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On some noetherian rings of C^∞ germs on a real closed field

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Abstract. Let R be a real closed field, and denote by $\mathcal{E}_{R,n}$ the ring of germs, at the origin of R^n, of C^∞ functions in a neighborhood of $0 \in R^n$. For each $n \in \mathbb{N}$, we construct a quasianalytic subring $\mathcal{A}_{R,n} \subset \mathcal{E}_{R,n}$ with some natural properties. We prove that, for each $n \in \mathbb{N}$, $\mathcal{A}_{R,n}$ is a noetherian ring and if $R = \mathbb{R}$ (the field of real numbers), then $\mathcal{A}_{\mathbb{R},n} = \mathcal{H}_n$, where $\mathcal{H}_n$ is the ring of germs, at the origin of $\mathbb{R}^n$, of real analytic functions. Finally, we prove the Real Nullstellensatz and solve Hilbert's 17th Problem for the ring $\mathcal{A}_{R,n}$.

1. Introduction. As is well known, real algebraic geometry is the study of sets of roots of polynomials and domains where they have a constant sign. For example, Hilbert's 17th Problem originally asked whether every nonnegative polynomial could be represented as a sum of squares of rational functions. Hilbert's 17th Problem was solved in the affirmative by Artin; the proof was based on the fact that a polynomial that is positive on $\mathbb{R}^n$ is also positive on other fields which contain the field of fractions $\mathbb{R}(X_1, \ldots, X_n)$. These fields have the same algebraic properties as $\mathbb{R}$. They are real closed (a *real closed field* is an ordered field in which every positive element is a square, and every odd-degree polynomial in one variable has a zero). However these fields can have infinitesimal points so they are not archimedean and in general the interval topology is not complete. Many problems in real algebraic geometry are solved by considering them over the elementary class of the field of reals, which is the class of real closed fields.

The definition of a real algebraic and semi-algebraic set over $\mathbb{R}$ can be extended to a general real closed field. Most of the properties can also be extended. A more detailed exposition can be found in [3].

The notion of analytic functions on $\mathbb{R}$ cannot be extended to a general real closed field. The main object of this paper is to introduce axiomatically an analogue of the ring of germs of real analytic functions on $\mathbb{R}^n$ for an arbitrary real closed field R.

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We let $\mathcal{E}_{R,n}$ denote the ring of germs, at the origin of R^n, of C^∞ functions in a neighborhood of $0 \in R^n$. A *differential system* is a sequence $\mathcal{A} = \{\mathcal{A}_{R,n} \mid n \geq 1\}$, where $\mathcal{A}_{R,n} \subset \mathcal{E}_{R,n}$ is a subring. We suppose that the system $\mathcal{A}$ has some natural properties. First we prove that the Weierstrass division and preparation theorems hold in $\mathcal{A}$. We then deduce that $\mathcal{A}_{R,n}$ is a noetherian ring for each $n \in \mathbb{N}$. We also prove that if $R = \mathbb{R}$ then, for each $n \in \mathbb{N}$, $\mathcal{A}_{\mathbb{R},n} = \mathcal{H}_n$, where $\mathcal{H}_n$ is the ring of germs, at the origin of $\mathbb{R}^n$, of real analytic functions. Using this system we extend to a real closed field the theory of semi-analytic sets developed by Łojasiewicz [7]. Finally, we give a Real Nullstellensatz and solve Hilbert's 17th Problem for the ring $\mathcal{A}_{R,n}$.

2. Preliminaries. Let R be a real closed field (possibly different from $\mathbb{R}$). For $x = (x_1, \ldots, x_n) \in R^n$, put $|x| = \sup(|x_1|, \ldots, |x_n|)$, where $|r| = \max(r, -r)$ for $r \in R$. For each $a \in R$ and $r \in R$, $r > 0$, we put $B_n(a, r) = \{x \in R^n \mid |x - a| < r\}$. We consider on R^n the *interval topology*, that is, a subset $U \subset R^n$ is open if $U = \emptyset$ or for each $a \in U$ there exists $r \in R$, $r > 0$, such that $B_n(a, r) \subset U$.

Consider an open set $U \subset R^n$ and a map $f : U \subset R^n \to R$. We call f *differentiable* at the point $a \in U$ if there is an R-linear map $T : R^n \to R$ such that for each $\varepsilon > 0$ in R we have $|f(x + a) - f(a) - T(x)| \leq \varepsilon |x|$ for all sufficiently small vectors x in R^n. (Such a linear map is necessarily unique.) Clearly if f is differentiable at a, then f is continuous at a and the partial derivatives $\frac{\partial f}{\partial x_i}(a), i = 1, \ldots, n$, exist. We call f a C^0 map if f is continuous, and inductively we define f to be a C^{k+1} map ($k \geq 0$) if f is differentiable at each point $a \in U$ and, for each $i = 1, \ldots, n$, the map $a \mapsto \frac{\partial f}{\partial x_i}(a)$ is a C^k map. Note that if f is C^{k+1}, then it is C^k. Finally, we call f a C^∞ map if it is a C^k map for all $k \in \mathbb{N}$; in that case all partial derivatives of f of all orders exist and are continuous on U. We leave to the reader the statement and proofs of the usual formal rules such as the chain rule for composition of C^1 maps.

If $U_1, U_2 \subset R^n$ are two open neighborhoods of $a \in R^n$, and $f_1 : U_1 \to R$, $f_2 : U_2 \to R$ are two C^∞ maps, then $(f_1, U_1) \sim (f_2, U_2)$ means that there exists some open neighborhood U of a with $U \subset U_1 \cap U_2$ such that $f_1(x) = f_2(x)$ for all $x \in U$. This is clearly an equivalence relation. We let $\mathcal{E}_{R,n,a}$ denote the set of equivalence classes, or *germs at* a. Clearly $\mathcal{E}_{R,n,a}$ is a ring. If a is the origin of R^n, we write $\mathcal{E}_{R,n}$ instead for $\mathcal{E}_{R,n,0}$.

3. Quasianalytic systems

3.1. Differentiable systems. Let R be a real closed field.

DEFINITION 3.1. A *differentiable system* is a sequence $\mathcal{A}_R = \{\mathcal{A}_{R,n} \mid n \in \mathbb{N}\}$ such that, for each $n \in \mathbb{N}$, $\mathcal{A}_{R,n}$ is a local subring of $\mathcal{E}_{R,n}$ closed under taking derivatives and satisfying:

(A_1) $R[x_1,\dots,x_n]\subset \mathcal{A}_{R,n}\subset \mathcal{E}_{R,n}$ for each $n\in\mathbb{N}$, where $R[x_1,\dots,x_n]$ is the ring of polynomials with coefficients in R.

(A_2) $\mathcal{A}_R$ is closed under composition: if $g\in\mathcal{A}_{R,k}$ and $f=(f_1,\dots,f_k)\in(\mathcal{A}_{R,n})^k$ with $f(0)=0$, then $g\circ f\in\mathcal{A}_{R,n}$.

