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Some further results on meromorphic functions that share two sets

by Qi Han and Hong-Xun Yi (Jinan)

Abstract. This paper concerns the uniqueness of meromorphic functions and shows that there exists a set $\mathbf{S} \subset \mathbb{C}$ of eight elements such that any two nonconstant meromorphic functions f and g in the open complex plane $\mathbb{C}$ satisfying $E_{3)}(\mathbf{S}, f) = E_{3)}(\mathbf{S}, g)$ and $\overline{E}(\infty, f) = \overline{E}(\infty, g)$ are identical, which improves a result of H. X. Yi. Also, some other related results are obtained, which generalize the results of G. Frank, E. Mues, M. Reinders, C. C. Yang, H. X. Yi, P. Li, M. L. Fang and H. Guo, and others.

1. Introduction and main results. In this paper, a *meromorphic function* will always mean meromorphic in the open complex plane $\mathbb{C}$. For any nonconstant meromorphic function f, we adopt the standard notations in *Nevanlinna's value distribution theory* of meromorphic functions such as the *characteristic function* $T(r, f)$, the *proximity function* $m(r, f)$ and the *counting function* $N(r, f)$ (reduced form $\overline{N}(r, f)$) of poles. Also, we denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$, possibly outside a set of finite linear measure in $\mathbb{R}$ that is not necessarily the same at each occurrence. We refer the reader to books [6] and [10] for more details on those notations.

Let f be a nonconstant meromorphic function, let $a \in \mathbb{C}$ be a finite value, and let k be a positive integer or infinity. We denote by $E(a, f)$ the set of zeros of $f - a$ (counting multiplicity), and by $\overline{E}(a, f)$ the set of zeros of $f - a$ (ignoring multiplicity). Also, we denote by $E_{k)}(a, f)$ the set of zeros of $f - a$ with multiplicities less than or equal to k (counting multiplicity). Obviously, $E(a, f) = E_{+\infty)}(a, f)$. If $a = \infty$, we define $E(\infty, f) := E(0, 1/f)$. $\overline{E}(\infty, f)$ and $E_{k)}(\infty, f)$ are similarly defined. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $N_{k)}(r, 1/(f - a))$ the counting function corresponding to the set $E_{k)}(a, f)$, and by $N_{(k+1}(r, 1/(f - a))$ the counting function corresponding to the set

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$E_{(k+1}(a,f) := E(a,f)/E_{k)}(a,f)$. Also, we denote by $\overline{N}_{k)}(r,1/(f-a))$ and $\overline{N}_{(k+1}(r,1/(f-a))$ the corresponding reduced forms of $N_{k)}(r,1/(f-a))$ and $N_{(k+1}(r,1/(f-a))$, respectively.

Let $\mathbf{S}$ be a subset in $\mathbb{C}\cup\{\infty\}$ of distinct elements, and let k be a positive integer or infinity. Set

$$E_{k)}(\mathbf{S},f) = \bigcup_{a\in\mathbf{S}} E_{k)}(a,f) \quad \text{and} \quad \overline{E}(\mathbf{S},f) = \bigcup_{a\in\mathbf{S}} \overline{E}(a,f). \tag{1.1}$$

Obviously, for $a \in \mathbb{C}\cup\{\infty\}$, $E_{k)}(\{a\},f) = E_{k)}(a,f)$ and $\overline{E}(\{a\},f) = \overline{E}(a,f)$. For another nonconstant meromorphic function g, we say that f and g *share the set* $\mathbf{S}$ *CM* (respectively, *IM*) provided that $E(\mathbf{S},f) = E(\mathbf{S},g)$ (respectively, $\overline{E}(\mathbf{S},f) = \overline{E}(\mathbf{S},g)$). Evidently, if $\mathbf{S}$ contains only one element, then it coincides with the usual definition of CM (respectively, IM) shared values. On the other hand, the condition $E_{k)}(\mathbf{S},f) = E_{k)}(\mathbf{S},g)$ obviously implies $E_{j)}(\mathbf{S},f) = E_{j)}(\mathbf{S},g)$ for all $1 \le j \le k$.

It was F. Gross [3] who first considered the uniqueness of meromorphic functions that share sets of distinct elements instead of values. In 1976, he proposed the following two questions in [4].

Question I. *Can one find two* (*or possibly even one*) *finite sets* $\mathbf{S}_j$ $(j=1,2)$ *such that any two nonconstant entire functions* f *and* g *satisfying* $E(\mathbf{S}_j,f) = E(\mathbf{S}_j,g)$ $(j=1,2)$ *are identical?*

Question II. *If the answer to Question I* (*for two finite sets*) *is positive, then how large have both sets to be?*

Further, in 1982, F. Gross and C. C. Yang [5] showed that any two nonconstant entire functions f and g satisfying $E(\mathbf{S},f) = E(\mathbf{S},g)$ are identical, where $\mathbf{S} := \{\omega\in\mathbb{C} \mid \omega + e^{\omega} = 0\}$. Only about twenty years later, were Questions I and II completely answered by H. X. Yi who proved the following three theorems.

