i \ne j\}$. Suppose that we have defined sets $J_+, J_- \subset J$ such that $J_+ \cup J_- = J,\ J_+ \cap J_- = \emptyset$. We assume that $(i,j) \in J_+$ if $(j,i) \in J_+$. In particular, it may happen that $J_+ = \emptyset$ or $J_- = \emptyset$. Relations between the sets $J_+,\ J_-$ and equation (1.1) are given in Section 3.

Given $z \in \mathcal{F}(E_{0.\mathbf{h}} \cup E_{\mathbf{h}}, \mathbb{R})$ and $(r,m) \in \mathbb{Z}^{1+n}$, $0 \le r \le K-1,\ -(M-1) \le m \le M-1$, where $M-1 = (M_1-1,\dots,M_n-1)$, write

$$\delta_0 z^{(r,m)} = \frac{1}{h_0}[z^{(r+1,m)} - z^{(r,m)}],$$
$$\delta_i^+ z^{(r,m)} = \frac{1}{h_i}[z^{(r,m+e_i)} - z^{(r,m)}],$$
$$\delta_i^- z^{(r,m)} = \frac{1}{h_i}[z^{(r,m)} - z^{(r,m-e_i)}], \qquad i = 1,\dots,n,$$

and $\delta z^{(r,m)} = (\delta_1 z^{(r,m)},\dots,\delta_n z^{(r,m)})$ where

$$\delta_i z^{(r,m)} = \frac{1}{2}[\delta_i^+ z^{(r,m)} + \delta_i^- z^{(r,m)}], \qquad i = 1,\dots,n.$$

The difference operator $\delta^{(2)} = [\delta_{ij}]_{i,j=1}^{n}$ is defined in the following way:

$$\delta_{ii}^{(2)} z^{(r,m)} = \delta_i^+ \delta_i^- z^{(r,m)} \quad \text{for } i = 1, \ldots, n$$

and

$$\delta_{ij}^{(2)} z^{(r,m)} = \frac{1}{2}[\delta_i^+ \delta_j^- z^{(r,m)} + \delta_i^- \delta_j^+ z^{(r,m)}] \quad \text{for } (i,j) \in J_-,$$
$$\delta_{ij}^{(2)} z^{(r,m)} = \frac{1}{2}[\delta_i^+ \delta_j^+ z^{(r,m)} + \delta_i^- \delta_j^- z^{(r,m)}] \quad \text{for } (i,j) \in J_+.$$

Solutions of difference functional equations are elements of the space $\mathcal{F}(E_{0.\mathbf{h}} \cup E_{\mathbf{h}}, \mathbb{R})$. Since equation (1.1) contains the functional variable $z_{\varphi(t,x)}$ which is an element of the space $C(D[\varphi(t,x)], \mathbb{R})$, we need an interpolating operator $T_{\mathbf{h}} : \mathcal{F}(E_{0.\mathbf{h}} \cup E_{\mathbf{h}}, \mathbb{R}) \to C(E_0 \cup E, \mathbb{R})$. We adopt additional assumptions on $T_{\mathbf{h}}$ in Section 3. Set

$$F_{\mathbf{h}}[z]^{(r,m)} = F(t^{(r)}, x^{(m)}, z^{(r,m)}, (T_{\mathbf{h}} z)_{\varphi^{(r,m)}}, \delta z^{(r+1,m)}, \delta^{(2)} z^{(r+1,m)})$$

and $\Lambda_{\mathbf{h}}[z]^{(r,m)} = (\Lambda_{\mathbf{h}}^+[z]^{(r,m)}, \Lambda_{\mathbf{h}}^-[z]^{(r,m)}$, where

$$\Lambda_{\mathbf{h}}^+[z]^{(r,m)} = (\Lambda_{\mathbf{h}.1}^+[z]^{(r,m)}, \ldots, \Lambda_{\mathbf{h}.n}^+[z]^{(r,m)}),$$
$$\Lambda_{\mathbf{h}}^-[z]^{(r,m)} = (\Lambda_{\mathbf{h}.1}^-[z]^{(r,m)}, \ldots, \Lambda_{\mathbf{h}.n}^-[z]^{(r,m)})$$

and

$$\Lambda_{\mathbf{h}.i}^+[z]^{(r,m)} = \beta^{(r,m)} z^{(r,m)} + \gamma^{(r,m)} \delta_i^- z^{(r,m)} \quad \text{on } \partial_0 E_{\mathbf{h}.i}^+,$$
$$\Lambda_{\mathbf{h}.i}^-[z]^{(r,m)} = \beta^{(r,m)} z^{(r,m)} - \gamma^{(r,m)} \delta_i^+ z^{(r,m)} \quad \text{on } \partial_0 E_{\mathbf{h}.i}^-,$$

where $i = 1, \ldots, n$.

Given $\Psi_{\mathbf{h}} : \partial_0 E_{\mathbf{h}} \to \mathbb{R}$ and $\psi_{\mathbf{h}} : E_{0.\mathbf{h}} \to \mathbb{R}$, we consider the difference functional equation

$$\delta_0 z^{(r,m)} = F_{\mathbf{h}}[z]^{(r,m)} \tag{2.1}$$

with the initial boundary conditions

$$z^{(r,m)} = \psi_{\mathbf{h}}^{(r,m)} \quad \text{on } E_{0.\mathbf{h}}, \tag{2.2}$$

$$\Lambda_{\mathbf{h}}[z]^{(r,m)} = \Psi_{\mathbf{h}}^{(r,m)} \quad \text{on } \partial_0 E_{\mathbf{h}}. \tag{2.3}$$

Note that the values $z^{(r+1,m+\lambda)}$ where $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\lambda_i \in \{-1, 0, 1\}$ for $i = 1, \ldots, n$ and $\|\lambda\| \le 2$ appear in the expressions $\delta z^{(r+1,m)}$ and $\delta^{(2)} z^{(r+1,m)}$. Then (2.1)–(2.3) is an implicit difference scheme for (1.1)–(1.4).

If we put $z^{(r+1,m)}$ and $(T_{\mathbf{h}} z)_{\varphi^{(r+1,m)}}$ instead of $z^{(r,m)}$ and $(T_{\mathbf{h}} z)_{\varphi^{(r,m)}}$ in the definition of $F_{\mathbf{h}}[z]$ then we obtain the following implicit difference method:

$$\delta_0 z^{(r,m)} = F(t^{(r)}, x^{(m)}, z^{(r+1,m)}, (T_{\mathbf{h}} z)_{\varphi^{(r+1,m)}}, \delta z^{(r+1,m)}, \delta^{(2)} z^{(r+1,m)}). \tag{2.4}$$

We do not discuss implicit difference schemes (2.4), (2.2), (2.3) in this paper. The papers [13], [14], [19] contain results on almost linear parabolic equations with a delay time variable and implicit difference equation of the form (2.4).

3. Solutions of difference functional problems. We give sufficient conditions for the existence and uniqueness of a solution to (2.1)–(2.3).

ASSUMPTION $\mathbf{H}_*[F]$. The function $F : \Xi \to \mathbb{R}$ of the variables (t, x, p, w, q, s), $q = (q_1, \dots, q_n)$, $s = [s_{ij}]_{i,j=1}^n$, satisfies condition (V) and

1) F is continuous, the derivatives
$$\partial_q F = (\partial_{q_1} F, \dots, \partial_{q_n} F), \qquad \partial_s F = [\partial_{s_{ij}} F]_{i,j=1}^n$$
exist and the functions $\partial_q F : \Xi \to \mathbb{R}^n$, $\partial_s F : \Xi \to M_{n\times n}$ are continuous and bounded,

2) the matrix $\partial_s F$ is symmetric and

$$\sum_{i,j=1}^n \partial_{s_{ij}} F(P)\lambda_i\lambda_j \ge 0 \quad \text{for } P \in \Xi,\ \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n, \tag{3.1}$$

$$\partial_{s_{ij}} F(P) \ge 0 \text{ for } (i,j) \in J_+, \qquad \partial_{s_{ij}} F(P) \le 0 \text{ for } (i,j) \in J_-, \tag{3.2}$$

where $P \in \Xi$,

3) the steps of the mesh satisfy the conditions

$$-\frac{1}{2}\,|\partial_{q_i} F(P)| + \frac{1}{h_i}\partial_{s_{ii}} F(P) - \sum_{\substack{j=1\\ j\ne i}}^n \frac{1}{h_j}\,\big|\partial_{s_{ij}} F(P)\big| \ge 0, \qquad i = 1, \dots, n, \tag{3.3}$$

where $P \in \Xi$.

REMARK 3.1. We have assumed that for each $(i,j) \in J$, the function
$$g_{ij}(P) = \operatorname{sign} \partial_{s_{ij}} F(P), \qquad P \in \Xi,$$
is constant. Relations (3.2) can be considered as definitions of the sets J_+ and J_-. For the symmetric matrix $\partial_s F$ the following condition is satisfied: $(j,i) \in J_+$ if $(i,j) \in J_+$.