(A_3) For each $n\in\mathbb{N}$, $\mathcal{A}_{R,n}$ is closed under division by coordinates: if $f\in\mathcal{A}_{R,n}$ with $f(x_1,\dots,x_{i-1},a_i,x_{i+1},\dots,x_n)$ is identically zero then $f(x)=(x_i-a_i)g(x)$ with $g\in\mathcal{A}_{R,n}$.

(A_4) The Implicit Function Theorem holds for $\mathcal{A}_R$: if $f=(f_1,\dots,f_m)\in(\mathcal{A}_{R,n+m})^m$ with $f(0,0)=0$, and $y=(y_1,\dots,y_m)$, and if

$$\det\left(\frac{\partial f_i}{\partial y_j}(0,0)\right)_{i,j=1,\dots,m}\neq 0,$$

then there is a (unique) $g=(g_1,\dots,g_m)\in(\mathcal{A}_{R,n})^m$ with $g(0)=0$ such that $f(x,g(x))=0$.

Let

$$\hat{\cdot}:\mathcal{A}_{R,n}\to R[[x_1,\dots,x_n]]$$

be the map which associates to each $f\in\mathcal{A}_{R,n}$ its Taylor expansion at the origin. We consider the following condition:

(A_5) $\hat{\cdot}$ is an injective homomorphism.

Definition 3.2. A differentiable system is called *quasianalytic* if the condition (A_5) holds.

Remark. In the following, for a differentiable quasianalytic system, we will not distinguish notationally between the germ and its image under $\hat{\cdot}$, i.e. its Taylor expansion at the origin.

In particular these conditions on a quasianalytic system imply that the maximal ideal of $\mathcal{A}_{R,n}$ is $\underline{m}=\{f\in\mathcal{A}_{R,n}\mid f(0)=0\}=(x_1,\dots,x_n)\mathcal{A}_{R,n}$ and its completion in the $\underline{m}$-adic topology is $R[[x_1,\dots,x_n]]$.

Recall that a subset of R^n is called *semi-algebraic* if it can be represented as a (finite) boolean combination of sets of the form $\{x\in R^n\mid P(x)=0\}$, $\{x\in R^n\mid Q(x)>0\}$ where $P(x),Q(x)$ are in $R[x_1,\dots,x_n]$.

Example 3.3.

(i) Let $U\subset R^n$ be an open semi-algebraic set and let $f:U\subset R^n\to R$ be a C^∞ map. We call f a *Nash map* if its graph $\Gamma_f=\{(x,f(x))\mid x\in U\}$ is a semi-algebraic set. We denote by $\mathcal{N}_{R,n}$ the set of all germs $\bar{f}\in\mathcal{E}_{R,n}$ such that $\bar{f}$ can be represented by a couple (U,f) where U is an open semi-algebraic set and f is a Nash map. Clearly $\mathcal{N}_{R,n}\subset\mathcal{E}_{R,n}$ is a subring [3, 8.1.8]. The system $\mathcal{N}_R=\{\mathcal{N}_{R,n}\mid n\in\mathbb{N}\}$ is a differentiable system [3]. By [3, 8.1.5] it is a quasianalytic system.

(ii) Suppose $R = \mathbb{R}$ and put $\mathcal{H}_\mathbb{R} = \{\mathcal{H}_{\mathbb{R},n} \mid n \in \mathbb{N}\}$ where $\mathcal{H}_{\mathbb{R},n}$ is the ring of germs, at the origin of $\mathbb{R}^n$, of real analytic functions. Clearly, $\mathcal{H}_\mathbb{R}$ is a quasianalytic system!

3.2. Strongly quasianalytic systems. Fix a differentiable quasianalytic system $\mathcal{A}_R = \{\mathcal{A}_{R,n} \mid n \in \mathbb{N}\}$. Let $\varphi = (\varphi_1, \dots, \varphi_m)$ with $\varphi_i \in \mathcal{A}_{R,n}$, $i = 1, \dots, m$. We suppose that $\varphi(0) = 0$, and let $x = (x_1, \dots, x_n)$, and $y = (y_1, \dots, y_m)$ be the coordinates in R^n and R^m respectively. The *generic rank* of φ, $\mathrm{rk}(\varphi)$, is the rank of the Jacobian matrix $\big[\frac{\partial \varphi_i}{\partial x_j}\big]_{i=1,\dots,m,j=1,\dots,n}$, considered as a matrix over the quotient field of $\mathcal{A}_{R,n}$ (recall that $\mathcal{A}_{R,n}$ is a domain by (A_5)). We consider the morphism $\phi : \mathcal{A}_{R,m} \to \mathcal{A}_{R,n}$ defined by composition with φ, and its extension to the completion, $\hat{\phi} : R[[y_1, \dots, y_m]] \to R[[x_1, \dots, x_n]]$.

DEFINITION 3.4. We say that the differentiable quasianalytic system $\mathcal{A}_R$ is *strongly quasianalytic* if for each φ as above with $\mathrm{rk}(\varphi) = m$ and for each $\psi \in R[[y_1, \dots, y_m]]$ such that $\hat{\phi}(\psi) = \hat{g}$ with $g \in \mathcal{A}_{R,n}$, there exists $\beta \in \mathcal{A}_{R,m}$ such that $\hat{\beta} = \psi$.

EXAMPLE 3.5.

(i) Suppose $R = \mathbb{R}$, and consider the quasianalytic system $\mathcal{H}_\mathbb{R}$ as in Example 3.3(ii). The main result of [5] says that this system is strongly quasianalytic.
(ii) By [3, 8.2.9], the Weierstrass division theorem holds in the quasianalytic system $\mathcal{N}_R$. By going through the proof of the strong quasianalyticity of the system $\mathcal{H}_\mathbb{R}$ in [5], we see that the system $\mathcal{N}_R$ is strongly quasianalytic.

4. Newton's theorem for a strongly quasianalytic system. In this section, $\mathcal{A}_R = \{\mathcal{A}_{R,n} \mid n \in \mathbb{N}\}$ is a strongly quasianalytic system. If $\lambda = (\lambda_1, \dots, \lambda_p) \in R^p$, then $P(x_n, \lambda) = x_n^p + \sum_{i=1}^p \lambda_i x_n^{p-i}$ is called a *generic polynomial* in x_n of degree p. If $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we put $x' = (x_1, \dots, x_{n-1})$.

Let $\sigma = (\sigma_1, \dots, \sigma_p) : R^p \to R^p$ be the polynomial map defined by $\sigma_i = (-1)^{-i}\tilde{\sigma}_i$, where the $\tilde{\sigma}_i$ are the elementary symmetric functions in the variables $\lambda_1, \dots, \lambda_p$. Then we have $P(x_n, \sigma(\lambda)) = (x_n - \lambda_1) \dots (x_n - \lambda_p)$. The function $\varphi(x, \lambda) = (x, \sigma(\lambda))$ defines, by substitution, a morphism $\phi : \mathcal{A}_{R,n+p} \to \mathcal{A}_{R,n+p}$. We can see that $\mathrm{rk}(\varphi) = n + p$ and ϕ is an injective morphism.