Theorem A (see [11]). *Let* $\mathbf{S}_1 = \{\omega \mid \omega^n - 1 = 0\}$ *and* $\mathbf{S}_2 = \{a\}$, *where* n *is a positive integer such that* $n \ge 5$, *and* a *is a constant such that* $a \ne 0$ *and* $a^{2n} \ne 1$. *If* f *and* g *are two nonconstant entire functions satisfying* $E(\mathbf{S}_j,f) = E(\mathbf{S}_j,g)$ *for* $j = 1,2$, *then* $f \equiv g$.

Theorem B (see [12]). *Let* $\mathbf{S} = \{\omega \mid \omega^n + a\omega^m + b = 0\}$, *where* n *and* m *are positive integers such that* $n \ge 15$, $n > m \ge 5$, *and* n *and* m *are relatively prime, and* a *and* b *are nonzero constants such that the algebraic equation* $\omega^n + a\omega^m + b = 0$ *has no multiple roots. If* f *and* g *are two nonconstant entire functions satisfying* $E(\mathbf{S},f) = E(\mathbf{S},g)$, *then* $f \equiv g$.

Theorem C (see [15]). *Let* $\mathbf{S}_1 = \{\omega \mid \omega^3 + a\omega^2 + b = 0\}$ *and* $\mathbf{S}_2 = \{0\}$, *where* a *and* b *are nonzero constants such that the equation* $\omega^3 + a\omega^2 + b$

$= 0$ *has no multiple roots. If f and g are two nonconstant entire functions satisfying $E(\mathbf{S}_j, f) = E(\mathbf{S}_j, g)$ for $j = 1, 2$, then $f \equiv g$. Obviously,* $\min\{\imath(\mathbf{S}_1), \imath(\mathbf{S}_2)\} \geq 1$ *and* $\max\{\imath(\mathbf{S}_1), \imath(\mathbf{S}_2)\} \geq 3$, *where $\imath(\mathbf{S})$ denotes the cardinality of* $\mathbf{S}$.

Clearly, Theorems A and B answer Question I affirmatively, while Theorem C answers Question II completely, since examples are given in [15] to show the sharpness. See also [14] for some related results.

In 1998, G. Frank and M. Reinders obtained the following result which extends and improves Theorem B.

Theorem D (see [2]). *Let n be a positive integer such that $n \geq 11$, and let c be a constant such that $c \neq 0, 1$. Then the polynomial*

$$P^{\mathrm{FR}}(\omega) = \frac{(n-1)(n-2)}{2}\,\omega^n - n(n-2)\omega^{n-1} + \frac{n(n-1)}{2}\,\omega^{n-2} - c \tag{1.2}$$

has no multiple roots. Let $\mathbf{S}$ denote the zero set of $P^{\mathrm{FR}}(\omega)$. If f and g are two nonconstant meromorphic functions satisfying $E(\mathbf{S}, f) = E(\mathbf{S}, g)$, then $f \equiv g$.

In particular, if f and g are entire, then $n \geq 7$ suffices, which coincides with the main result of E. Mues and M. Reinders [9] on entire functions.

It is natural to consider the uniqueness of two nonconstant meromorphic functions that satisfy $E(\mathbf{S}, f) = E(\mathbf{S}, g)$ and $E(\infty, f) = E(\infty, g)$, a problem clearly inspired by Theorems B and D. The first result on this problem was obtained by P. Li and C. C. Yang:

Theorem E (see [7]). *Let $\mathbf{S} = \{\omega \mid \omega^n + a\omega^{n-m} + b = 0\}$, where n and m are two positive integers such that $m \geq 2$, $n > 4m + 6$, and n and m are relatively prime, and a and b are nonzero constants such that the equation $\omega^n + a\omega^{n-m} + b = 0$ has no multiple roots. If f and g are two nonconstant meromorphic functions satisfying $E(\mathbf{S}, f) = E(\mathbf{S}, g)$ and $E(\infty, f) = E(\infty, g)$, then $f \equiv g$.*

From then on, many results on that problem have been obtained. In 1995, H. X. Yi [13], and independently P. Li and C. C. Yang [8], showed that there exists a set $\mathbf{S}$ of 11 elements such that any two nonconstant meromorphic functions f and g satisfying $E(\mathbf{S}, f) = E(\mathbf{S}, g)$ and $E(\infty, f) = E(\infty, g)$ are identical. In 1997, M. L. Fang and H. Guo in [1] exhibited a set $\mathbf{S}$ of nine elements with this property.

In 2002, H. X. Yi proved the following theorem.

Theorem F (see [16]). *Let n be a positive integer such that $n \geq 8$, and let a, b be nonzero constants such that $ab^{n-2} \neq 2$. Then the polynomial*

$$P(\omega) = a\omega^n - n(n-1)\omega^2 + 2n(n-2)b\omega - (n-1)(n-2)b^2 \tag{1.3}$$

has no multiple roots. Let $\mathbf{S}$ *denote the zero set of* $P(\omega)$*. If* f *and* g *are two nonconstant meromorphic functions satisfying* $E(\mathbf{S}, f) = E(\mathbf{S}, g)$ *and* $\overline{E}(\infty, f) = \overline{E}(\infty, g)$*, then* $f \equiv g$*.*

In this paper, we prove the following five results, which are improvements on and supplements to the above theorems.