REMARK 3.2. Suppose that there is $\tilde{c} > 0$ such that
$$\partial_{s_{ii}} F(P) - \sum_{\substack{j=1\\ j\ne i}}^n |\partial_{s_{ij}} F(P)| \ge \tilde{c}, \qquad P \in \Xi,\ i = 1, \dots, n.$$
Then condition (3.1) is satisfied (see [18]) and there is $\varepsilon_0 > 0$ such that inequalities (3.3) hold for $\|h\| < \varepsilon_0$ and for $h_1 = \dots = h_n$.

For $z \in C(E_0 \cup E, \mathbb{R})$ and $u \in \mathcal{F}(E_{0.\mathbf{h}} \cup E_{\mathbf{h}}, \mathbb{R})$ we define the seminorms
$$\|z\|_t = \max\{|z(\tau, x)| : (\tau, x) \in (E_0 \cup E) \cap ([-b_0, t] \times \mathbb{R}^n)\},$$
$$\|u\|_{\mathbf{h}.r} = \max\{|u^{(i,m)}| : (t^{(i)}, x^{(m)}) \in (E_{0.\mathbf{h}} \cup E_{\mathbf{h}}) \cap ([-b_0, t^{(r)}] \times \mathbb{R}^n)\},$$
for $0 \le t \le a$ and $0 \le r \le K$.

THEOREM 3.3. *Suppose that Assumption* $\mathbf{H}_*[F]$ *is satisfied and* $\psi_{\mathbf{h}} : E_{0.\mathbf{h}} \to \mathbb{R}$, $\Psi_{\mathbf{h}} : \partial_0 E_{\mathbf{h}} \to \mathbb{R}$, $\beta : \partial_0 E_{\mathbf{h}} \to (0,+\infty)$, $\gamma : \partial_0 E_{\mathbf{h}} \to \mathbb{R}_+$. *Then there is exactly one solution* $u_{\mathbf{h}} : E_{0.\mathbf{h}} \cup E_{\mathbf{h}} \to \mathbb{R}$ *of problem* (2.1)–(2.3).

Proof. Suppose that $0 \le r < K$ is fixed and that the solution $u_{\mathbf{h}}$ to problem (2.1)–(2.3) is given on the set $(E_{0.\mathbf{h}} \cup E_{\mathbf{h}}) \cap ([-b_0, t^{(r)}] \times \mathbb{R}^n)$. We prove that the values $u_{\mathbf{h}}^{(r+1,m)}$, $-M \le m \le M$, exist and that they are unique. It is sufficient to show that there exists exactly one solution of the system of equations

(3.4)
$$z^{(r+1,m)} = u_{\mathbf{h}}^{(r,m)} + h_0 F(t^{(r)}, x^{(m)}, u_{\mathbf{h}}^{(r,m)}, (T_{\mathbf{h}} u_{\mathbf{h}})_{\varphi^{(r,m)}}, \delta z^{(r+1,m)}, \delta^{(2)} z^{(r+1,m)}),$$

where $-(M-1) \le m \le M-1$, and

$$\Lambda[z]^{(r+1,m)} = \Psi_{\mathbf{h}}^{(r+1,m)} \quad \text{on } \partial_0 E_{\mathbf{h}}. \tag{3.5}$$

There is $A_{\mathbf{h}} \in \mathbb{R}_+$ such that

$$A_{\mathbf{h}} \ge 2h_0 \sum_{i=1}^{n} \frac{1}{h_i^2} \partial_{s_{ii}} F(P) - h_0 \sum_{(i,j)\in J} \frac{1}{h_i h_j} |\partial_{s_{ij}} F(P)|, \qquad P \in \Xi. \tag{3.6}$$

Write

$$\mathbb{G}_i^{(r,m)}[\tau] = \frac{h_i \Psi_{\mathbf{h}}^{(r,m)}}{h_i \beta^{(r,m)} + \gamma^{(r,m)}} + \frac{\gamma^{(r,m)}}{h_i \beta^{(r,m)} + \gamma^{(r,m)}} \tau, \qquad i = 1, \ldots, n,$$

where $\tau \in \mathbb{R}$, and

$$\mathcal{G}_{\mathbf{h}}^{(r,m)}[q,s] = F(t^{(r)}, x^{(m)}, u_{\mathbf{h}}^{(r,m)}, (T_{\mathbf{h}} u_{\mathbf{h}})_{\varphi^{(r,m)}}, q, s),$$

where $q \in \mathbb{R}^n$, $s \in M_{n\times n}$. The difference problem (3.4), (3.5) is equivalent to the system

(3.7)
$$z^{(r+1,m)} = \frac{1}{1+A_{\mathbf{h}}} \big[A_{\mathbf{h}} z^{(r+1,m)} + u_{\mathbf{h}}^{(r,m)} + h_0 \mathcal{G}_{\mathbf{h}}^{(r,m)}[\delta z^{(r+1,m)}, \delta^{(2)} z^{(r+1,m)}]\big],$$

where $-(M-1) \le m \le M-1$ and

$$\begin{aligned} z^{(r+1,m)} &= \mathbb{G}_i^{(r+1,m)}[z^{(r+1,m-e_i)}] \quad \text{on } \partial_0 E_{\mathbf{h}.i}^{+}, \\ z^{(r+1,m)} &= \mathbb{G}_i^{(r+1,m)}[z^{(r+1,m+e_i)}] \quad \text{on } \partial_0 E_{\mathbf{h}.i}^{-}, \end{aligned} \tag{3.8}$$

where $i = 1, \ldots, n$. Set $X_{\mathbf{h}} = \{x^{(m)} : -M \le m \le M\}$. For $\chi \in \mathcal{F}(X_{\mathbf{h}}, \mathbb{R})$ we write $\chi^{(m)} = \chi(x^{(m)})$ and

$$\delta\chi^{(m)} = (\delta_1 \chi^{(m)}, \ldots, \delta_n \chi^{(m)}), \qquad \delta^{(2)}\chi^{(m)} = [\delta_{ij}\chi^{(m)}]_{i,j=1}^n,$$

where δ_i, δ_{ij}, $1 \le i \le n$, are defined in Section 2. The norm in the space $\mathcal{F}(X_{\mathbf{h}}, \mathbb{R})$ is defined by

$$\|\chi\|_{X_{\mathbf{h}}} = \max\{|\chi^{(m)}| : x^{(m)} \in X_{\mathbf{h}}\}.$$

Set

$$Y_{\mathbf{h}} = \{\chi \in \mathcal{F}(X_{\mathbf{h}}, \mathbb{R}) : \chi^{(m)} = \mathbb{G}_i^{(r+1,m)}[\chi^{(m-e_i)}] \text{ on } \partial_0 E^+_{\mathbf{h}.i} \text{ and } \chi^{(m)} = \mathbb{G}_i^{(r+1,m)}[\chi^{(m+e_i)}] \text{ on } \partial_0 E^-_{\mathbf{h}.i},\ i = 1, \ldots, n\}.$$

Let $W_{r.\mathbf{h}}$ be an operator defined on $Y_{\mathbf{h}}$ in the following way:

$$W_{r.\mathbf{h}}[\chi]^{(m)} = \frac{1}{1 + A_{\mathbf{h}}}\big[A_{\mathbf{h}}\chi^{(m)} + u_{\mathbf{h}}^{(r,m)} + h_0 \mathcal{G}_{\mathbf{h}}^{(r,m)}[\delta\chi^{(r+1,m)}, \delta^{(2)}\chi^{(r+1,m)}]\big],$$

where $-(M-1) \le m \le M-1$ and

$$W_{r.\mathbf{h}}[\chi]^{(m)} = \mathbb{G}_i^{(r+1,m)}[W_{r.\mathbf{h}}[\chi]^{(m-e_i)}] \quad \text{on } \partial_0 E^+_{\mathbf{h}.i}, \tag{3.9}$$

$$W_{r.\mathbf{h}}[\chi]^{(m)} = \mathbb{G}_i^{(r+1,m)}[W_{r.\mathbf{h}}[\chi]^{(m+e_i)}] \quad \text{on } \partial_0 E^-_{\mathbf{h}.i}, \tag{3.10}$$

where $i = 1, \ldots, n$. It follows that $W_{r.\mathbf{h}} : Y_{\mathbf{h}} \to Y_{\mathbf{h}}$. It is clear that problem (3.7), (3.8) is equivalent to the equation

$$\chi = W_{r.\mathbf{h}}[\chi]. \tag{3.11}$$

Now we prove that for $\chi, \bar{\chi} \in Y_{\mathbf{h}}$ we have

$$\|W_{r.\mathbf{h}}[\chi] - W_{r.\mathbf{h}}[\bar{\chi}]\|_{X_{\mathbf{h}}} \le \frac{A_{\mathbf{h}}}{1 + A_{\mathbf{h}}}\|\chi - \bar{\chi}\|_{X_{\mathbf{h}}}. \tag{3.12}$$