LEMMA 4.1. *Let $f(x, \lambda) \in \mathcal{A}_{R,n+p}$ be symmetric with respect to the variables λ_i, $i = 1, \dots, p$. Then there exists $g \in \mathcal{A}_{R,n+p}$ such that $g(x, \sigma(\lambda)) = f(x, \lambda)$.*

Proof. By hypothesis $\hat{f}(x,\lambda) \in R[[x,\lambda]]$ is symmetric with respect to the variables $\lambda_i, i = 1,\ldots,p$. By Newton's theorem (for formal series, see [4]), there exists $h \in R[[x,\lambda]]$ such that $h(x,\sigma(\lambda)) = \hat{f}(x,\lambda)$, i.e. $\hat{\phi}(h) = \hat{f}$. Since $\mathrm{rk}(\varphi) = n+p$ $(\varphi(x,\lambda) = (x,\sigma(\lambda)))$ and the system $\mathcal{A}_R$ is strongly quasianalytic, there exists $g \in \mathcal{A}_{R,n+p}$ such that $\hat{g} = h$; hence, by quasianalyticity, $\phi(g) = f$, i.e. $g(x,\sigma(\lambda)) = f(x,\lambda)$. ∎

5. Generic division

Lemma 5.1. *Let $P(x_n,\lambda)$ be a generic polynomial in x_n of degree p. If $f \in \mathcal{A}_{R,n}$ then there exist unique $q \in \mathcal{A}_{R,n+p}$ and $r_i \in \mathcal{A}_{R,n-1+p}$, $1 \le i \le p$, such that*

$$f(x) = P(x_n,\lambda)q(x,\lambda) + \sum_{i=1}^{p} r_i(x',\lambda)x_n^{p-i}.$$

Proof. We consider $f(x',x_n)$ and $f(x',\lambda_1)$ as elements of $\mathcal{A}_{R,n+p}$. Since $\mathcal{A}_{R,n+p}$ is closed under division by coordinates,

$$f(x',x_n) - f(x',\lambda_1) = (x_n-\lambda_1)f_1(x,\lambda)$$

with $f_1 \in \mathcal{A}_{R,n+p}$ again. By repeating this process with f_1,

$$f_1(x',x_n,\lambda) - f_1(x',\lambda_2,\lambda) = (x_n-\lambda_2)f_2(x,\lambda)$$

with $f_2 \in \mathcal{A}_{R,n+p}$. Finally, we get

$$f(x) = g(x,\lambda)(x_n-\lambda_1)\ldots(x_n-\lambda_p) + \sum_{i=1}^{p} g_i(x',\lambda)x_n^{p-i}$$

with $g \in \mathcal{A}_{R,n+p}$ and $g_i \in \mathcal{A}_{R,n-1+p}$, $1 \le i \le p$.

We see that g and g_i, $1 \le i \le p$, are symmetric with respect to $\lambda_1,\ldots,\lambda_p$, since g and the g_i are uniquely determined in the above representation of $f(x)$, by the quasianalyticity of the system $\mathcal{A}_R$. By Lemma 4.1, there exist $q \in \mathcal{A}_{R,n+p}$ and $r_i \in \mathcal{A}_{R,n-1+p}$, $1 \le i \le p$, such that

$$g = \phi(q), \quad g_i = \phi(r_i).$$

Then

$$f = \phi(f) = \phi(q)\phi(P) + \sum_i \phi(r_i)x_n^{p-i},$$

and this implies that

$$f(x) = P(x,\lambda)q(x,\lambda) + \sum_{i=1}^{p} r_i(x',\lambda)x_n^{p-i},$$

since ϕ is injective. The uniqueness follows from the quasianalyticity of the system $\mathcal{A}_R$. ∎

6. Weierstrass division theorem. Let $f \in \mathcal{A}_{R,n} - \{0\}$. We say that f is *regular of order* p with respect to x_n if the formal series $\hat{f}$ is regular of order p with respect to x_n. For any $f \in \mathcal{A}_{R,n} - \{0\}$, after making a linear transformation of $(x_1, \dots, x_n)$, there is an integer $p \in \mathbb{N}$ such that f is regular of order p with respect to x_n.

THEOREM 6.1. *Let $f \in \mathcal{A}_{R,n} - \{0\}$ be regular of order p with respect to x_n. Then for any $g \in \mathcal{A}_{R,n}$, there exist unique $q \in \mathcal{A}_{R,n}$ and $r_i \in \mathcal{A}_{R,n-1}$, $1 \le i \le p$, such that*

$$g(x) = f(x)q(x) + \sum_{i=1}^{p} r_i(x')x_n^{p-i}.$$

Proof. By Lemma 5.1, we divide f by the generic polynomial $P(x_n, \lambda) = x_n^p + \sum_{i=1}^{p} \lambda_i x_n^{p-i}$. We have

$$(*) \qquad f(x) = P(x_n, \lambda)q(x, \lambda) + \sum_{i=1}^{p} r_i(x', \lambda)x_n^{p-i}$$

with $q(x, \lambda) \in \mathcal{A}_{R,n+p}$ and $r_i(x', \lambda) \in \mathcal{A}_{R,n-1+p}$, $1 \le i \le p$. Since f is regular of order p with respect x_n, we can easily see that

$$q(0,0) \neq 0, \qquad r_i(0,0) = 0, \qquad 1 \le i \le p.$$

We shall show that the $p \times p$ matrix $\left[\frac{\partial r_i}{\partial \lambda_j}(0,0)\right]_{i,j=1,\dots,p}$ is nonsingular. By applying $\frac{\partial}{\partial \lambda_k}$, $k = 1, \dots, p$, to $(*)$, we find

$$q(0, x_n, 0)x_n^{p-k} + x_n^p \frac{\partial q(0, x_n, 0)}{\partial \lambda_k} + \sum_{i=1}^{p} \frac{\partial r_i(0,0)}{\partial \lambda_k} x_n^{p-i} = 0.$$

Since $q(0,0) \neq 0$, we see that

$$\frac{\partial r_i(0,0)}{\partial \lambda_k} = -\delta_{ik} q(0,0), \qquad i, k = 1, \dots, p,$$

where δ_{ik} is the Kronecker symbol. We then see that the determinant of the matrix $\left[\frac{\partial r_i}{\partial \lambda_j}(0,0)\right]_{i,j=1,\dots,p}$ is

$$\frac{D(r_1, \dots, r_p)}{D(\lambda_1, \dots, \lambda_p)}(0,0) = (-1)^p (q(0,0))^p.$$

By the implicit function theorem (A_4), there is $\psi(x') = (\psi_1(x') \dots, \psi_p(x')) \in (\mathcal{A}_{R,n-1})^p$, $\psi(0) = 0$, such that $r_i(x', \psi(x')) = 0$, $1 \le i \le p$. Hence $f(x) = q(x, \psi(x'))P(x_n, \psi(x'))$. Since $q(x, \psi(x'))$ is invertible in $\mathcal{A}_{R,n}$, f is equivalent in $\mathcal{A}_{R,n}$ to the polynomial $x_n^p + \sum_{i=1}^{p} \psi_i(x')x_n^{p-i}$.