THEOREM 1. *Let* $\mathbf{S}$ *be the zero set of* $P(\omega)$ *given by* (1.3) *with* $n \geq 7$*. If* f *and* g *are two nonconstant entire functions satisfying* $E_{2)}(\mathbf{S}, f) = E_{2)}(\mathbf{S}, g)$*, then* $f \equiv g$*.*

THEOREM 2. *Let* $\mathbf{S}$ *be the zero set of* $P(\omega)$ *given by* (1.3) *with* $n \geq 9$*. If* f *and* g *are two nonconstant entire functions satisfying* $E_{1)}(\mathbf{S}, f) = E_{1)}(\mathbf{S}, g)$*, then* $f \equiv g$*.*

THEOREM 3. *Let* $\mathbf{S}$ *be the zero set of* $P(\omega)$ *given by* (1.3) *with* $n \geq 8$*. If* f *and* g *are two nonconstant meromorphic functions satisfying* $E_{3)}(\mathbf{S}, f) = E_{3)}(\mathbf{S}, g)$ *and* $\overline{E}(\infty, f) = \overline{E}(\infty, g)$*, then* $f \equiv g$*.*

THEOREM 4. *Let* $\mathbf{S}$ *be the zero set of* $P(\omega)$ *given by* (1.3) *with* $n \geq 9$*. If* f *and* g *are two nonconstant meromorphic functions satisfying* $E_{2)}(\mathbf{S}, f) = E_{2)}(\mathbf{S}, g)$ *and* $\overline{E}(\infty, f) = \overline{E}(\infty, g)$*, then* $f \equiv g$*.*

THEOREM 5. *Let* $\mathbf{S}$ *be the zero set of* $P(\omega)$ *given by* (1.3) *with* $n \geq 12$*. If* f *and* g *are two nonconstant meromorphic functions satisfying* $E_{1)}(\mathbf{S}, f) = E_{1)}(\mathbf{S}, g)$ *and* $\overline{E}(\infty, f) = \overline{E}(\infty, g)$*, then* $f \equiv g$*.*

REMARK 6. Let $P(\omega)$ be given by (1.3), and let n, a and b be as in Theorem F. Set

$$R(\omega) = \frac{a\omega^n}{n(n-1)(\omega-\alpha)(\omega-\beta)}, \tag{1.4}$$

where α and β are the two distinct roots of the algebraic equation

$$n(n-1)\omega^2 - 2n(n-2)b\omega + (n-1)(n-2)b^2 = 0. \tag{1.5}$$

Then

$$R'(\omega) = \frac{a(n-2)\omega^{n-1}(\omega-b)^2}{n(n-1)(\omega-\alpha)^2(\omega-\beta)^2}. \tag{1.6}$$

So $\omega = 0$ is an n-fold root of $R(\omega) = 0$, while $\omega = b$ is a triple root of $R(\omega) - ab^{n-2}/2 = 0$. Since $ab^{n-2} \neq 0$, 2, we have $R(\omega) - 1 = 0$, and thus $P(\omega) = 0$ has no multiple roots.

2. Lemmas

LEMMA 1 (see [2]). *Let*

$$Q(\omega) = (n-1)^2(\omega^n - 1)(\omega^{n-2} - 1) - n(n-2)(\omega^{n-1} - 1)^2. \tag{2.1}$$

Then

$$Q(\omega) = (n-1)^2(\omega-1)^4(\omega-\gamma_1)(\omega-\gamma_2)\cdots(\omega-\gamma_{2n-6}), \tag{2.2}$$

where $\gamma_j \in \mathbb{C} \setminus \{0,1\}$ *are pairwise distinct for* $j = 1, \dots, 2n-6$.

LEMMA 2. *Define*

$$\varphi := \left(\frac{F'}{F-1} - \frac{F'}{F}\right) - \left(\frac{G'}{G-1} - \frac{G'}{G}\right), \tag{2.3}$$

where $F := R(f)$ *and* $G := R(g)$, *and* $R(\omega)$ *is given by* (1.4). *Suppose* $n \geq 8$ *if* f *and* g *are meromorphic, and* $n \geq 6$ *if* f *and* g *are entire. Then* $\varphi \equiv 0$ *implies* $F \equiv G$.

Proof. Since we assume $\varphi \equiv 0$, integrating (2.3) yields

$$\frac{1}{F} - \frac{A}{G} \equiv 1 - A, \tag{2.4}$$

where $A \neq 0$ is a constant. As $F = R(f)$ and $G := R(g)$, we know that $T(r,F) = nT(r,f) + O(1)$ and $T(r,G) = nT(r,g) + O(1)$, which together with (2.4) yields

$$T(r,f) = T(r,g) + O(1). \tag{2.5}$$

Define $h_1 := 1/F$ and $h_2 := -A/G$. So, $h_1 + h_2 \equiv 1 - A$. If $A \neq 1$, applying the second main theorem to h_1, plus (2.5), yields

$$\begin{aligned} T(r,h_1) &\leq \overline{N}(r,h_1) + \overline{N}\left(r,\frac{1}{h_1}\right) + \overline{N}\left(r,\frac{1}{h_1-(1-A)}\right) + S(r,f) \\ &\leq \overline{N}(r,F) + \overline{N}(r,G) + \overline{N}\left(r,\frac{1}{F}\right) + S(r,f) \\ &\leq \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f-\alpha}\right) + \overline{N}\left(r,\frac{1}{f-\beta}\right) + \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}(r,g) \\ &\quad + \overline{N}\left(r,\frac{1}{g-\alpha}\right) + \overline{N}\left(r,\frac{1}{g-\beta}\right) + S(r,f) \leq 7T(r,f) + S(r,f), \end{aligned}$$

which implies that $nT(r,f) \leq 7T(r,f) + S(r,f)$, a contradiction to our assumption that $n \geq 8$ if f and g are meromorphic.