Suppose that $-(M-1) \le m \le M-1$. Set

$$P[\tau, \chi, \bar{\chi}]^{(r,m)} = \big(t^{(r)}, x^{(m)}, u_{\mathbf{h}}^{(r,m)}, (T_{\mathbf{h}}u_{\mathbf{h}})_{\varphi^{(r,m)}}, (1-\tau)\delta\bar{\chi}^{(m)} + \tau\delta\chi^{(m)}, (1-\tau)\delta^{(2)}\bar{\chi}^{(m)} + \tau\delta^{(2)}\chi^{(m)}\big)$$

and

$$S_0[\chi, \bar{\chi}]^{(r,m)} = h_0 \sum_{(i,j)\in J} \frac{1}{h_i h_j} \int_0^1 |\partial_{s_{ij}} F(P[\tau, \chi, \bar{\chi}]^{(r,m)})|\, d\tau - 2h_0 \sum_{i=1}^n \frac{1}{h_i^2} \int_0^1 \partial_{s_{ii}} F(P[\tau, \chi, \bar{\chi}]^{(r,m)})\, d\tau,$$

$$S_{i.+}[\chi, \bar{\chi}]^{(r,m)} = \frac{h_0}{2h_i} \int_0^1 \partial_{q_i} F(P[\tau, \chi, \bar{\chi}]^{(r,m)})\, d\tau + \frac{h_0}{h_i^2} \int_0^1 \partial_{s_{ii}} F(P[\tau, \chi, \bar{\chi}]^{(r,m)})\, d\tau - h_0 \sum_{\substack{j=1\\ j\ne i}}^n \frac{1}{h_i h_j} \int_0^1 |\partial_{s_{ij}} F(P[\tau, \chi, \bar{\chi}]^{(r,m)})|\, d\tau,$$

$$S_{i.-}[\chi,\bar{\chi}]^{(r,m)} = -\frac{h_0}{2h_i}\int_0^1 \partial_{q_i}F(P[\tau,\chi,\bar{\chi}]^{(r,m)})\,d\tau$$
$$+\frac{h_0}{h_i^2}\int_0^1 \partial_{s_{ii}}F(P[\tau,\chi,\bar{\chi}]^{(r,m)})\,d\tau - h_0\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{h_ih_j}\int_0^1 |\partial_{s_{ij}}F(P[\tau,\chi,\bar{\chi}]^{(r,m)})|\,d\tau,$$

where $i=1,\dots,n$ and

$$S_{ij}[\chi,\bar{\chi}]^{(r,m)} = \frac{h_0}{2h_ih_j}\int_0^1 \partial_{s_{ij}}F(P[\tau,\chi,\bar{\chi}]^{(r,m)})\,d\tau, \qquad (i,j)\in J.$$

By using the Hadamard mean value theorem for the difference

$$\mathcal{G}_{\mathbf{h}}^{(r,m)}[\delta\chi^{(m)},\delta^{(2)}\chi^{(m)}] - \mathcal{G}_{\mathbf{h}}^{(r,m)}[\delta\bar{\chi}^{(m)},\delta^{(2)}\bar{\chi}^{(m)}]$$

we get

$$\big[W_{r.\mathbf{h}}[\chi]^{(m)} - W_{r.\mathbf{h}}[\bar{\chi}]^{(m)}\big](1+A_{\mathbf{h}}) = (A_{\mathbf{h}} + S_0[\chi,\bar{\chi}]^{(r,m)})(\chi-\bar{\chi})^{(m)}$$
$$+\sum_{i=1}^{n} S_{i.+}[\chi,\bar{\chi}]^{(r,m)}(\chi-\bar{\chi})^{(m+e_i)} + \sum_{i=1}^{n} S_{i.-}[\chi,\bar{\chi}]^{(r,m)}(\chi-\bar{\chi})^{(m-e_i)}$$
$$+\sum_{(i,j)\in J_+} S_{ij}[\chi,\bar{\chi}]^{(r,m)}\big[(\chi-\bar{\chi})^{(m+e_i+e_j)} + (\chi-\bar{\chi})^{(m-e_i-e_j)}\big]$$
$$-\sum_{(i,j)\in J_-} S_{ij}[\chi,\bar{\chi}]^{(r,m)}\big[(\chi-\bar{\chi})^{(m+e_i-e_j)} + (\chi-\bar{\chi})^{(m-e_i+e_j)}\big].$$

We conclude from (3.1), (3.2) and (3.6) that

$$A_{\mathbf{h}} + S_0[\chi,\bar{\chi}]^{(r,m)} \ge 0,\ \ S_{i.+}[\chi,\bar{\chi}]^{(r,m)} \ge 0,\ \ S_{i.-}[\chi,\bar{\chi}]^{(r,m)} \ge 0,\ \ i=1,\dots,n,$$
$$S_{ij}[\chi,\bar{\chi}]^{(r,m)} \ge 0 \quad \text{for } (i,j)\in J_+, \qquad S_{ij}[\chi,\bar{\chi}]^{(r,m)} \le 0 \quad \text{for } (i,j)\in J_-$$

and

$$S_0[\chi,\bar{\chi}]^{(m)} + \sum_{i=1}^{n} S_{i.+}[\chi,\bar{\chi}]^{(r,m)} + \sum_{i=1}^{n} S_{i.-}[\chi,\bar{\chi}]^{(r,m)} + 2\sum_{(i,j)\in J} |S_{ij}[\chi,\bar{\chi}]^{(r,m)}| = 0.$$

Thus we get

(3.13)
$$\|W_{r.\mathbf{h}}[\chi]^{(m)} - W_{r.\mathbf{h}}[\bar{\chi}]^{(m)}\| \le \frac{A_{\mathbf{h}}}{1+A_{\mathbf{h}}}\|\chi-\bar{\chi}\|_{X_{\mathbf{h}}} \quad \text{for } -(M-1)\le m\le M-1.$$

Suppose that $(t^{(r+1)},x^{(m)})\in\partial_0E_{\mathbf{h}}$. It follows from (3.9), (3.10) that

$$W_{r.\mathbf{h}}[\chi]^{(m)} - W_{r.\mathbf{h}}[\bar{\chi}]^{(m)} = \frac{\gamma^{(r,m)}}{h_i\beta^{(r,m)}+\gamma^{(r,m)}}\{W_{r.\mathbf{h}}[\chi]^{(m-e_i)} - W_{r.\mathbf{h}}[\bar{\chi}]^{(m-e_i)}\}$$

for $(t^{(r+1)}, x^{(m)}) \in \partial_0 E^+_{\mathbf{h}.i}$ and

$$W_{r.\mathbf{h}}[\chi]^{(m)} - W_{r.\mathbf{h}}[\bar{\chi}]^{(m)} = \frac{\gamma^{(r,m)}}{h_i \beta^{(r,m)} + \gamma^{(r,m)}} \{W_{r.\mathbf{h}}[\chi]^{(m+e_i)} - W_{r.\mathbf{h}}[\bar{\chi}]^{(m+e_i)}\}$$

for $(t^{(r+1)}, x^{(m)}) \in \partial_0 E^-_{\mathbf{h}.i}$, where $i = 1, \ldots, n$. The result is

$$\|W_{r.\mathbf{h}}[\chi]^{(m)} - W_{r.\mathbf{h}}[\bar{\chi}]^{(m)}\| \le \frac{A_{\mathbf{h}}}{1 + A_{\mathbf{h}}} \|\chi - \bar{\chi}\|_{X_{\mathbf{h}}} \tag{3.14}$$

for $(t^{(r+1)}, x^{(m)}) \in \partial_0 E_{\mathbf{h}}$. Relations (3.13) and (3.14) imply (3.12). The Banach fixed point theorem implies that there exists exactly one solution to (3.11). It follows that the values $u_{\mathbf{h}}^{(r+1,m)}$, $-M \le m \le M$, exist and that they are unique. Since $u_{\mathbf{h}}$ is given on $E_{0.\mathbf{h}}$, the proof is completed by induction.