Now let $g \in \mathcal{A}_{R,n}$. We divide g by the generic polynomial $P(x_n, \lambda)$ and hence by $P(x_n, \psi(x'))$ after the substitution $\lambda \mapsto \psi(x')$. Since $q(x', \psi(x'))$

is invertible in $\mathcal{A}_{R,n}$, we have proved the existence. The uniqueness follows from the quasianalyticity of the system $\mathcal{A}_R$. ∎

From Theorem 6.1, we have:

COROLLARY 6.2. *$\mathcal{A}_{R,n}$ is a regular local ring of dimension n. In particular it is a noetherian unique factorization domain.*

Proof. Using Theorem 6.1 we prove, by induction on $n \in \mathbb{N}$, that $\mathcal{A}_{R,n}$ is a noetherian ring. The proof is the same as that given in [9, II.1.5, 1.6] for real analytic systems. ∎

The following consequence of Theorem 6.1 is the approximation theorem of M. Artin [1]. It was announced for $\mathcal{N}_R$ in [3, 8.3.1]. The proof for $\mathcal{A}_R$ is the same as that given in [9, III.4.2] for real analytic systems. This proof uses the implicit function theorem and Weierstrass's division theorem.

THEOREM 6.3. *Let $f = (f_1, \ldots, f_q) \in (\mathcal{A}_{R,n+m})^q$ be such that $f(0,0) = 0$. Consider a solution $\hat{y}(x) = (\hat{y}_1(x), \ldots, \hat{y}_m(x)) \in R[[x_1, \ldots, x_n]]$ of the equation $f(x,y) = 0$. Then for any integer $s \geq 1$ there exists a solution $y(x) = (y_1(x), \ldots, y_m(x)) \in (\mathcal{A}_{R,n})^m$ of $f(x,y) = 0$ such that $\hat{y}_i(x) - y_i(x) \in \underline{m}^s$ for $i = 1, \ldots, m$.*

7. Strongly quasianalytic systems associated to a real closed field. If $\mathcal{A}_R$ and $\mathcal{A}'_R$ are two differentiable systems, we will write $\mathcal{A}_R \subset \mathcal{A}'_R$ to mean that $\mathcal{A}_{R,n} \subset \mathcal{A}'_{R,n}$ for all $n \in \mathbb{N}$.

Let R be a real closed field and consider the family

$$\mathcal{F}_R = \{\mathcal{A}'_R \mid \mathcal{A}'_R \text{ is a strongly quasianalytic system}, \mathcal{N}_R \subset \mathcal{A}'_R\}.$$

DEFINITION 7.1. For a real closed field R, a maximal element of $\mathcal{F}_R$ will be called a *strongly quasianalytic system associated to R*.

In the following, we prove that if R is the field of reals, there exists a unique strongly quasianalytic system associated to $\mathbb{R}$, which is the system given in Example 3.3(ii). The following result, proved in [6], will be used.

THEOREM 7.2. *Let $\mathcal{C}_\mathbb{R}$ be a quasianalytic system such that the Weierstrass division theorem holds in the ring $\mathcal{C}_{\mathbb{R},3}$. Then $\mathcal{C}_\mathbb{R} \subset \mathcal{H}_\mathbb{R}$.*

We recall that if $\mathcal{H}_n$ is the ring of germs, at the origin of $\mathbb{R}^n$, of analytic functions, then we put $\mathcal{H}_\mathbb{R} = \{\mathcal{H}_n \subset \mathcal{E}_{\mathbb{R},n}\}$.

COROLLARY 7.3. *Let $\mathcal{A}_\mathbb{R}$ be a strongly quasianalytic system associated to $\mathbb{R}$. Then $\mathcal{A}_\mathbb{R} = \mathcal{H}_\mathbb{R}$.*

Proof. First, by Theorem 6.1, Weierstrass's division theorem holds in $\mathcal{A}_\mathbb{R}$, hence by Theorem 7.2, we have $\mathcal{A}_\mathbb{R} \subset \mathcal{H}_\mathbb{R}$. Moreover $\mathcal{N}_\mathbb{R} \subset \mathcal{H}_\mathbb{R}$ and by [5] we know that $\mathcal{H}_\mathbb{R}$ is a strongly quasianalytic system, hence $\mathcal{H}_\mathbb{R} = \mathcal{A}_\mathbb{R}$. ∎

8. Semi-$\mathcal{A}_R$ germs

DEFINITION 8.1.

(i) A *neighborhood system* of the origin in R^n (n.s.) is a collection of open subsets of R^n which contains the origin.
(ii) If $\mathcal{U}$ is a n.s. in R^n, then $\tilde{\mathcal{A}}_{R,n}(\mathcal{U})$ denotes the set of all pairs (U, f) where $U \in \mathcal{U}$ and $f : U \to R$ is a C^∞ map such that $\hat{f} \in \mathcal{A}_{R,n}$.
(iii) A *semi-$\mathcal{A}_R$ germ* is a germ of a set, at the origin of R^n, which can be represented as a (finite) boolean combination of germs of sets of the form $\{x \in U_1 \mid f_1(x) = 0\}$, $\{x \in U_2 \mid f_2(x) > 0\}$ where (U_1, f_1), $(U_2, f_2) \in \tilde{\mathcal{A}}_{R,n}(\mathcal{U})$.
(iv) A semi-$\mathcal{A}_R$ germ, A, is called *semi-$\mathcal{A}_R$ connected* if there do not exist disjoint semi-$\mathcal{A}_R$ germs A_1, A_2 both open in A such that $A = A_1 \cup A_2$.

As for germs of semi-analytic sets in $\mathbb{R}^n$, we prove some properties of semi-$\mathcal{A}_R$ germs. The proof is the same as in the case of $R = \mathbb{R}$, but we give a brief summary for completeness.

THEOREM 8.2. *Let $P_1(x, t), \dots, P_l(x, t)$ be polynomials of $n+1$ variables, where $x = (x_1, \dots, x_n)$. Then there is a semi-algebraic partition $\{A_1, \dots, A_m\}$ of R^n such that, for each $k = 1, \dots, m$, the zeros of $P_1(x, t), \dots, P_l(x, t)$ on A_k are given by continuous semi-algebraic functions $\xi_1 < \dots < \xi_{r_k}$ and the sign of each $P_j(x, y)$ on A_k depends only on the sign of $y - \xi_i(x)$, $i = 1, \dots, r_k$.*

For the proof see [3, 2.3.1].

DEFINITION 8.3. Let $U \in \mathcal{U}$ and $f : U \subset R^n \to R$ a map such that $f(0) = 0$. We say that f is a *semi-$\mathcal{A}_R$ map* if the germ of its graph, at $(0, 0) \in R^n \times R$, is a semi-$\mathcal{A}_R$ germ.

