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On the zero set of the Kobayashi–Royden pseudometric of the spectral unit ball

by Nikolai Nikolov (Sofia) and Pascal J. Thomas (Toulouse)

Abstract. Given $A \in \Omega_n$, the n^2-dimensional spectral unit ball, we show that if B is an $n \times n$ complex matrix, then B is a "generalized" tangent vector at A to an entire curve in Ω_n if and only if B is in the tangent cone C_A to the isospectral variety at A. In the case of Ω_3, the zero set of the Kobayashi–Royden pseudometric is completely described.

1. Introduction and results. Let $\mathcal{M}_n$ be the set of all $n \times n$ complex matrices. For $A \in \mathcal{M}_n$ denote by $\mathrm{sp}(A)$ and $r(A) = \max_{\lambda \in \mathrm{sp}(A)} |\lambda|$ the spectrum and the spectral radius of A, respectively. The *spectral ball* Ω_n is the set

$$\Omega_n := \{A \in \mathcal{M}_n : r(A) < 1\}.$$

The *spectral Nevanlinna–Pick problem* is the following: given N points $a_1, \dots, a_N$ in the unit disk $\mathbb{D} \subset \mathbb{C}$ and N matrices $A_1, \dots, A_N \in \Omega_n$ decide whether there is a mapping $\varphi \in \mathcal{O}(\mathbb{D}, \Omega_n)$ such that $\varphi(a_j) = A_j$, $1 \le j \le N$ (cf. [1, 2, 4, 7, 8] and the references there).

The study of the Nevanlinna–Pick problem in the case $N = 2$ reduces to the computation of the *Lempert function*, defined as follows for a domain $D \subset \mathbb{C}^m$:

$$l_D(z, w) := \inf\{|\alpha| : \exists \varphi \in \mathcal{O}(\mathbb{D}, D) : \varphi(0) = z,\ \varphi(\alpha) = w\}, \qquad z, w \in D.$$

The infinitesimal version of the above is the *Carathéodory–Fejér problem of order* 1: given matrices $A_0, A_1 \in \mathcal{M}_n$, decide whether there is a mapping $\varphi \in \mathcal{O}(\mathbb{D}, \Omega_n)$ such that $A_0 = \varphi(0)$, $A_1 = \varphi'(0)$. This problem has been studied in [10].

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Its study reduces to the computation of the *Kobayashi–Royden pseudometric*, defined as follows for a domain $D \subset \mathbb{C}^m$:

$$k_D(z;X) := \inf\{|\alpha| : \exists \varphi \in \mathcal{O}(\mathbb{D}, D) : \\ \varphi(0) = z,\, \alpha\varphi'(0) = X\}, \quad z \in D,\, X \in \mathbb{C}^m.$$

To each matrix A we associate its characteristic polynomial

$$P_A(t) := \det(tI - A) = t^n + \sum_{j=1}^{n} (-1)^j \sigma_j(A) t^{n-j},$$

where $I \in \mathcal{M}_n$ is the unit matrix,

$$\sigma_j(A) := \sigma_j(\lambda_1, \ldots, \lambda_n) := \sum_{1 \le k_1 < \cdots < k_j \le n} \lambda_{k_1} \ldots \lambda_{k_j}$$

and $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A.

Put $\sigma := (\sigma_1, \ldots, \sigma_n) : \mathcal{M}_n \to \mathbb{C}^n$. The set

$$\mathbb{G}_n := \{\sigma(A) : A \in \Omega_n\}$$

is a taut (even hyperconvex) domain called the *symmetrized n-disk* (cf. [3, 8, 11] and references there). Explicit formulas for $l_{\mathbb{G}_2}$ and $k_{\mathbb{G}_2}$ can be found in [2] (see also [11]) and [10], respectively.

Recall now that a matrix $A \in \mathcal{M}_n$ is called *nonderogatory* if all the blocks in the Jordan form of A have distinct eigenvalues. Many properties equivalent to this definition may be found in [13, Proposition 3]. We point out one of them: A is nonderogatory if and only if $\operatorname{rank}(\sigma_{*,A}) = n$, where $\sigma_{*,A}$ stands for the differential of σ at the point A.