4. Estimates of solutions. Now we give estimates of solutions to (1.1)–(1.4) and (2.1)–(2.3).

ASSUMPTION $\mathbf{H}[F, \sigma_0]$. Suppose that Assumption $\mathbf{H}_*[F]$ is satisfied and

1) the function $\sigma_0 : [0, a] \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, nondecreasing with respect to both variables, and for each $\eta \in \mathbb{R}_+$ there exists on $[0, a]$ the maximal solution of the Cauchy problem

$$\omega'(t) = \sigma_0(t, \omega(t)), \qquad \omega(t) = \eta, \tag{4.1}$$

2) the estimate

$$|F(t, x, p, w, \mathbf{0}_n, \mathbf{0}_{[n]})| \le \sigma_0(t, \max\{|p|, \|w\|_B\}) \tag{4.2}$$

is satisfied for $(t, x) \in E$, $p \in \mathbb{R}$, $w \in C(B, \mathbb{R})$, where $\mathbf{0}_n = (0, \ldots, 0) \in \mathbb{R}^n$ and $\mathbf{0}_{[n]} \in M_{n\times n}$ is the zero matrix,

3) the functions $\beta : \partial_0 E \to (0, \infty)$, $\gamma : \partial_0 E \to \mathbb{R}_+$ are continuous and the constant $\tilde{B} > 0$ is such that $\beta(t, x) \ge \tilde{B}$ on $\partial_0 E$,

4) the following initial boundary estimates are satisfied:

$$|\psi(t, x)| \le \tilde{\eta} \text{ on } E_0 \qquad \text{and} \qquad |\Psi(t, x)| \le \tilde{B}\omega(t, \tilde{\eta}) \text{ on } \partial_0 E, \tag{4.3}$$

$$|\psi_{\mathbf{h}}^{(r,m)}| \le \tilde{\eta} \text{ on } E_{0.\mathbf{h}} \quad \text{and} \quad |\Psi_{\mathbf{h}}^{(r,m)}| \le \tilde{B}\omega(t^{(r)}, \tilde{\eta}) \text{ on } \partial_0 E_{\mathbf{h}}, \tag{4.4}$$

where $\omega(\cdot, \tilde{\eta})$ is a maximal solution to problem (4.1) with $\eta = \tilde{\eta}$,

5) there is $c_0 > 0$ such that $h_i \le c_0 h_0$ for $i = 1, \ldots, n$.

LEMMA 4.1. *Suppose that Assumption* $\mathbf{H}[F, \sigma_0]$ *is satisfied and* v *is a solution of problem* (1.1)–(1.4) *and* v *is of class* C_*. *Then*

$$|v(t, x)| \le \omega(t, \tilde{\eta}) \quad \text{on } E. \tag{4.5}$$

Proof. For $\varepsilon > 0$ we denote by $\omega(\cdot, \tilde{\eta}, \varepsilon)$ the right hand maximal solution of the Cauchy problem

$$\omega'(t) = \sigma_0(t, \omega(t)) + \varepsilon, \qquad \omega(0) = \tilde{\eta} + \varepsilon. \tag{4.6}$$

There exists $\tilde{\varepsilon} > 0$ such that for $0 < \varepsilon < \tilde{\varepsilon}$ the solution $\omega(\cdot, \tilde{\eta}, \varepsilon)$ is defined on $[0, a]$ and $\lim_{\varepsilon \to 0} \omega(t, \tilde{\eta}, \varepsilon) = \omega(t, \tilde{\eta})$ uniformly on $[0, a]$. Write $\zeta(t) = \|v\|_t$, $t \in [0, a]$. We prove that

$$\zeta(t) < \omega(t, \tilde{\eta}, \varepsilon) \quad \text{for } t \in [0, a]. \tag{4.7}$$

Suppose for contradiction that assertion (4.7) fails to be true. Then the set $\Sigma_+ = \{t \in [0, a] : \zeta(t) \geq \omega(t, \tilde{\eta}, \varepsilon)\}$ is not empty. If we put $\tilde{t} = \min \Sigma_+$, from condition 4) it is clear that $\tilde{t} > 0$ and there exists $\tilde{x} \in [-b, b]$ such that

$$\omega(\tilde{t}, \tilde{\eta}, \varepsilon) = \zeta(\tilde{t}) = \left| v(\tilde{t}, \tilde{x}) \right|.$$

There are two possibilities: either (i) $v(\tilde{t}, \tilde{x}) = \omega(\tilde{t}, \tilde{\eta}, \varepsilon)$ or (ii) $v(\tilde{t}, \tilde{x}) = -\omega(\tilde{t}, \tilde{\eta}, \varepsilon)$. Let us consider the first case. We conclude from conditions 3) and 4) of Assumption $\mathbf{H}[F, \sigma_0]$ that $(\tilde{t}, \tilde{x}) \notin \partial_0 E$. Hence $(\tilde{t}, \tilde{x}) \in E \setminus \partial_0 E$. We have

$$D_- \zeta(\tilde{t}) \geq \omega'(\tilde{t}, \tilde{\eta}, \varepsilon). \tag{4.8}$$

Set

$$\begin{aligned} A(t,x) &\\ = F(t, x, v(t, x), v_{\varphi(t,x)}, \partial_x v(t, x), \partial_{xx} v(t, x)) &- F(t, x, v(t, x), v_{\varphi(t,x)}, \mathbf{0}_n, \mathbf{0}_{[n]}), \\ B(t, x) = F(t, x, v(t, x), v_{\varphi(t,x)}, \mathbf{0}_n, \mathbf{0}_{[n]}). & \end{aligned}$$

It follows from Hadamard's mean value theorem that

$$A(t,x) = \sum_{i=1}^{n} \int_0^1 \partial_{q_i} F(\tilde{P}[\tau, v])\, d\tau\, \partial_{x_i} v(t, x) + \sum_{i,j=1}^{n} \int_0^1 \partial_{s_{ij}} F(\tilde{P}[\tau, v])\, d\tau\, \partial_{x_i x_j} v(t, x),$$

where $\tilde{P}[\tau, v]$, $0 \leq \tau \leq 1$, are intermediate points defined by the theorem. Since $\tilde{x} \in (-b, b)$, we have $\partial_x v(\tilde{t}, \tilde{x}) = 0$ and

$$\sum_{i,j=1}^{n} \partial_{x_i x_j} v(\tilde{t}, \tilde{x}) \lambda_i \lambda_j \leq 0 \quad \text{for } \lambda \in \mathbb{R}^n. \tag{4.9}$$

We have

$$D_- \zeta(\tilde{t}) \leq \partial_t v(\tilde{t}, \tilde{x}) = A(\tilde{t}, \tilde{x}) + B(\tilde{t}, \tilde{x})$$

From conditions 1), 2) of Assumption $\mathbf{H}[F, \sigma_0]$ and from (4.9) we conclude that

$$D_- \zeta(\tilde{t}) \leq \sigma_0(\tilde{t}, \omega(\tilde{t}, \tilde{\eta}, \varepsilon)) < \omega'(\tilde{t}, \tilde{\eta}, \varepsilon),$$

which contradicts (4.8). The case $v(\tilde{t}, \tilde{x}) = -\omega(\tilde{t}, \tilde{\eta}, \varepsilon)$ can be treated in a similar way. Hence Σ_+ is empty and inequality (4.5) is proved on E. This completes the proof.

Assumption $\mathbf{H}[T_{\mathbf{h}}]$. The operator $T_{\mathbf{h}} : \mathcal{F}(E_{0.\mathbf{h}} \cup E_{\mathbf{h}}, \mathbb{R}) \to C(E_0 \cup E, \mathbb{R})$ satisfies the conditions:

1) for $z, \tilde{z} \in \mathcal{F}(E_{0.\mathbf{h}} \cup E_{\mathbf{h}}, \mathbb{R})$ we have
$$\|T_{\mathbf{h}}[z] - T_{\mathbf{h}}[\tilde{z}]\|_{t^{(r)}} \le \|z - \tilde{z}\|_{\mathbf{h}.r}, \quad 0 \le r \le K,$$
2) if $z : E_0 \cup E \to \mathbb{R}$ is of class C^2 then there is $\gamma_* : \mathbb{H} \to \mathbb{R}_+$ such that
$$\|T_{\mathbf{h}}[z_{\mathbf{h}}] - z\|_t \le \gamma_*(\mathbf{h}),\ 0 \le t < a, \quad \lim_{\mathbf{h}\to 0} \gamma_*(\mathbf{h}) = 0$$
where $z_{\mathbf{h}}$ is the restriction of z to the set $E_{0.\mathbf{h}} \cup E_{\mathbf{h}}$,
3) if $\mathbb{O}_{\mathbf{h}} \in \mathcal{F}(E_{0.\mathbf{h}} \cup E_{\mathbf{h}}, \mathbb{R})$ is given by $\mathbb{O}_{\mathbf{h}}(t,x) = 0$ for $(t,x) \in E_{0.\mathbf{h}} \cup E_{\mathbf{h}}$ then $T_{\mathbf{h}}[\mathbb{O}_{\mathbf{h}}](t,x) = 0$ for $(t,x) \in E_0 \cup E$.

Remark 4.2. The operator $T_{\mathbf{h}}$ given in [3, Chapter V], is such that for each $z : E_0 \cup E \to \mathbb{R}$ of class C^2 there is $\bar{C} \in \mathbb{R}_+$ such that
$$\|z_{\mathbf{h}} - T_{\mathbf{h}} z_{\mathbf{h}}\|_t \le \bar{C} \|\mathbf{h}\|^2, \quad 0 \le t \le a,$$
(see [3, Theorem 5.27, p. 159]) and conditions 1), 3) of Assumption $\mathbf{H}[T_{\mathbf{h}}]$ are satisfied.