PROPOSITION 8.4. *Let $f_1(x, y), \dots, f_q(x, y) \in \tilde{\mathcal{A}}_{R,n}(\mathcal{U})[y]$. Then there is $U \in \mathcal{U}$ in which all f_i are defined and a partition $\{A_1, \dots, A_s\}$ of U such that, for each $k = 1, \dots, s$:*

(1) *The germ of A_k at $0 \in R^n$ is a semi-$\mathcal{A}_R$ germ.*
(2) *The zeros of $f_1, \dots, f_q$ on A_k are given by continuous semi-$\mathcal{A}_R$ maps $\xi_1 < \dots < \xi_{r_k}$.*
(3) *The sign of each $f_j(x, y)$ on A_k depends only on the sign of the $y - \xi_i(x)$.*

Proof. Let $f_j(x, y) = \sum_{k=1}^{N} \lambda_{jk}(x) y^k$, $j = 1, \dots, t$, where each $\lambda_{jk} \in \tilde{\mathcal{A}}_{R,n}(\mathcal{U})$. Each λ_{jk} is a C^∞ map on $U_{jk} \in \mathcal{U}$. We put $U = \bigcap_{j,k} U_{jk}$. Define the polynomials $P_j(Z, y) = \sum_{k=1}^{N} Z_{jk} y^k$ where $Z = (Z_{jk})$ are variables. For the finite family of polynomials $P_1(Z, y), \dots, P_t(Z, y)$, consider the semi-algebraic partition $\{A'_k\}$ and, for each k, the continuous semi-algebraic functions $\xi'_i(Z)$ given by Theorem 8.2. Then $f_j(x, \xi'_i(Z(x))) = P_j(Z(x), \xi'_i(Z(x))) = 0$ whenever $Z(x) \in A'_k$. Take $A_k = Z^{-1}(A'_k)$ and $\xi_i = \xi'_i \circ Z$. ■

DEFINITION 8.5. A finite family $f_1, \dots, f_m \in \tilde{\mathcal{A}}_{R,n}(\mathcal{U})$ is *separating* if there exists an open neighborhood $U \subset R^n$ of the origin such that all the f_j are defined and C^∞ maps on U, and for any subset $A \subset U$ of the form

$$A = \bigcap_{i=1}^{m} \{x \in U \mid f_i(x)\, \sigma_i\, 0\},$$

where each σ_i is in $\{<, >, =\}$, we have:

(i) The germ of A at $0 \in R^n$ is either empty or semi-$\mathcal{A}_R$ connected.

(ii) If $A \neq \emptyset$, then the germ at $0 \in R^n$ of the set

$$\bigcap_{i=1}^{m} \{x \in U \mid f_i(x)\, \overline{\sigma_i}\, 0\},$$

where $\overline{\sigma_i} \in \{\leq, \geq, =\}$, is exactly the germ at $0 \in R^n$ of the closure of A in U.

THEOREM 8.6. *Any finite family of elements in $\tilde{\mathcal{A}}_{R,n}(\mathcal{U})$ can be completed to a separating family.*

Proof. By induction on n, the proof is the same as for $R = \mathbb{R}$ by using Weierstrass's division theorem (see [2, 2.6]). ■

COROLLARY 8.7. *Each semi-$\mathcal{A}_R$ germ is the union of a finite number of semi-$\mathcal{A}_R$ connected germs. The closure, and thus the interior, of a semi-$\mathcal{A}_R$ germ is also a semi-$\mathcal{A}_R$ germ.*

9. The real spectrum of the ring $\mathcal{A}_{R,n}$

9.1. The real spectrum of a ring. The notion of the real spectrum of a ring was introduced by M. Coste and M. F. Roy in the late 1970's and soon became a fundamental tool in real geometry. A more detailed exposition can be found in [3], where the proofs are also given.

In this section we shall interpret the elements of the real spectrum of $\mathcal{A}_{R,n}$ by equivalence classes of certain points with coordinates in a real closed field R^* which is an extension of R.

Let A be a unital commutative ring. We consider the set of all nontrivial homomorphisms α from A into some real closed field R_α. We define an equivalence relation on this set as follows: if $\alpha : A \to R_\alpha$ and $\beta : A \to R_\beta$ are ring homomorphisms from A into real closed fields R_α and R_β respectively, we say that α and β are equivalent, and write $\alpha \sim \beta$, if $\alpha^{-1}(R_\alpha^2) = \beta^{-1}(R_\beta^2)$

where R^2 is the set of all nonnegative elements of the real closed field R. We denote by $[\alpha]$ the class of α.

Now the *real spectrum* of A is defined as

$$\operatorname{sper} A = \{[\alpha] \mid \alpha : A \to R,\ R \text{ a real closed field}\}.$$

9.2. Structure. A *language* $\mathcal{L}$ consists of:

(i) a set $\mathcal{F}$ of function symbols and a positive integer n_f for each $f \in \mathcal{F}$,
(ii) a set $\mathcal{R}$ of relation symbols and a positive integer n_R for each $R \in \mathcal{R}$,
(iii) a set $\mathcal{C}$ of constant symbols.

The numbers n_f and n_R tell us that f is a function of n_f variables and R is an n_R-ary relation. Some or all of the sets $\mathcal{F}$, $\mathcal{R}$ and $\mathcal{C}$ may be empty. Examples of languages:

(i) the language $\mathcal{L}_{\mathrm{r}}$ of rings where $\mathcal{F} = \{+, -, \cdot\}$, $\mathcal{R} = \emptyset$, $\mathcal{C} = \{0, 1\}$,
(ii) the language of ordered rings $\mathcal{L}_{\mathrm{or}} = \mathcal{L}_{\mathrm{r}} \cup \{<\}$ where $\mathcal{R} = \{<\}$.

Definition 9.1. An $\mathcal{L}$-*structure* $\mathcal{M}$ is given by the following data:

(i) a nonempty set M called the domain of $\mathcal{M}$,
(ii) a function $f^{\mathcal{M}} : M^{n_f} \to M$ for each $f \in \mathcal{F}$,
(iii) a set $R^{\mathcal{M}} \subset M^{n_R}$ for each $R \in \mathcal{R}$,
(iv) an element $c^{\mathcal{M}} \in M$ for each $c \in \mathcal{C}$.

An $\mathcal{L}$-*term* is a finite sequence of symbols obtained by repeated application of the following rules:

(i) Each constant $c \in \mathcal{C}$ is an $\mathcal{L}$-term.
(ii) Variables are $\mathcal{L}$-terms.
(iii) If $f \in \mathcal{F}$ and $t_1, \ldots, t_{n_f}$ are $\mathcal{L}$-terms, then $f(t_1, \ldots, t_{n_f})$ is an $\mathcal{L}$-term.