If f and g are entire, then $\overline{N}(r,f) = O(1)$ and $\overline{N}(r,g) = O(1)$. A contradiction follows immediately, since now $nT(r,f) \leq 5T(r,f) + S(r,f)$ but $n \geq 6$. ∎

LEMMA 3. *Let* φ, F *and* G *be as in Lemma* 2. *If* $\varphi \not\equiv 0$, *and if two nonconstant meromorphic functions* f *and* g *satisfy* $E_{k)}(\mathbf{S},f) = E_{k)}(\mathbf{S},g)$ *for a positive integer* k *and* $\overline{E}(\infty,f) = \overline{E}(\infty,g)$, *then*

$$N(r,f) \leq \frac{k+1}{kn-3k-2}\left(T(r,f) + T(r,g)\right) + S(r,f) + S(r,g). \tag{2.6}$$

Proof. Since $E_{k)}(\mathbf{S}, f) = E_{k)}(\mathbf{S}, g)$ for a positive integer k, we have $E_{k)}(1, F) = E_{k)}(1, G)$.

It is not difficult to show that

$$N(r, 1/h') \leq \overline{N}(r, h) + N(r, 1/h) + S(r, h), \tag{2.7}$$

where h is a nonconstant meromorphic function.

In fact, from the lemma of logarithmic derivative, we have

$$m(r, 1/h) \leq m(r, 1/h') + m(r, h'/h) \leq m(r, 1/h') + S(r, h).$$

Combining the above inequality with the first main theorem yields

$$\begin{aligned} N(r, 1/h') &= T(r, h') - m(r, 1/h') + O(1) \\ &\leq m(r, h') + N(r, h') - m(r, 1/h) + S(r, h) \\ &\leq m(r, h) + N(r, h) + \overline{N}(r, h) + m(r, h'/h) - m(r, 1/h) + S(r, h) \\ &\leq T(r, h) + \overline{N}(r, h) - m(r, 1/h) + S(r, h) \\ &\leq N(r, 1/h) + \overline{N}(r, h) + S(r, h), \end{aligned}$$

which implies (2.7).

Obviously, $m(r, \varphi) = S(r, f) + S(r, g)$. Rewrite φ as

$$\varphi = \frac{F'}{F(F-1)} - \frac{G'}{G(G-1)}. \tag{2.8}$$

From (2.8) and the assumption $E_{k)}(1, F) = E_{k)}(1, G)$, it is easy to see that all the poles of φ are simple, and derive from the zeros of F or G and the sets $E_{(k+1}(1, F)$ and $E_{(k+1}(1, G)$, respectively, that is,

$$\begin{aligned} N(r, \varphi) &= \overline{N}(r, \varphi) + O(1) \\ &\leq \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}_{(k+1}\left(r, \frac{1}{F-1}\right) + \overline{N}\left(r, \frac{1}{G}\right) \\ &\quad + \overline{N}_{(k+1}\left(r, \frac{1}{G-1}\right) + S(r, f) + S(r, g) \\ &\leq \overline{N}\left(r, \frac{1}{f}\right) + \sum_{j=1}^{n} \overline{N}_{(k+1}\left(r, \frac{1}{f-\omega_j}\right) + \overline{N}\left(r, \frac{1}{g}\right) \\ &\quad + \sum_{j=1}^{n} \overline{N}_{(k+1}\left(r, \frac{1}{g-\omega_j}\right) + S(r, f) + S(r, g) \\ &\leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}_{(k}\left(r, \frac{1}{f'}\right) + \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}_{(k}\left(r, \frac{1}{g'}\right) + S(r, f) + S(r, g) \\ &\leq \overline{N}\left(r, \frac{1}{f}\right) + \frac{1}{k}N\left(r, \frac{1}{f'}\right) + \overline{N}\left(r, \frac{1}{g}\right) + \frac{1}{k}N\left(r, \frac{1}{g'}\right) + S(r, f) + S(r, g), \end{aligned} \tag{2.9}$$

where ω_j for $j = 1, \dots, n$ are the pairwise distinct roots of $P(\omega)$ given by (1.3).

Combining (2.7), (2.9) with the condition $\overline{E}(\infty, f) = \overline{E}(\infty, g)$ yields

$$N(r,\varphi) \le \frac{k+1}{k}\,(T(r,f)+T(r,g)) + \frac{2}{k}\,\overline{N}(r,f) + S(r,f) + S(r,g). \tag{2.10}$$

Now, let z_0 be a pole of f with multiplicity p, and hence a pole of g with multiplicity q. Then z_0 is a pole of F with multiplicity $(n-2)p$ and a pole of G with multiplicity $(n-2)q$. By (2.8), z_0 is a zero of φ with multiplicity at least $n-3$. Since we assume $\overline{E}(\infty, f) = \overline{E}(\infty, g)$, it follows that

$$(n-3)\overline{N}(r,f) \le N(r,1/\varphi) \le T(r,\varphi) + O(1). \tag{2.11}$$

Hence, (2.10), (2.11) and $m(r,\varphi) = S(r,f) + S(r,g)$ yield (2.6). ∎

Lemma 4 (see [16]). *Define*

$$\psi := \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right), \tag{2.12}$$

where F and G are nonconstant meromorphic functions. If $\psi \equiv 0$, and

$$\overline{N}(r,F) + \overline{N}(r,G) + \overline{N}\left(r,\frac{1}{F}\right) \le \lambda T(r,F) + S(r,F) \tag{2.13}$$

and

$$\overline{N}(r,F) + \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}\left(r,\frac{1}{F}\right) \le \mu T(r,F) + S(r,F) \tag{2.14}$$

for some real numbers $\lambda, \mu < 1$, then either $F \equiv G$ or $FG \equiv 1$.