Lemma 4.3. *Suppose that Assumptions* $\mathbf{H}[T_{\mathbf{h}}]$, $\mathbf{H}[F, \sigma_0]$ *are satisfied and* $v_{\mathbf{h}}$ *is a solution of problem* (2.1)–(2.3)*. Then*
$$|v_{\mathbf{h}}^{(r,m)}| \le \omega(t^{(r)}, \tilde{\eta}) \quad \text{on } E_{0.\mathbf{h}} \cup E_{\mathbf{h}}. \tag{4.10}$$

Proof. For $\varepsilon > 0$ we denote by $\omega(\cdot, \tilde{\eta}, \varepsilon)$ the right hand maximal solution of the Cauchy problem (4.6). There exists $\tilde{\varepsilon} > 0$ such that for $0 < \varepsilon < \tilde{\varepsilon}$ the solution $\omega(\cdot, \tilde{\eta}, \varepsilon)$ is defined on $[0, a]$ and $\lim_{\varepsilon\to 0} \omega(t, \tilde{\eta}, \varepsilon) = \omega(t, \tilde{\eta})$ uniformly on $[0, a]$. Write
$$\varepsilon_{\mathbf{h}}^{(r)} = \max\{|v_{\mathbf{h}}^{(i,m)}| : (t^{(i)}, x^{(m)}) \in E_{0.\mathbf{h}} \cup E_{\mathbf{h}},\ i \le r\}, \quad 0 \le r \le K.$$
We now prove that
$$\varepsilon_{\mathbf{h}}^{(r)} < \omega(t^{(r)}, \tilde{\eta}, \varepsilon) \tag{4.11}$$
where $0 \le r \le K$. From (4.4) we deduce that (4.11) is true for $r = 0$. Assuming that $\varepsilon_{\mathbf{h}}^{(j)} < \omega(t^{(j)}, \tilde{\eta}, \varepsilon)$ for $0 \le j \le r$ we will prove (4.11) for $r+1$. There exists $(t^{(i)}, x^{(m)}) \in E_{\mathbf{h}}$ such that $\varepsilon_{\mathbf{h}}^{(r+1)} = |v_{\mathbf{h}}^{(i,m)}|$. If $i < r+1$ then $|v_{\mathbf{h}}^{(i,m)}| \le \varepsilon_{\mathbf{h}}^{(r)} < \omega(t^{(r)}, \tilde{\eta}, \varepsilon)$ and consequently $\varepsilon_{\mathbf{h}}^{(r+1)} < \omega(t^{(r+1)}, \tilde{\eta}, \varepsilon)$. Suppose that $\varepsilon_{\mathbf{h}}^{(r+1)} = |v_{\mathbf{h}}^{(r+1,m)}|$. There are two possibilities: either $\varepsilon_{\mathbf{h}}^{(r+1)} = v_{\mathbf{h}}^{(r+1,m)}$ or $\varepsilon_{\mathbf{h}}^{(r+1)} = -v_{\mathbf{h}}^{(r+1,m)}$. Let us consider the first case. We conclude from conditions 3), 4) of Assumption $\mathbf{H}[F, \sigma_0]$ that $(t^{(r+1)}, x^{(m)}) \notin \partial_0 E_{\mathbf{h}}$.

Write

$$(4.12)\qquad \tilde{A}^{(r,m)} = F(t^{(r)}, x^{(m)}, v_{\mathbf{h}}^{(r,m)}, (T_{\mathbf{h}} v_{\mathbf{h}})_{\varphi^{(r,m)}}, \delta v_{\mathbf{h}}^{(r+1,m)}, \delta^{(2)} v_{\mathbf{h}}^{(r+1,m)}) - F(t^{(r)}, x^{(m)}, v_{\mathbf{h}}^{(r,m)}, (T_{\mathbf{h}} v_{\mathbf{h}})_{\varphi^{(r,m)}}, \mathbf{0}_n, \mathbf{0}_{[n]}),$$

and

$$\tilde{B}^{(r,m)} = F(t^{(r)}, x^{(m)}, v_{\mathbf{h}}^{(r,m)}, (T_{\mathbf{h}} v_{\mathbf{h}})_{\varphi^{(r,m)}}, \mathbf{0}_n, \mathbf{0}_{[n]}).$$

By applying the Hadamard mean value theorem to the difference (4.12) we conclude that there are

$$S_0^{(r,m)},\ S_{i.+}^{(r,m)},\ S_{i.-}^{(r,m)},\ i = 1, \ldots, n, \quad \text{and} \quad S_{ij}^{(r,m)},\ (i,j) \in J,$$

such that

$$v_{\mathbf{h}}^{(r+1,m)}(1 + S_0^{(r,m)}) = v_{\mathbf{h}}^{(r,m)} + h_0 \tilde{B}^{(r,m)} + \Big[\sum_{i=1}^{n} S_{i.+}^{(r,m)} v_{\mathbf{h}}^{(r+1,m+e_i)} + \sum_{i=1}^{n} S_{i.-}^{(r,m)} v_{\mathbf{h}}^{(r+1,m-e_i)} + \sum_{(i,j)\in J_+} S_{ij}^{(r,m)} [v_{\mathbf{h}}^{(r+1,m+e_i+e_j)} + v_{\mathbf{h}}^{(r+1,m-e_i-e_j)}] - \sum_{(i,j)\in J_-} S_{ij}^{(r,m)} [v_{\mathbf{h}}^{(r+1,m-e_i+e_j)} + v_{\mathbf{h}}^{(r+1,m+e_i-e_j)}] \Big].$$

Moreover we have

$$S_{i.+}^{(r,m)} \ge 0, \qquad S_{i.-}^{(r,m)} \ge 0 \quad \text{for } i = 1, \ldots, n,$$

$$S_{ij}^{(r,m)} \ge 0 \quad \text{for } (i,j) \in J_+, \qquad S_{ij}^{(r,m)} \le 0 \quad \text{for } (i,j) \in J_-$$

and

$$S_0^{(r,m)} + \sum_{i=1}^{n} S_{i.+}^{(r,m)} + \sum_{i=1}^{n} S_{i.-}^{(r,m)} + 2 \sum_{(i,j)\in J} |S_{ij}^{(r,m)}| = 0.$$

The above relations and Assumption $\mathbf{H}[F, \sigma_0]$ imply

$$\varepsilon_{\mathbf{h}}^{(r+1)} \le \varepsilon_{\mathbf{h}}^{(r)} + h_0 \sigma_0(t^{(r)}, \varepsilon_{\mathbf{h}}^{(r)})$$

and consequently

$$\varepsilon_{\mathbf{h}}^{(r+1)} < \omega(t^{(r)}, \tilde{\eta}, \varepsilon).$$

The case $\varepsilon_{\mathbf{h}}^{(r)} = -v_{\mathbf{h}}^{(r+1,m)}$ can be treated in a similar way. Hence the proof of (4.11) is complete. From (4.11), letting ε tend to 0, we obtain inequality (4.10). This proves the lemma.

5. Convergence of implicit difference methods. Write $A = \omega(a, \tilde{\eta})$, where $\omega(\cdot, \tilde{\eta})$ is the maximal solution to (4.1) and $K_{C(B,\mathbb{R})}[A] = \{\omega \in C(B, \mathbb{R}) : \|\omega\|_B \le A\}$. Set

$$\varXi_A = E \times [-A, A] \times K_{C(B,\mathbb{R})}[A] \times \mathbb{R}^n \times M_{n\times n}.$$

Assumption $\mathbf{H}[\sigma]$. Assumption $\mathbf{H}[F, \sigma_0]$ is satisfied and there is a function $\sigma : [0, a] \times \mathbb{R}_+ \to \mathbb{R}_+$ such that:

1) σ is continuous and it is nondecreasing with respect to both variables,
2) the estimate

$$|F(t, x, p, w, q, s) - F(t, x, \bar{p}, \bar{w}, q, s)| \le \sigma(t, \max\{|p - \bar{p}|, \|w - \bar{w}\|_B\}) \tag{5.1}$$

is satisfied on Ξ_A,

3) $\sigma(t, 0) = 0$ for $t \in [0, a)$ and function $\bar{\omega} \equiv 0$ is the maximal solution of the Cauchy problem

$$\omega'(t) = \sigma(t, \omega(t)), \qquad \omega(0) = 0.$$

In the theorems on convergence of difference methods in [4], [6]–[9], [16], [17], it is assumed that the corresponding functions F of the variables (t, x, p, w, q, s) satisfy estimates of the Perron type globally with respect to (p, w). Note that we have assumed estimate (5.1) for $p \in [-A, A]$ and $w \in K_{C(B,\mathbb{R})}[A]$ only. It is clear that there are differential equations with deviated variables and differential integral equations such that the corresponding functions F satisfy condition (5.1) on Ξ_A but not on Ξ.