Suppose $\mathcal{M}$ is an $\mathcal{L}$-structure and s is a term built using variables from $\overline{v} = (v_1, \ldots, v_{i_m})$. We want to interpret s as a function $s^{\mathcal{M}} : M^{i_m} \to M$. For $\overline{a} = (a_1, \ldots, a_{i_m}) \in M^{i_m}$, we inductively define $s^{\mathcal{M}}(\overline{a})$:

(i) If s is a constant symbol $c \in \mathcal{C}$, then $s^{\mathcal{M}}(\overline{a}) = c^{\mathcal{M}}$.
(ii) If s is the variable v_{i_j}, then $s^{\mathcal{M}}(\overline{a}) = a_{i_j}$.
(iii) If s is the term $f(t_1, \ldots, t_{n_f})$, where f is a function symbol and $t_1, \ldots, t_{n_f}$ are terms, then $s^{\mathcal{M}}(\overline{a}) = f^{\mathcal{M}}(t_1^{\mathcal{M}}(\overline{a}), \ldots, t_{n_f}^{\mathcal{M}}(\overline{a}))$.

We are now ready to define $\mathcal{L}$-formulas. We say that ϕ is an *atomic* $\mathcal{L}$-*formula* if ϕ is either

(i) $t_1 = t_2$ where t_1, t_2 are terms, or
(ii) $R(t_1, \ldots, t_{n_R})$ where $R \in \mathcal{R}$ and $t_1, \ldots, t_{n_R}$ are terms.

The set of $\mathcal{L}$-*formulas* is the smallest set $\mathcal{W}$ containing the atomic formulas such that:

(i) If $\phi \in \mathcal{W}$, then $\neg\phi$ is in $\mathcal{W}$.

(ii) If ϕ and ψ are in $\mathcal{W}$, then $\phi \wedge \psi$ and $\phi \vee \psi$ are in $\mathcal{W}$.
(iii) If $\phi \in \mathcal{W}$, then $\exists v_i \phi$ and $\forall v_i \phi$ are in $\mathcal{W}$.

We say that a variable *occurs freely* in a formula ϕ if it is not inside the scope of a quantifier, otherwise we say it is *bound*. We call an $\mathcal{L}$-formula an $\mathcal{L}$-*sentence* if it has no free variables. If $v_1, \ldots, v_m$ are distinct variables, we write $\phi(v_1, \ldots, v_m)$ to indicate a formula ϕ all of whose freely occurring variables are among $v_1, \ldots, v_m$.

Let $\mathcal{M}$ be an $\mathcal{L}$-structure and ϕ an $\mathcal{L}$-sentence. Then ϕ is either true or false in $\mathcal{M}$. If ϕ is true in $\mathcal{M}$, we write $\mathcal{M} \models \phi$.

We can extend the language $\mathcal{L}$ to the language $\mathcal{L}_M$ by adding a new constant symbol a for each element $a \in M$. We consider $\mathcal{M}$ as an $\mathcal{L}_M$-structure in the obvious way. Given an $\mathcal{L}$-formula $\phi(v_1, \ldots, v_m)$ and elements $a_1, \ldots, a_m \in M$ we let $\phi(a_1, \ldots, a_m)$ be an $\mathcal{L}_M$-sentence, hence $\phi(a_1, \ldots, a_m)$ is either true or false in M.

Let $\mathcal{M}$ and $\mathcal{N}$ be two $\mathcal{L}$-structures with domains respectively M and N, and suppose that $M \subset N$. If for all $\mathcal{L}$-formulas $\phi(v_1, \ldots, v_n)$ and all $a_1, \ldots, a_m \in M$, we have

$$\mathcal{M} \models \phi(a_1, \ldots, a_m) \;\Leftrightarrow\; \mathcal{N} \models \phi(a_1, \ldots, a_m),$$

we say that $\mathcal{N}$ is an *elementary extension* of $\mathcal{M}$.

Let $\mathcal{L}$ be a language and $\mathcal{M}$ an $\mathcal{L}$-structure with domain M. Let $Y \subset M$. We denote by $\mathrm{Fml}_n(\mathcal{L}_Y)$ the set of all $\mathcal{L}_Y$-formulas ϕ with free variables among $v_1, \ldots, v_n$. A subset $\Phi \subset \mathrm{Fml}_n(\mathcal{L}_Y)$ is said to be *realized* in $\mathcal{M}$ if there exists an element $\underline{a} = (a_1, \ldots, a_n) \in M^n$ such that, for all $\phi \in \Phi$, $\mathcal{M} \models \phi(\underline{a})$. Clearly if $\Phi \subset \mathrm{Fml}_n(\mathcal{L}_Y)$ is realized in $\mathcal{M}$, then every finite subset $\{\phi_1, \ldots, \phi_q\} \subset \Phi$ is realized in $\mathcal{M}$. The converse is not true.

We say that $\Phi \subset \mathrm{Fml}_n(\mathcal{L}_Y)$ is *finitely realized* in $\mathcal{M}$ if every finite subset of Φ is realized in $\mathcal{M}$. By definition, an $\mathcal{L}_Y$-*type* of $\mathcal{M}$ is a finitely realized subset of $\mathrm{Fml}_n(\mathcal{L}_Y)$.

Definition 9.2. Let κ be an infinite cardinal. An $\mathcal{L}$-structure $\mathcal{M}$ is called κ-*saturated* if for each subset $Y \subset M$ such that $\operatorname{card} Y < \kappa$, every $\mathcal{L}_Y$-type of $\mathcal{M}$ is realized in $\mathcal{M}$.

The following theorem is usually proved for saturated structures.

Theorem 9.3. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure and let κ be the cardinal of $\mathcal{L}$ ($\kappa = \max\{\aleph_0, \operatorname{card}\mathcal{F}, \operatorname{card}\mathcal{R}, \operatorname{card}\mathcal{C}\}$). Then there exists a κ^+-saturated elementary extension $\mathcal{M}^*$ of $\mathcal{M}$.*

Recall that κ^+ is the next cardinal after κ.

Now let R be a real closed field. We consider the language of ordered rings, $\mathcal{L}_{\mathrm{or}}$. We denote by $\overline{R}$ the $\mathcal{L}_{\mathrm{or}}$-structure whose domain is R. We extend the language $\mathcal{L}_{\mathrm{or}}$ by introducing for each $f \in \tilde{\mathcal{A}}_{R,n}(\mathcal{U})$ a function symbol, denoted again by f. We write $\mathcal{L}$ for the extended language. Our structure is $(\overline{R}, (f), f \in \tilde{\mathcal{A}}_R(\mathcal{U}))$. Let us recall that for each $f \in \tilde{\mathcal{A}}_{R,n}(\mathcal{U})$ there is an open neighborhood U of the origin in R^n such that f is a C^∞ map on U with values in R and the germ of f at $0 \in R^n$ is in $\mathcal{A}_{R,n}$. As a function on an $\mathcal{L}$-structure is a total function, we extend f to all R^n by setting it to 0 outside of U.

We let κ be the cardinality of R. The extended language $\mathcal{L}$ then has cardinality $\nu = \max\{\aleph_0, 2^\kappa\}$. By Theorem 9.3, there is an elementary extension $\tilde{R}^*$ of $\overline{R}$ that is ν^+-saturated. Let R^* be its domain.