Proof. For convenience, we outline a proof. Since we assume $\psi \equiv 0$, integrating (2.12) yields

$$\frac{1}{G-1} = \frac{A}{F-1} + B, \tag{2.15}$$

where A $(\ne 0)$ and B are two constants. Rewrite it as

$$G = \frac{(B+1)F + (A-B-1)}{BF + (A-B)}. \tag{2.16}$$

We now distinguish the following three cases.

Case (i): $B \ne 0, -1$. If $A - B - 1 \ne 0$, then

$$\overline{N}\left(r, \frac{1}{F + \frac{A-B-1}{B+1}}\right) = \overline{N}\left(r, \frac{1}{G}\right) + O(1)$$

by (2.16). Applying the second main theorem to F yields

$$T(r,F) \le \overline{N}(r,F) + \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{F+\frac{A-B-1}{B+1}}\right) + S(r,F)$$
$$\le \overline{N}(r,F) + \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}\left(r,\frac{1}{F}\right) \le \mu T(r,F) + S(r,F),$$

a contradiction to (2.14) since $\mu < 1$.

Hence, $A - B - 1 = 0$. From (2.16), we see that $\overline{N}(r, 1/(F + 1/B)) = \overline{N}(r,G) + O(1)$. Similarly, we derive a contradiction to (2.13).

Case (ii): $B = 0$. If $A \ne 1$, we see that $\overline{N}(r, 1/(F+A-1)) = \overline{N}(r, 1/G) + O(1)$. Analogously, we derive a contradiction to (2.14). So, $A = 1$, and thus $F \equiv G$.

Case (iii): $B = -1$. If $A \ne -1$, then $\overline{N}(r, 1/(F - A - 1)) = \overline{N}(r,G) + O(1)$. Analogously, a contradiction to (2.13) follows. Therefore, $A = -1$ and $FG \equiv 1$. ■

3. Proof of Theorem 3. Define $F := R(f)$ and $G := R(g)$, where $R(\omega)$ is given by (1.4). Since we assume $E_{3)}(\mathbf{S}, f) = E_{3)}(\mathbf{S}, g)$, it follows that $E_{3)}(1,F) = E_{3)}(1,G)$. Let ψ be defined by (2.12).

Firstly, we assume that $\psi \not\equiv 0$. Obviously, $E_{1)}(1,F) = E_{1)}(1,G)$ anyway. Let z_0 be a simple zero of $F-1$, and hence a simple zero of $G-1$. A routine calculation leads to $\psi(z_0) = 0$. Therefore, noting that $m(r,\psi) = S(r,f) + S(r,g)$ and that all the poles of ψ are simple, we have

$$\overline{N}_{1)}\left(r,\frac{1}{F-1}\right) = \overline{N}_{1)}\left(r,\frac{1}{G-1}\right) \le N\left(r,\frac{1}{\psi}\right) \le \overline{N}(r,\psi) + S(r,f) + S(r,g). \tag{3.1}$$

From the expression of F, we know that the poles of F arise from the poles of f and the zeros of $(f-\alpha)(f-\beta)$. Let z_∞ be a simple pole of F. A routine calculation leads to

$$\left.\left(\frac{F''}{F'} - \frac{2F'}{F-1}\right)\right|_{z_\infty} = O(1).$$

Obviously, the simple zeros of $(f-\alpha)(f-\beta)$ are simple poles of F, while the multiple zeros of $(f-\alpha)(f-\beta)$ are zeros of f'.

Differentiating F and G yields

$$F' = \frac{a(n-2)f^{n-1}(f-b)^2 f'}{n(n-1)(f-\alpha)^2(f-\beta)^2}, \qquad G' = \frac{a(n-2)g^{n-1}(g-b)^2 g'}{n(n-1)(g-\alpha)^2(g-\beta)^2}. \tag{3.2}$$

Combining (2.12) and (3.2), together with the assumptions that $\psi \not\equiv 0$, $E_{3)}(1,F) = E_{3)}(1,G)$, $\overline{E}(\infty,f) = \overline{E}(\infty,g)$ and similar discussions on G, yields

$$\begin{aligned}(3.3)\quad \overline{N}(r,\psi) \le{}& \overline{N}(r,f) + \overline{N}_{(4}\left(r,\frac{1}{F-1}\right) + \overline{N}_{(4}\left(r,\frac{1}{G-1}\right)\\ &+ \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f-b}\right) + \overline{N}\left(r,\frac{1}{g}\right) + \overline{N}\left(r,\frac{1}{g-b}\right)\\ &+ N_0\left(r,\frac{1}{f'}\right) + N_0\left(r,\frac{1}{g'}\right) + S(r,f) + S(r,g),\end{aligned}$$

where $N_0(r,1/f')$ denotes the counting function of the zeros of f' that are not the zeros of $f(f-b)$ and $F-1$, and $N_0(r,1/g')$ denotes the counting function of the zeros of g' that are not the zeros of $g(g-b)$ and $G-1$.