Theorem 5.1. *Suppose that Assumptions $\mathbf{H}[\sigma]$, $\mathbf{H}[T_\mathbf{h}]$ are satisfied and*

1) $u_\mathbf{h} : E_{0.\mathbf{h}} \cup E_\mathbf{h} \to \mathbb{R}$ *is a solution of* (2.1)–(2.3),
2) $v : E_0 \cup E \to \mathbb{R}$ *is a solution of* (1.1)–(1.4) *and v is of class C_* and $v_\mathbf{h}$ is restriction of v to the set $E_{0.\mathbf{h}} \cup E_\mathbf{h}$,*
3) *there is $\alpha_0 : \mathbb{H} \to \mathbb{R}_+$ such that*

$$|\psi_\mathbf{h}^{(r,m)} - \psi^{(r,m)}| \le \alpha_0(\mathbf{h}) \qquad \text{on } E_{0.\mathbf{h}},$$
$$|\Psi_\mathbf{h}^{(r,m)} - \Psi^{(r,m)}| \le h_0\alpha_0(\mathbf{h}) \qquad \text{on } \partial_0 E_\mathbf{h}$$

and $\lim_{\mathbf{h}\to 0} \alpha_0(\mathbf{h}) = 0$.

Then there is $\alpha : \mathbb{H} \to \mathbb{R}_+$ such that

$$|(u_\mathbf{h} - v_\mathbf{h})^{(r,m)}| \le \alpha(\mathbf{h}) \quad \text{on } E_\mathbf{h} \quad \text{and} \quad \lim_{\mathbf{h}\to 0} \alpha(\mathbf{h}) = 0. \tag{5.2}$$

Proof. The proof starts with the observation that

$$\delta_{ij} v^{(r,m)} = \frac{1}{2}\int_0^1\!\!\int_0^1 \partial_{x_i x_j} v(t^{(r)}, x^{(m)} + sh_i e_i + \tau h_j e_j)\, ds\, d\tau$$
$$+ \frac{1}{2}\int_0^1\!\!\int_0^1 \partial_{x_i x_j} v(t^{(r)}, x^{(m)} - sh_i e_i - \tau h_j e_j)\, ds\, d\tau \quad \text{for } (i, j) \in J_+$$

and

$$\delta_{ij}v^{(r,m)} = \frac{1}{2}\int_0^1\int_0^1 \partial_{x_ix_j}v(t^{(r)},x^{(m)}+sh_ie_i-\tau h_je_j)\,ds\,d\tau$$
$$+\frac{1}{2}\int_0^1\int_0^1 \partial_{x_ix_j}v(t^{(r)},x^{(m)}-sh_ie_i+\tau h_je_j)\,ds\,d\tau \quad \text{for } (i,j)\in J_-.$$

It follows from the above relations and from Assumptions $\mathbf{H}_*[F]$, $\mathbf{H}[T_{\mathbf{h}}]$ that there are $\mathit{\Gamma}_{\mathbf{h}}: E_{\mathbf{h}}\to\mathbb{R}$ and $\gamma_1:\mathbb{H}\to\mathbb{R}_+$ such that

$$\delta_0 v_{\mathbf{h}}^{(r,m)} = F_{\mathbf{h}}[v_{\mathbf{h}}]^{(r,m)} + \mathit{\Gamma}_{\mathbf{h}}^{(r,m)} \tag{5.3}$$

for $-(M-1)\le m\le M-1,\ 0\le r\le K-1$ and

$$\begin{aligned}\delta_i^+ v_{\mathbf{h}}^{(r,m)} &= \partial_{x_i}v^{(r,m)} + \mathit{\Gamma}_{\mathbf{h}}^{(r,m)} \quad \text{on } \partial_0 E_{\mathbf{h}.i}^+,\\ \delta_i^- v_{\mathbf{h}}^{(r,m)} &= \partial_{x_i}v^{(r,m)} - \mathit{\Gamma}_{\mathbf{h}}^{(r,m)} \quad \text{on } \partial_0 E_{\mathbf{h}.i}^-,\end{aligned} \tag{5.4}$$

where $i=1,\dots,n$ and

$$|\mathit{\Gamma}_{\mathbf{h}}^{(r,m)}|\le\gamma_1(\mathbf{h}) \quad \text{on } E_{\mathbf{h}}, \quad \lim_{\mathbf{h}\to 0}\gamma_1(\mathbf{h})=0.$$

Write $z_{\mathbf{h}} = v_{\mathbf{h}} - u_{\mathbf{h}}$ and

$$\varepsilon_{\mathbf{h}}^{(r)} = \max\{|z_{\mathbf{h}}^{(i,m)}| : (t^{(i)},x^{(m)})\in E_{0.\mathbf{h}}\cup E_{\mathbf{h}},\ i\le r\}, \quad 0\le r\le K.$$

We prove that the function $\varepsilon_{\mathbf{h}}^{(r)}$ satisfies the recurrent inequality

$$\varepsilon_{\mathbf{h}}^{(r+1)} \le \varepsilon_{\mathbf{h}}^{(r)} + h_0\sigma(t^{(r)},\varepsilon_{\mathbf{h}}^{(r)}) + h_0\tilde{\gamma}(\mathbf{h}), \quad 0\le r\le K-1, \tag{5.5}$$

where $\tilde{\gamma}(\mathbf{h}) = \gamma_1(\mathbf{h})(1+c_0) + \tilde{B}^{-1}\alpha_0(\mathbf{h})$. Suppose that $(t^{(r+1)},x^{(m)})\in E_{\mathbf{h}}$ and $-(M-1)\le m\le M-1$. Then

$$z_{\mathbf{h}}^{(r+1,m)} = z_{\mathbf{h}}^{(r,m)} + h_0 A_{\mathbf{h}}[u_{\mathbf{h}},v_{\mathbf{h}}]^{(r,m)} + h_0 B_{\mathbf{h}}[u_{\mathbf{h}},v_{\mathbf{h}}]^{(r,m)}, \tag{5.6}$$

where

$$A_{\mathbf{h}}[u_{\mathbf{h}},v_{\mathbf{h}}]^{(r,m)} =$$
$$F_{\mathbf{h}}[v_{\mathbf{h}}]^{(r,m)} - F(t^{(r)},x^{(m)},u_{\mathbf{h}}^{(r,m)},(T_{\mathbf{h}}u_{\mathbf{h}})_{\varphi^{(r,m)}},\delta v_{\mathbf{h}}^{(r+1,m)},\delta^{(2)}v_{\mathbf{h}}^{(r+1,m)}) + \mathit{\Gamma}_{\mathbf{h}}^{(r,m)}$$

and

$$B_{\mathbf{h}}[u_{\mathbf{h}},v_{\mathbf{h}}]^{(r,m)} =$$
$$F(t^{(r)},x^{(m)},u_{\mathbf{h}}^{(r,m)},(T_{\mathbf{h}}u_{\mathbf{h}})_{\varphi^{(r,m)}},\delta v_{\mathbf{h}}^{(r+1,m)},\delta^{(2)}v_{\mathbf{h}}^{(r+1,m)}) - F_{\mathbf{h}}[u_{\mathbf{h}}]^{(r,m)}.$$

It follows from condition 2) of Assumption $\mathbf{H}[\sigma]$ that

$$|A_{\mathbf{h}}[u_{\mathbf{h}},v_{\mathbf{h}}]^{(r,m)}| \le \sigma(t^{(r)},\varepsilon_{\mathbf{h}}^{(r)}) + \gamma_1(\mathbf{h}). \tag{5.7}$$

We conclude from the Hadamard mean value theorem that there are

$$\tilde{S}_0^{(r,m)},\ \tilde{S}_{i.+}^{(r,m)},\ \tilde{S}_{i.-}^{(r,m)},\ i=1,\dots,n,\ \tilde{S}_{ij}^{(r,m)},(i,j)\in J,$$

such that

$$h_0 B_{\mathbf{h}}[u_{\mathbf{h}}, v_{\mathbf{h}}]^{(r,m)} = \tilde{S}_0^{(r,m)} z_{\mathbf{h}}^{(r+1,m)} + \sum_{i=1}^{n} \tilde{S}_{i.+}^{(r,m)} z_{\mathbf{h}}^{(r+1,m+e_i)}$$
$$+ \sum_{i=1}^{n} \tilde{S}_{i.-}^{(r,m)} z_{\mathbf{h}}^{(r+1,m-e_i)} + \sum_{(i,j)\in J_+} \tilde{S}_{ij}^{(r,m)} [z_{\mathbf{h}}^{(r+1,m+e_i+e_j)} + z_{\mathbf{h}}^{(r+1,m-e_i-e_j)}]$$
$$- \sum_{(i,j)\in J_-} \tilde{S}_{ij}^{(r,m)} [z_{\mathbf{h}}^{(r+1,m-e_i+e_j)} + z_{\mathbf{h}}^{(r+1,m+e_i-e_j)}].$$

Moreover

$$\tilde{S}_{i.+}^{(r,m)} \geq 0, \qquad \tilde{S}_{i.-}^{(r,m)} \geq 0 \quad \text{for } i = 1, \ldots, n,$$
$$\tilde{S}_{ij}^{(r,m)} \geq 0 \quad \text{for } (i,j) \in J_+, \qquad \tilde{S}_{ij}^{(r,m)} \leq 0 \quad \text{for } (i,j) \in J_-$$

and

$$\tilde{S}_0^{(r,m)} + \sum_{i=1}^{n} \tilde{S}_{i.+}^{(r,m)} + \sum_{i=1}^{n} \tilde{S}_{i.-}^{(r,m)} + 2 \sum_{(i,j)\in J} |\tilde{S}_{ij}^{(r,m)}| = 0.$$