Let $f \in \tilde{\mathcal{A}}_{R,n}(\mathcal{U})$. We denote by $\bar{f}$ its germ at the origin of R^n (recall that $\bar{f} \in \mathcal{A}_{R,n}$), and by f^* its extension to R^{*n}. Let

$$M_R = \{a \in R^* \mid |a| < \epsilon \text{ for all } \epsilon \in R,\ \epsilon > 0\}$$

be the ideal of infinitesimals. If $x = (x_1, \dots, x_n) \in R^n$, we put $\|x\| = \max_{i=1}^n |x_i|$.

LEMMA 9.4. *For all $f, g \in \tilde{\mathcal{A}}_{R,n}(\mathcal{U})$ we have*

$$\bar{f} = \bar{g} \ \Leftrightarrow\ \forall a \in M_R^n,\ f^*(a) = g^*(a).$$

Proof. If $\bar{f} = \bar{g}$, then for some positive $\epsilon \in R$ we have

$$\overline{R} \models \forall x(\|x\| < \epsilon \rightarrow f(x) = g(x)).$$

Since $\tilde{R}^*$ is an elementary extension of $\overline{R}$, the same $\mathcal{L}_R$-sentence holds in $\tilde{R}^*$. Hence $f^*(x) = g^*(x)$ for all $x \in R^{*n}$ such that $\|x\| < \epsilon$. Thus, in particular, $f^* = g^*$ on M_R^n.

If, conversely, $f^*(a) = g^*(a)$ for all $a \in M_R^n$, we find

$$\tilde{R}^* \models \exists \epsilon \forall x(\|x\| < \epsilon \rightarrow f^*(x) = g^*(x)),$$

taking for ϵ any positive element in M_R^n. Since $\tilde{R}^*$ is an elementary extension of $\overline{R}$, this sentence is also true in $\overline{R}$, hence

$$\overline{R} \models \exists \epsilon \forall x(\|x\| < \epsilon \rightarrow f(x) = g(x)),$$

hence $f(x) = g(x)$ for all x in the ball $B(0, \epsilon)$; thus $\bar{f} = \bar{g}$. ∎

As a consequence of Lemma 9.4, for each $\alpha \in M_R^n$, the assignment $\bar{f} \mapsto f^*(\alpha)$ is well defined and yields a ring homomorphism from $\mathcal{A}_{R,n}$ to R^*. This corresponds to a point in $\operatorname{sper} \mathcal{A}_{R,n}$.

We define an equivalence relation on M_R^n as follows: for $\alpha, \beta \in M_R^n$,

$$\alpha \sim \beta \ \Leftrightarrow\ (f^*(\alpha) \geq 0 \Leftrightarrow f^*(\beta) \geq 0),\ \forall \bar{f} \in \mathcal{A}_{R,n}.$$

We shall see that the equivalence classes $[\alpha]$ of $\alpha \in M_R^n$ give the full real spectrum of the ring $\mathcal{A}_{R,n}$.

REMARK. Let $\sigma : \mathcal{A}_{R,n} \rightarrow R_\sigma$ be a homomorphism into a real closed field R_σ. Let R'_σ be the real closure of the quotient field, say $[\sigma(\mathcal{A}_{R,n})]$, of the ring $\sigma(\mathcal{A}_{R,n})$. Consider the homomorphism $\sigma' : \mathcal{A}_{R,n} \rightarrow [\sigma(\mathcal{A}_{R,n})] \hookrightarrow R'_\sigma$; clearly we have $\sigma \sim \sigma'$. Since $\operatorname{card} R = \kappa$ and the elementary extension $\tilde{R}^*$ of $\overline{R}$ is ν^+-saturated, where $\nu = \max\{\aleph_0, 2^\kappa\}$, we see, by [8, 7.7, 9.7], that R'_α can be embedded in R^*. Thus

$$\operatorname{sper} \mathcal{A}_{R,n} = \{[\alpha] \mid \alpha : \mathcal{A}_{R,n} \to R^*\}.$$

The following theorem can be seen as a weak version, for $\mathcal{A}_{R,n}$, of the substitution theorem for Nash functions [3, 8.5.2]. We will use it to transfer some algebraic properties to geometric ones.

THEOREM 9.5. *Let* $[\sigma] \in \operatorname{sper} \mathcal{A}_{R,n}$. *Then there exists* $\alpha \in M_R^n$ *such that* $\sigma(\bar{f}) \geq 0 \Leftrightarrow f^*(\alpha) \geq 0$, *for all* $\bar{f} \in \mathcal{A}_{R,n}$.

Proof. For the proof, we need the following lemma proved for $\mathcal{A}_{\mathbb{R},n} = \mathcal{H}_n$ in [8, 10.3] by using the Weierstrass division theorem and Tarski's quantifier elimination for real closed fields. The proof for $\mathcal{A}_{R,n}$ is the same.

LEMMA 9.6. *Let* $[\sigma] \in \mathcal{A}_{R,n}$ *and* $\bar{f}, \overline{g}_1, \ldots, \overline{g}_m \in \mathcal{A}_{R,n}$ *such that* $\sigma(\bar{f}) = 0$, $\sigma(\overline{g}_1) > 0, \ldots, \sigma(\overline{g}_m) > 0$. *Then there exists some* $\alpha \in M_R^n$ *satisfying* $f^*(\alpha) = 0, g_1^*(\alpha) > 0, \ldots, g_m^*(\alpha) > 0$.

Let $[\sigma] \in \mathcal{A}_{R,n}$. We consider the set of $\mathcal{L}_R$-formulas

$$\begin{aligned} \Phi = \{&f(x_1, \ldots, x_n) \geq 0 \mid \sigma(\bar{f}) \geq 0,\ \bar{f} \in \mathcal{A}_{R,n}\} \\ &\cup \{f(x_1, \ldots, x_n) < 0,\ \sigma(\bar{f}) < 0,\ \bar{f} \in \mathcal{A}_{R,n}\} \\ &\cup \{|x_i| < \epsilon, 1 \leq i \leq n,\ 0 < \epsilon \in R\}. \end{aligned}$$

By the above lemma, the set Φ is finitely realized in R^*. Since $\tilde{R}^*$ is ν^+-saturated, where $\nu = \max\{\aleph_0, 2^\kappa\}$, and $\operatorname{card} R = \kappa$, the $\mathcal{L}_R$-type Φ is realized in $\tilde{R}^*$. Hence there is $\alpha \in R^{*n}$ satisfying Φ. Thus in particular $\alpha \in M_R^n$ and $f^*(\alpha) \geq 0 \Leftrightarrow \sigma(\bar{f}) \geq 0$. ∎

We mention a formal theorem proved in [3, 4.4.1] for a general ring A with unit. We need some terminology and notation.