Applying the second main theorem to f and g jointly with $n+3$ values 0, b, ∞ and the zeros of $P(\omega)$ defined by (1.3) yields

$$\begin{aligned}(3.4)\quad & (n+1)(T(r,f)+T(r,g))\\ &\le \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{F-1}\right) + \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f-b}\right)\\ &\quad + \overline{N}(r,g) + \overline{N}\left(r,\frac{1}{G-1}\right) + \overline{N}\left(r,\frac{1}{g}\right) + \overline{N}\left(r,\frac{1}{g-b}\right)\\ &\quad - N_0\left(r,\frac{1}{f'}\right) - N_0\left(r,\frac{1}{g'}\right) + S(r,f) + S(r,g).\end{aligned}$$

From (3.1), (3.3), (3.4) and $\overline{E}(\infty,f) = \overline{E}(\infty,g)$, we have

$$\begin{aligned}(3.5)\quad & (n+1)(T(r,f)+T(r,g))\\ &\le 3\overline{N}(r,f) + 2\overline{N}\left(r,\frac{1}{f}\right) + 2\overline{N}\left(r,\frac{1}{f-b}\right) + 2\overline{N}\left(r,\frac{1}{g}\right)\\ &\quad + 2\overline{N}\left(r,\frac{1}{g-b}\right) + \overline{N}\left(r,\frac{1}{F-1}\right) + \overline{N}\left(r,\frac{1}{G-1}\right) + \overline{N}_{(4}\left(r,\frac{1}{F-1}\right)\\ &\quad + \overline{N}_{(4}\left(r,\frac{1}{G-1}\right) - N_{1)}\left(r,\frac{1}{F-1}\right) + S(r,f) + S(r,g).\end{aligned}$$

It is not difficult to see that

$$\overline{N}\left(r,\frac{1}{F-1}\right) - \frac{1}{2}N_{1)}\left(r,\frac{1}{F-1}\right) + \overline{N}_{(4}\left(r,\frac{1}{F-1}\right) \le \frac{1}{2}N\left(r,\frac{1}{F-1}\right),$$
$$\overline{N}\left(r,\frac{1}{G-1}\right) - \frac{1}{2}N_{1)}\left(r,\frac{1}{G-1}\right) + \overline{N}_{(4}\left(r,\frac{1}{G-1}\right) \le \frac{1}{2}N\left(r,\frac{1}{G-1}\right),$$

which implies that

$$\overline{N}\left(r, \frac{1}{F-1}\right) + \overline{N}\left(r, \frac{1}{G-1}\right) + \overline{N}_{(4}\left(r, \frac{1}{F-1}\right) + \overline{N}_{(4}\left(r, \frac{1}{G-1}\right) - N_{1)}\left(r, \frac{1}{F-1}\right) \leq \frac{n}{2}\,(T(r,f)+T(r,g)) + O(1). \tag{3.6}$$

Noting that

$$2\overline{N}\left(r, \frac{1}{f}\right) + 2\overline{N}\left(r, \frac{1}{f-b}\right) \leq 4T(r,f) + O(1),$$
$$2\overline{N}\left(r, \frac{1}{g}\right) + 2\overline{N}\left(r, \frac{1}{g-b}\right) \leq 4T(r,g) + O(1),$$

we deduce from (3.5) and (3.6) that

$$(n-6)(T(r,f)+T(r,g)) \leq 6\overline{N}(r,f) + S(r,f) + S(r,g). \tag{3.7}$$

We have $\varphi \not\equiv 0$. Indeed, otherwise, from the conclusions of Lemma 2, we have $F \equiv G$, and hence $\psi \equiv 0$. From (2.6) and (3.7), and noting $k = 3$, we conclude that

$$\frac{3n^2 - 29n + 42}{3n - 11}\,(T(r,f)+T(r,g)) \leq S(r,f) + S(r,g), \tag{3.8}$$

a contradiction to the assumption $n \geq 8$.

Therefore, $\psi \equiv 0$. By (2.15), we know that $T(r,F) = T(r,G) + O(1)$. So,

$$\overline{N}(r,F) + \overline{N}(r,G) + \overline{N}\left(r, \frac{1}{F}\right) \leq \frac{7}{n}\,T(r,F) + S(r,F),$$
$$\overline{N}(r,F) + \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}\left(r, \frac{1}{F}\right) \leq \frac{5}{n}\,T(r,F) + S(r,F).$$

By the conclusions of Lemma 4, we have either $F \equiv G$ or $FG \equiv 1$.

We now distinguish the following two cases.