We conclude from the above relations and from (5.6), (5.7) that

$$|z_{\mathbf{h}}^{(r+1,m)}| \leq \varepsilon_{\mathbf{h}}^{(r)} + h_0 \sigma(t^{(r)}, \varepsilon_{\mathbf{h}}^{(r)}) + h_0 \gamma_1(\mathbf{h}). \tag{5.8}$$

Suppose that $(t^{(r+1)}, x^{(m)}) \in \partial_0 E_{\mathbf{h}.i}^{+}$. Then

$$z_{\mathbf{h}}^{(r+1,m)} = \frac{\gamma^{(r+1,m)}}{h_i \beta^{(r+1,m)} + \gamma^{(r+1,m)}} z_{\mathbf{h}}^{(r+1,m-e_i)}$$
$$+ \frac{\gamma^{(r+1,m)}}{h_i \beta^{(r+1,m)} + \gamma^{(r+1,m)}} h_i \Gamma_{\mathbf{h}}^{(r+1,m)}$$
$$+ \frac{h_i}{h_i \beta^{(r+1,m)} + \gamma^{(r+1,m)}} [\Psi^{(r+1,m)} - \Psi_{\mathbf{h}}^{(r+1,m)}]$$

and consequently

$$|z_{\mathbf{h}}^{(r+1,m)}| \leq \varepsilon_{\mathbf{h}}^{(r)} + h_0 \sigma(t^{(r)}, \varepsilon_{\mathbf{h}}^{(r)}) + h_0 \gamma_1(\mathbf{h}) + c_0 h_0 \gamma_1(\mathbf{h}) + \frac{1}{\tilde{B}} h_0 \alpha_0(\mathbf{h}). \tag{5.9}$$

In a similar way we prove (5.9) for $(t^{(r+1)}, x^{(m)}) \in \partial_0 E_{\mathbf{h}.i}^{-}$. Estimates (5.8) and (5.9) imply (5.5). Let us consider the Cauchy problem

$$\omega'(t) = \sigma(t, \omega(t)) + \tilde{\gamma}(\mathbf{h}), \qquad \omega(0) = \alpha_0(\mathbf{h}). \tag{5.10}$$

There is $\varepsilon_0 > 0$ such that the maximal solution $\omega(\cdot, \mathbf{h})$ to (5.10) is defined on $[0, a]$ for $0 \leq \|\mathbf{h}\| < \varepsilon_0$ and $\lim_{\mathbf{h}\to 0} \omega(t, \mathbf{h}) = 0$ uniformly on $[0, a]$. We conclude from Assumption $\mathbf{H}[\sigma]$ that

$$\omega(t^{(r+1)}, \mathbf{h}) \geq \omega(t^{(r)}, \mathbf{h}) + h_0 \sigma(t^{(r)}, \omega(t^{(r)}, \mathbf{h})) + h_0 \tilde{\gamma}(\mathbf{h}), \qquad 0 \leq r \leq K-1.$$

This gives

$$\varepsilon_{\mathbf{h}}^{(r)} \le \omega(t^{(r)}, \mathbf{h}) \quad \text{for } 0 \le r \le K.$$

Thus assertion (5.2) is satisfied with $\alpha(\mathbf{h}) = \omega(a, \mathbf{h})$.

Remark 5.2. Suppose that all the assumptions of Theorem 5.1 are satisfied and $\sigma : [0, a] \times \mathbb{R}_+ \times C(I, \mathbb{R}_+) \to \mathbb{R}_+$ is given by

$$\sigma(t, p) = Lp \quad \text{on } [0, a] \times \mathbb{R}_+,$$

where $L \in \mathbb{R}_+$. Then

$$|u_{\mathbf{h}}^{(r,m)} - v_{\mathbf{h}}^{(r,m)}| \le \tilde{\alpha}(\mathbf{h}) \quad \text{on } E_{\mathbf{h}}$$

where

$$\tilde{\alpha}(\mathbf{h}) = \alpha_0(\mathbf{h}) e^{La} + \frac{\tilde{\gamma}(\mathbf{h})}{L}(e^{La} - 1) \quad \text{if } L > 0, \tag{5.11}$$

$$\tilde{\alpha}(\mathbf{h}) = \alpha_0(\mathbf{h}) + a\tilde{\gamma}(\mathbf{h}) \quad \text{if } L = 0. \tag{5.12}$$

We obtain the above estimates by solving problem (5.10) with $\sigma(t, p) = Lp$.

We consider problem (2.1)–(2.3) with $T_{\mathbf{h}}$ defined in [3].

Lemma 5.3. *Suppose that all assumptions of Theorem* 5.1 *are satisfied with* $\sigma(t, p) = Lp$ *on* $[0, a] \times \mathbb{R}_+$, *where* $L \in \mathbb{R}_+$ *and*

1) $\psi_{\mathbf{h}} = \psi$ *on* $E_{0.\mathbf{h}}$ *and* $\Psi_{\mathbf{h}} = \Psi$ *on* $\partial_0 E_{\mathbf{h}}$,
2) *the interpolating operator* $T_{\mathbf{h}} : \mathcal{F}_C(E_{0.\mathbf{h}} \cup E_{\mathbf{h}}, \mathbb{R}) \to C(E_0 \cup E, \mathbb{R})$ *is given in* [3, Chapter 5],
3) $v : E_0 \cup E \to \mathbb{R}$ *is a solution of* (1.1)–(1.4) *and* v *is of class* C^3.

Then there are $C_1, C_2 \in \mathbb{R}_+$ *such that*

$$|(u_{\mathbf{h}} - v_{\mathbf{h}})^{(r,m)}| \le C_1 \|\mathbf{h}\| + C_2 \|\mathbf{h}\|^2 \quad \text{on } E_{\mathbf{h}}, \tag{5.13}$$

where $u_{\mathbf{h}} : E_{0.\mathbf{h}} \cup E_{\mathbf{h}} \to \mathbb{R}$ *is a solution to* (2.1)–(2.3) *and* $v_{\mathbf{h}}$ *is the restriction of* v *to the set* $E_{0.\mathbf{h}} \cup E_{\mathbf{h}}$.

Proof. It follows from Theorem 5.1 and from condition 3) that there are $\tilde{C}_0, \tilde{C}_1 \in \mathbb{R}_+$ such that the function $\Gamma_{\mathbf{h}}$ defined by (5.3), (5.4) satisfies the condition

$$|\Gamma_{\mathbf{h}}^{(r,m)}| \le \tilde{C}_0 \|\mathbf{h}\| + \tilde{C}_1 \|\mathbf{h}\|^2 \quad \text{on } E_{\mathbf{h}}.$$

Then we obtain (5.13) from (5.11), (5.12).

6. Comments and examples. Let us assume that the solution $u_{\mathbf{h}}$ to problem (2.1)–(2.3) is known on the set $(E_{0.\mathbf{h}} \cup E_{\mathbf{h}}) \cap ([-b_0, t^{(r)}] \times \mathbb{R}^n)$. Then computing the values $u_{\mathbf{h}}^{(r+1,m)}$ for $-M \le m \le M$ leads to the system (3.4), (3.5). We will show that the Newton method is a natural way for computing approximate solutions to the above problem. Write

$$k = \prod_{i=1}^{n} (2M_i + 1).$$

Elements of the space $\mathbb{R}^k$ will be denoted by $y = \{y^{(m)}\}_{-M \le m \le M}$. Consider the operator $\mathcal{G} : \mathbb{R}^k \to \mathbb{R}^k$, $\mathcal{G}(y) = \{\mathcal{G}^{(m)}(y)\}_{-M \le m \le M}$, where

$$\mathcal{G}^{(m)}(y) = u_{\mathbf{h}} + h_0 F(t^{(r)}, x^{(m)}, u_{\mathbf{h}}^{(r,m)}, (T_{\mathbf{h}} u_{\mathbf{h}})_{\psi^{(r,m)}}, \delta y^{(m)}, \delta^{(2)} y^{(m)}),$$

$-(M-1) \le m \le M-1$, and

$$\mathcal{G}^{(m)}(y) = \frac{h_i \Psi^{(r+1,m)}}{h_i \beta^{(r+1,m)} + \gamma^{(r+1,m)}} + \frac{\gamma^{(r+1,m)}}{h_i \beta^{(r+1,m)} + \gamma^{(r+1,m)}} y^{(m-e_i)}$$

for $(t^{(r+1)}, x^{(m)}) \in \partial_0 E^{+}_{\mathbf{h}.i}$, while

$$\mathcal{G}^{(m)}(y) = \frac{h_i \Psi^{(r+1,m)}}{h_i \beta^{(r+1,m)} + \gamma^{(r+1,m)}} + \frac{\gamma^{(r+1,m)}}{h_i \beta^{(r+1,m)} + \gamma^{(r+1,m)}} y^{(m+e_i)}$$

for $(t^{(r+1)}, x^{(m)}) \in \partial_0 E^{-}_{\mathbf{h}.i}$, where $i = 1, \ldots, n$. System (3.4), (3.5) is equivalent to the equation

$$y = \mathcal{G}(y) \tag{6.1}$$

Let $E_{k\times k}$ denote the identity matrix in $M_{k\times k}$. We approximate the solution $u_{\mathbf{h}}^{(r+1,m)}$, $-M \le m \le M$, of (6.1) with the sequence $\{y_{(i)}\}$, $y_{(i)} \in \mathbb{R}^k$, defined by the following Newton method:

$$y_{(0)} = \{u_{\mathbf{h}}^{(r,m)}\}_{-M \le m \le M},$$
$$y_{(i+1)} = y_{(i)} - [E_{k\times k} - \mathcal{G}'(y_{(i)})]^{-1} [y_{(i)} - \mathcal{G}(y_{(i)})] \quad \text{for } i \ge 0.$$

The next remark states that the Newton sequence $\{y_{(i)}\}$ exists.