Let A be a commutative ring with unit. We denote by $\sum A^2$ the set of all finite sums $\sum a_i^2$, $a_i \in A$. If $a_1, \ldots, a_t \in A$, we denote by $T(a_1, \ldots, a_t)$ the smallest subset of A containing the elements $a_1, \ldots, a_t$ and $\sum A^2$, and closed under addition and multiplication. Clearly $T(a_1, \ldots, a_t)$ is exactly the set

$$\left\{p + q_1 g_1 + \cdots + q_s g_s \;\middle|\; p, q_1, \ldots, q_s \in \sum A^2\right\}$$

where $g_1, \ldots, g_s$ are finite products of $\overline{a}_1, \ldots, \overline{a}_t$.

THEOREM 9.7 ([3, 4.4.1]). *Let* $\overline{h}_1, \ldots, \overline{h}_r, \overline{g}_1, \ldots, \overline{g}_s, \bar{f}_1, \ldots, \bar{f}_t \in \mathcal{A}_{R,n}$. *Then the following assertions are equivalent:*

(i) $\{\sigma \in \operatorname{sper} \mathcal{A}_{R,n} \mid \sigma(\overline{h}_1) = 0, \ldots, \sigma(\overline{h}_r) = 0,\ \sigma(\overline{g}_1) \neq 0, \ldots, \sigma(\overline{g}_s) \neq 0,$ $\sigma(\bar{f}_1) \geq 0, \ldots, \sigma(\bar{f}_t) \geq 0\} = \emptyset$.

(ii) *There exists an equation of the form*

$$\overline{g}_1^{2m_1}.\dots.\overline{g}_s^{2m_s} + \overline{a} = \overline{b}_1\overline{h}_1 + \cdots + \overline{b}_r\overline{h}_r$$

for suitable $m_1, \dots, m_s \in \mathbb{N}$, $\overline{b}_1, \dots, \overline{b}_r \in \mathcal{A}_{R,n}$ *and* $\overline{a} \in T(\bar{f}_1, \dots, \bar{f}_t)$.

Let $I \subset \mathcal{A}_{R,n}$ be an ideal. We denote by $V_I(R)$ the germ at $0 \in R^n$ of the zero-set of I. For $\overline{g}_1, \dots, \overline{g}_s$, $\bar{f}_1, \dots, \bar{f}_t \in \mathcal{A}_{R,n}$, we denote by $S_{R,g,f}$ the semi-$\mathcal{A}_R$ germ defined by

$$x \in S_{R,g,f} \Leftrightarrow g_1(x) \neq 0, \dots, g_s(x) \neq 0,\ f_1(x) \geq 0, \dots, f_t(x) \geq 0.$$

Proposition 9.8. *With the notation above, the following are equivalent:*

(i) $V_I(R) \cap S_{R,g,f} = \emptyset$.
(ii) *There exist* $n_1, \dots, n_s \in \mathbb{N}$ *and* $\overline{q} \in T(\bar{f}_1, \dots, \bar{f}_t)$ *such that*

$$\overline{g}_1^{2n_1} \dots \overline{g}_s^{2n_s} + \overline{q} \in I.$$

Proof. Let $(\overline{h}_1, \dots, \overline{h}_r)$ be a system of generators of the ideal I. It is clear that (ii)⇒(i). For the nontrivial implication (i)⇒(ii), assume that no $n_1, \dots, n_s \in \mathbb{N}$ and $\overline{q} \in T(\bar{f}_1, \dots, \bar{f}_t)$ exist. Then by Theorem 9.7, there exists $\sigma \in \operatorname{sper} \mathcal{A}_{R,n}$ such that

$$\sigma(\overline{h}_1) = \cdots = \sigma(\overline{h}_r) = 0,\ \sigma(\overline{g}_1) \neq 0, \dots, \sigma(\overline{g}_s) \neq 0,\ \sigma(\bar{f}_1) \geq 0, \dots, \sigma(\bar{f}_t) \geq 0.$$

By Lemma 9.6, there exists $\alpha \in M_R^n$ such that

$$h_1^*(\alpha) = \cdots = h_r^*(\alpha) = 0,\ g_1^*(\alpha) \neq 0, \dots, g_s^*(\alpha) \neq 0,\ f_1^*(\alpha) \geq 0, \dots, f_t^*(\alpha) \geq 0.$$

Hence $\alpha \in V_I(R^*) \cap S_{R^*,g,f}$. Since $\tilde{R}^*$ is an elementary extension of $\overline{R}$, there exists $x \in V_I(R) \cap S_{R,g,f}$, which is a contradiction. ∎

Theorem 9.9 (Positivstellensatz). *Let* $\bar{f} \in \mathcal{A}_{R,n}$. *Then* f *is strictly positive on the semi-*$\mathcal{A}_R$ *germ* $\{f_1(x) \geq 0, \dots, f_t(x) \geq 0\} \cap V_I(R)$ *if and only if there are* $\overline{q}_1, \overline{q}_2 \in T(\bar{f}_1, \dots, \bar{f}_t)$ *such that* $\overline{q}_1\bar{f} - (1 + \overline{q}_2) \in I$.

Proof. For the nontrivial implication, we can apply Proposition 9.8 to $s = 1, \overline{g}_1 = 1$, and add $\bar{f}_{t+1} = \overline{-f}$. This gives $1 + (\overline{q}_2 - \bar{f}\overline{q}_1) \in I$, where $\overline{q}_1, \overline{q}_2 \in T(\bar{f}_1, \dots, \bar{f}_t)$. ∎

Theorem 9.10 (Real Nullstellensatz). *Let* $\bar{f} \in \mathcal{A}_{R,n}$. *Then* f *is zero on the germ* $V_I(R)$ *if and only if there exist* $m \in \mathbb{N}$ *and* $\overline{\varphi}_1, \dots, \overline{\varphi}_q \in \mathcal{A}_{R,n}$ *such that* $\bar{f}^{2m} + \overline{\varphi}_1^2 + \ldots + \overline{\varphi}_q^2 \in I$.

Proof. We take $s = 1$, $g_1 = f$, $t = 1$, and $f_1 = 0$ in Proposition 9.8. Now $T(0)$ is exactly the set $\sum \mathcal{A}_{R,n}^2$. For the nontrivial implication assume that f is zero on $V_I(R)$. Then $V_I(R) \cap \{x \mid f(x) \neq 0\} = \emptyset$. Now Theorem 9.9 yields the result. ∎

Theorem 9.11 (Hilbert's 17th Problem). *Let* $\bar{f}, \overline{\varphi}_1, \dots, \overline{\varphi}_q \in \mathcal{A}_{R,n}$. *If on a neighborhood of* $0 \in R^n$, $f(x) \geq 0$ *whenever* $f_1(x) \geq 0, \dots, f_t(x) \geq 0$, *then there exist* $m \in \mathbb{N}$ *and* $\overline{q}_1, \overline{q}_2 \in T(\bar{f}_1, \dots, \bar{f}_t)$ *such that* $\bar{f}\overline{q}_1 = \bar{f}^{2m} + \overline{q}_2$.

Proof. We apply Proposition 9.8 to $s = 1$, $g_1 = f$, I the null ideal and add $f_{t+1} = -f$. ∎

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