Case (i): $FG \equiv 1$. It is obvious that

$$\frac{f^n}{(f-\alpha)(f-\beta)}\,\frac{g^n}{(g-\alpha)(g-\beta)} \equiv \frac{n^2(n-1)^2}{a^2}. \tag{3.9}$$

By (3.9) and noting $\overline{E}(\infty, f) = \overline{E}(\infty, g)$, we find that ∞ is a Picard value of f, and zeros of $(f-\alpha)(f-\beta)$ are of multiplicities at least n. Therefore, applying the second main theorem to f yields

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f-\alpha}\right) + \overline{N}\left(r, \frac{1}{f-\beta}\right) + S(r,f) \\ &\leq \frac{1}{n}N\left(r, \frac{1}{f-\alpha}\right) + \frac{1}{n}N\left(r, \frac{1}{f-\beta}\right) + S(r,f) \leq \frac{2}{n}\,T(r,f) + S(r,f), \end{aligned}$$

a contradiction to the assumption $n \geq 8$.

Case (ii): $F \equiv G$. It is obvious that

$$n(n-1)f^2g^2(f^{n-2}-g^{n-2}) - 2bn(n-2)fg(f^{n-1}-g^{n-1}) + b^2(n-1)(n-2)(f^n-g^n) \equiv 0.$$

Set $h = f/g$. Then substituting $f = hg$ into the above equation yields

$$n(n-1)h^2(h^{n-2}-1)g^2 - 2bn(n-2)h(h^{n-1}-1)g + b^2(n-1)(n-2)(h^n-1) \equiv 0. \tag{3.10}$$

If h is not a constant, then from (3.10) we have

$$\{n(n-1)h(h^{n-2}-1)g - bn(n-2)(h^{n-1}-1)\}^2 \equiv -b^2n(n-2)Q(h), \tag{3.11}$$

where $Q(\omega)$ is given by (2.1). Applying the conclusions of Lemma 1 to (3.11) yields

$$\{n(n-1)h(h^{n-2}-1)g - bn(n-2)(h^{n-1}-1)\}^2 \equiv -b^2n(n-2)(n-1)^2(h-1)^4(h-\gamma_1)(h-\gamma_2)\cdots(h-\gamma_{2n-6}), \tag{3.12}$$

where $\gamma_j \in \mathbb{C}\backslash\{0,1\}$ are pairwise distinct and $Q(\gamma_j) = 0$ for $j = 1, \ldots, 2n-6$.

It is easily seen from (3.12) that zeros of $h - \gamma_j$ $(j = 1, \ldots, 2n-6)$ are of multiplicities at least 2. Therefore, applying the second main theorem to h yields

$$\begin{aligned}(2n-8)T(r,h) &\le \sum_{j=1}^{2n-6} \overline{N}\left(r, \frac{1}{h-\gamma_j}\right) + S(r,h)\\ &\le \frac{1}{2}\sum_{j=1}^{2n-6} N\left(r, \frac{1}{h-\gamma_j}\right) + S(r,h)\\ &\le (n-3)T(r,h) + S(r,h),\end{aligned}$$

a contradiction to the assumption $n \ge 8$.

Hence, h is a constant. From (3.10), we see that $h^{n-2}-1 = 0$, $h^{n-1}-1 = 0$ and $h^n - 1 = 0$ simultaneously, which means $h = 1$. So, $f \equiv g$. ∎

4. Proof of Theorem 4. Similar to the proof Theorem 3, we have

$$\begin{aligned}&(n+1)(T(r,f)+T(r,g))\\ &\le 3\overline{N}(r,f) + 2\overline{N}\left(r,\frac{1}{f}\right) + 2\overline{N}\left(r,\frac{1}{f-b}\right) + 2\overline{N}\left(r,\frac{1}{g}\right)\\ &\quad + 2\overline{N}\left(r,\frac{1}{g-b}\right) + \overline{N}\left(r,\frac{1}{F-1}\right) + \overline{N}\left(r,\frac{1}{G-1}\right) + \overline{N}_{(3}\left(r,\frac{1}{F-1}\right)\\ &\quad + \overline{N}_{(3}\left(r,\frac{1}{G-1}\right) - N_{1)}\left(r,\frac{1}{F-1}\right) + S(r,f) + S(r,g).\end{aligned} \tag{4.1}$$

It is not difficult to see that

$$\overline{N}\left(r,\frac{1}{F-1}\right)-\frac{1}{2}N_{1)}\left(r,\frac{1}{F-1}\right)+\frac{1}{2}\overline{N}_{(3}\left(r,\frac{1}{F-1}\right)\leq\frac{1}{2}N\left(r,\frac{1}{F-1}\right),\tag{4.2}$$

$$\overline{N}\left(r,\frac{1}{G-1}\right)-\frac{1}{2}N_{1)}\left(r,\frac{1}{G-1}\right)+\frac{1}{2}\overline{N}_{(3}\left(r,\frac{1}{G-1}\right)\leq\frac{1}{2}N\left(r,\frac{1}{G-1}\right).\tag{4.3}$$

From (2.7), we have

$$\begin{aligned}\overline{N}_{(3}\left(r,\frac{1}{F-1}\right)&\leq\sum_{j=1}^{n}\overline{N}_{(3}\left(r,\frac{1}{f-\omega_j}\right)\leq\frac{1}{2}\,N\left(r,\frac{1}{f'}\right)\\&\leq\frac{1}{2}\,\overline{N}(r,f)+\frac{1}{2}N\left(r,\frac{1}{f}\right)+S(r,f)\\&\leq\frac{1}{2}\,\overline{N}(r,f)+\frac{1}{2}\,T(r,f)+S(r,f),\end{aligned}$$

where ω_j for $j=1,\dots,n$ are the pairwise distinct roots of $P(\omega)$ given by (1.3).