Remark 6.1. Suppose that Assumption $\mathbf{H}_*[F]$ is satisfied and

1) the solution $u_{\mathbf{h}}$ of (2.1)–(2.3) is given on the set $(E_{0.\mathbf{h}} \cup E_{\mathbf{h}}) \cap ([-b_0, t^{(r)}] \times \mathbb{R}^n)$,
2) $T_{\mathbf{h}} : \mathcal{F}(E_{0.\mathbf{h}} \cup E_{\mathbf{h}}, \mathbb{R}) \to C(E_0 \cup E, \mathbb{R})$,
3) the functions $\beta : \partial_0 E \to (0, \infty)$, $\gamma : \partial_0 E \to \mathbb{R}_+$ are continuous and $\tilde{B}$ is such that $\beta(t,x) \ge \tilde{B}$ on $\partial_0 E$.

Then the matrix $C = E_{k\times k} - \mathcal{G}'(y)$ is invertible.

Indeed, it follows from Assumption $\mathbf{H}_*[F]$ that the matrix $C = E_{k\times k} - \mathcal{G}'(y)$, $C = [c_{ij}]_{i,j=1}^{k}$, satisfies the condition

$$c_{ii} > \sum_{\substack{j=1 \\ j \ne i}}^{k} |c_{ij}|, \qquad i = 1, \ldots, k,$$

and consequently $\det C \ne 0$. Thus the Newton sequence for (6.1) exists.

It is easy to formulate a Kantorovich type theorem on the convergence of the Newton method for (6.1). In order to formulate such a theorem one would need the following additional assumption: the functions $\partial_q F$, $\partial_s F$ of variables (t, x, p, w, q, s) satisfy the Lipschitz condition with respect to (q, s) on Ξ_A.

Now we give numerical examples. We begin with a differential integral equation.

EXAMPLE 6.2. Let $n = 2$, $I = [-0.5, 0.5]$, $I_+ = [0, 0.5]$, $E = I_+ \times I \times I$ and $E_0 = \{0\} \times I \times I$. Consider the differential integral equation

$$\begin{aligned}(6.2)\qquad \partial_t z(t,x,y) &= \partial_{xx} z(t,x,y) - \frac{1}{2}\partial_{xy} z(t,x,y) + \partial_{yy} z(t,x,y)\\ &+ y\int_0^x z(t,s,y)ds + x\int_0^y z(t,x,s)ds + f(x,y)z(t,x,y) + g(t,x,y)\end{aligned}$$

where

$$f(x,y) = 2(xy - 2x^2 - 2y^2), \qquad g(t,x,y) = e^{2xy} + (e^t - 1),$$

and the initial boundary conditions

$$(6.3)\qquad z(0,x,y) = 0, \qquad (x,y) \in I \times I,$$

$$(6.4)\qquad \begin{aligned}\partial_x z(t,0.5,y) + z(t,0.5,y) &= \psi_1^{(+)}(t,y), \quad (t,y) \in I_+ \times I,\\ -\partial_x z(t,-0.5,y) + z(t,-0.5,y) &= \psi_1^{(-)}(t,y), \quad (t,y) \in I_+ \times I,\end{aligned}$$

$$(6.5)\qquad \begin{aligned}\partial_y z(t,x,0.5) + z(t,x,0.5) &= \psi_2^{(+)}(t,x), \quad (t,x) \in I_+ \times I,\\ \partial_y z(t,x,-0.5) + z(t,x,-0.5) &= \psi_2^{(-)}(t,x), \quad (t,x) \in I_+ \times I,\end{aligned}$$

where

$$\begin{aligned}\psi_1^{(+)}(t,y) &= e^{y}(e^t - 1)(1 + 2y), \quad \psi_1^{(-)}(t,y) = e^{-y}(e^t - 1)(1 - 2y),\\ \psi_2^{(+)}(t,x) &= e^{x}(e^t - 1)(1 + 2x), \quad \psi_2^{(-)}(t,x) = e^{-x}(e^t - 1)(1 - 2x).\end{aligned}$$

The solution of (6.2)–(6.5) is known to be $v(t,x,y) = (e^t - 1)e^{2xy}$.

Let us denote by $u_{\mathbf{h}} : E_{\mathbf{h}} \to \mathbb{R}$ the solution of the implicit difference problem corresponding to (6.2)–(6.5). Write

$$(6.6)\quad \varepsilon_h^{(r)} = \frac{1}{(2M_1+1)(2M_2+1)} \sum_{m=-(M_1,M_2)}^{(M_1,M_2)} |u_{\mathbf{h}}^{(r,m)} - v^{(r,m)}|, \ 0 \le r \le K,$$

where $M_1 h_1 = 0.5$, $M_2 h_2 = 0.5$. The numbers $\varepsilon_h^{(r)}$ are the arithmetical means of the errors. In Table I we give experimental values for $\varepsilon_h^{(r)}$ and $h_0 = \frac{1}{128}$, $h_1 = h_2 = \frac{1}{64}$, where (h_0, h_1, h_2) are steps of the mesh with respect to (t, x, y), respectively.

Table I

$t^{(r)}$	0.125	0.188	0.250	0.312	0.438
$\varepsilon_h^{(r)}$	$2.96e-04$	$3.73e-04$	$4.16e-04$	$4.35e-04$	$4.25e-04$

Note that condition (1.5) is not satisfied in the above example and the explicit difference method is not convergent. The average errors of the explicit difference method exceed 10^6.

Now we consider a differential equation with deviated variables.

EXAMPLE 6.3. Let $n = 2$, $E = [0, 0.5] \times [-1, 1] \times [-1, 1]$ and $E_0 = \{0\} \times [-1, 1] \times [-1, 1]$. Consider the differential equation with deviated variables

$$\begin{aligned}\partial_t z(t,x,y) = {} & 2\partial_{xx} z(t,x,y) + 2\partial_{yy} z(t,x,y) \\ & - \cos[\partial_{xx} z(t,x,y) - \partial_{yy} z(t,x,y)] \\ & + z(t, 0.5(x+y), 0.5(x-y)) \\ & - 4z(t,x,y) + e^{x-y} + 1 - te^{y}\end{aligned} \tag{6.7}$$

and the initial boundary conditions

$$z(0,x,y) = 0, \qquad (x,y) \in [-1,1] \times [-1,1], \tag{6.8}$$

$$\begin{aligned}&\partial_x z(t,1,y) + z(t,1,y) = 2te^{1-y}, \ -\partial_x z(t,-1,y) + z(t,-1,y) = 0, \\ &\partial_y z(t,x,1) + z(t,x,1) = 0, \ -\partial_y z(t,x,-1) + z(t,x,-1) = 2te^{x+1}.\end{aligned} \tag{6.9}$$

The solution of (6.7)–(6.9) is known to be $v(t,x,y) = te^{x-y}$. Let us denote by $\tilde{u}_{\mathbf{h}} : E_{\mathbf{h}} \to \mathbb{R}$ the solution of the implicit difference problem corresponding to (6.7)–(6.9). Let $u_{\mathbf{h}} : E_{\mathbf{h}} \to \mathbb{R}$ be a numerical approximation of $\tilde{u}_{\mathbf{h}}$ which is obtained by using the Newton method. We have calculated three Newton iterations. Let $\varepsilon_{\mathbf{h}}^{(r)}$ be defined by (6.6) with $M_1 h_1 = 1$, $M_2 h_2 = 1$. In Table II we give experimental values of $\varepsilon_{\mathbf{h}}^{(r)}$ for $h_0 = \frac{1}{64}$, $h_1 = h_2 = \frac{1}{64}$.

Table II

$t^{(r)}$	0.125	0.188	0.250	0.312	0.438
$\varepsilon_h^{(r)}$	$4.15e-03$	$5.64e-03$	$6.90e-03$	$8.01e-03$	$9.95e-03$

Note that condition (1.5) is not satisfied in the above example and the explicit difference method is not convergent.

Our results show that there are implicit difference methods for nonlinear parabolic functional differential equations with general initial boundary conditions which are convergent while the corresponding explicit difference schemes are not convergent.

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