Similarly, we have

$$\overline{N}_{(3}\left(r,\frac{1}{G-1}\right)\leq\frac{1}{2}\,\overline{N}(r,g)+\frac{1}{2}\,T(r,g)+S(r,g).\tag{4.5}$$

Combining (4.2)–(4.5) with the assumptions $E_{2)}(1,F)=E_{2)}(1,G)$ and $\overline{E}(\infty,f)=\overline{E}(\infty,g)$ yields

$$\begin{aligned}\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)+\overline{N}_{(3}\left(r,\frac{1}{F-1}\right)&\\+\overline{N}_{(3}\left(r,\frac{1}{G-1}\right)-N_{1)}\left(r,\frac{1}{F-1}\right)&\\\leq\frac{1}{2}\,\overline{N}(r,f)+\frac{2n+1}{4}\,(T(r,f)+T(r,g))+O(1).&\end{aligned}$$

Substituting the above inequality into (4.1) yields

$$(2n-13)(T(r,f)+T(r,g))\leq 14\overline{N}(r,f)+S(r,f)+S(r,g).\tag{4.6}$$

Applying the conclusions of Lemma 2 to (4.6) with $k=2$ yields

$$\frac{2n^2-21n+31}{n-4}\,(T(r,f)+T(r,g))\leq S(r,f)+S(r,g),$$

a contradiction to the assumption $n\geq 9$. ■

5. Proof of Theorem 5. Analogously to the proof Theorem 3, we have

$$(5.1)\qquad (n+1)(T(r,f)+T(r,g))$$
$$\leq 3\overline{N}(r,f)+2\overline{N}\left(r,\frac{1}{f}\right)+2\overline{N}\left(r,\frac{1}{f-b}\right)+2\overline{N}\left(r,\frac{1}{g}\right)$$
$$+2\overline{N}\left(r,\frac{1}{g-b}\right)+\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)+\overline{N}_{(2}\left(r,\frac{1}{F-1}\right)$$
$$+\overline{N}_{(2}\left(r,\frac{1}{G-1}\right)-N_{1)}\left(r,\frac{1}{F-1}\right)+S(r,f)+S(r,g).$$

It is not difficult to see that

$$(5.2)\qquad \overline{N}\left(r,\frac{1}{F-1}\right)-\frac{1}{2}N_{1)}\left(r,\frac{1}{F-1}\right)\leq\frac{1}{2}N\left(r,\frac{1}{F-1}\right),$$

$$(5.3)\qquad \overline{N}\left(r,\frac{1}{G-1}\right)-\frac{1}{2}N_{1)}\left(r,\frac{1}{G-1}\right)\leq\frac{1}{2}N\left(r,\frac{1}{G-1}\right).$$

From (2.7) and (4.4), we have

$$(5.4)\qquad \overline{N}_{(2}\left(r,\frac{1}{F-1}\right)\leq\sum_{j=1}^{n}\overline{N}_{(2}\left(r,\frac{1}{f-\omega_j}\right)\leq N\left(r,\frac{1}{f'}\right)$$
$$\leq\overline{N}(r,f)+T(r,f)+S(r,f),$$

where ω_j are as before for $j=1,\ldots,n$.

Analogously, we have

$$(5.5)\qquad \overline{N}_{(2}\left(r,\frac{1}{G-1}\right)\leq\overline{N}(r,g)+T(r,g)+S(r,g).$$

Combining (5.2)–(5.5) with the assumptions $E_{1)}(1,F)=E_{1)}(1,G)$ and $\overline{E}(\infty,f)=\overline{E}(\infty,g)$ yields

$$\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)+\overline{N}_{(2}\left(r,\frac{1}{F-1}\right)$$
$$+\overline{N}_{(2}\left(r,\frac{1}{G-1}\right)-N_{1)}\left(r,\frac{1}{F-1}\right)$$
$$\leq 2\overline{N}(r,f)+\frac{n+2}{2}(T(r,f)+T(r,g))+O(1).$$

Substituting the above inequality into (5.1) yields

$$(5.6)\qquad (n-8)(T(r,f)+T(r,g))\leq 10\overline{N}(r,f)+S(r,f)+S(r,g).$$

Applying the conclusions of Lemma 2 to (5.6) with $k=1$ yields

$$\frac{n^2-13n+20}{n-5}(T(r,f)+T(r,g))\leq S(r,f)+S(r,g),$$

a contradiction to the assumption $n\geq 12$. ∎

6. Proof of Theorems 1 and 2. Analogously to the proof of Theorem 4, we have

$$(2n-13)(T(r,f)+T(r,g)) \le S(r,f)+S(r,g), \tag{6.1}$$

since the terms $\overline{N}(r,f)$ and $\overline{N}(r,g)$ are $O(1)$. Clearly, this contradicts the assumption that $n \ge 7$ and terminates the proof of Theorem 1. ■

For the proof of Theorem 2, by the proofs of Theorems 1 and 5, we see that

$$(n-8)(T(r,f)+T(r,g)) \le S(r,f)+S(r,g), \tag{6.2}$$

which immediately yields a contradiction to $n \ge 9$. ■

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School of Mathematics and System Sciences
Shandong University
Jinan 250100, Shandong, People's Republic of China
E-mail: k.l.han@tom.com
hxyi@sdu.edu.cn

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