Denote by $\mathcal{C}_n$ the open and dense set of all nonderogatory matrices in Ω_n.

If $A_1, \ldots, A_N \in \mathcal{C}_n$, then any mapping $\varphi \in \mathcal{O}(\mathbb{D}, \mathbb{G}_n)$ with $\varphi(\alpha_j) = \sigma(A_j)$ can be lifted to a mapping $\widetilde{\varphi} \in \mathcal{O}(\mathbb{D}, \Omega_n)$ with $\widetilde{\varphi}(\alpha_j) = A_j$, $1 \le j \le N$ (see [1]). This means that in a generic case the spectral Nevanlinna–Pick problem for Ω_n (with dimension n^2) can be reduced to the standard Nevanlinna–Pick problem for $\mathbb{G}_n$ (with dimension n).

By the contractibility of the Lempert function, we have

$$l_{\Omega_n}(A, B) \ge l_{\mathbb{G}_n}(\sigma(A), \sigma(B)), \quad A, B \in \Omega_n,$$

and the lifting above implies that equality holds when $A, B \in \mathcal{C}_n$.

From this and the fact that $\mathbb{G}_n$ is a bounded domain, $l_{\Omega_n}(A, B) > 0$ if $\operatorname{sp}(A) \ne \operatorname{sp}(B)$. On the other hand, if $\operatorname{sp}(A) = \operatorname{sp}(B)$, then there is an entire mapping $\varphi : \mathbb{C} \to \Omega_n$ with $\varphi(0) = A$ and $\varphi(1) = B$ (see [9]); note that $\operatorname{sp}(\varphi(\zeta)) = \operatorname{sp}(A)$ for all $\zeta \in \mathbb{C}$, since by Liouville's theorem, whenever $\varphi(\mathbb{C}) \subset \Omega$, then $\sigma \circ \varphi$ is constant. This situation is similar to that of Brody's theorem for compact manifolds [5]: failure of hyperbolicity (that is, vanishing of the pseudodistance) can be explained by the presence of a (nonconstant) entire curve in the manifold.

Restricting again to nonderogatory matrices, a similar lifting [10] implies in the Carathéodory–Fejér case

$$k_{\Omega_n}(A;B) = k_{\mathbb{G}_n}(A, \sigma_{*,A}(B)), \qquad A \in \mathcal{C}_n, B \in \mathcal{M}_n;$$

in particular, $k_{\Omega_n}(A;B) = 0$ if and only if $\sigma_{*,A}(B) = 0$. On the other hand, if $\sigma_{*,A}(B) = 0$, then there is an entire mapping $\varphi : \mathbb{C} \to \Omega_n$ with $\varphi(0) = A$ and $\varphi'(0) = B$ (indeed, $B = [Y, A] := YA - AY$ for some $Y \in \mathcal{M}_n$ [13, proof of Proposition 3] and then the mapping $\zeta \mapsto e^{\zeta Y} A e^{-\zeta Y}$ does the job).

The aim of this paper is to study the zeros of $B \mapsto k_{\Omega_n}(A;B)$ in the remaining case, where A is a derogatory matrix, and to relate it to the existence of entire curves tangent to B at the point A (which is an obvious sufficient condition for $k_{\Omega_n}(A;B) = 0$).

For $A \in \Omega_n$ denote by C_A the tangent cone (cf. [6, p. 79] for this notion) to the isospectral variety

$$L_A := \{C \in \Omega_n : \operatorname{sp}(C) = \operatorname{sp}(A)\},$$

that is,

$$C_A := \{B \in \mathcal{M}_n : \exists 0 < c_j \to 0,\ C_j \in L_A \text{ with } c_j(C_j - A) \to B\}.$$

Observe that L_A is smooth at A if $A \in \mathcal{C}_n$; then $C_A = \ker \sigma_{*,A}$. When $A \notin \mathcal{C}_n$, the rank of $\sigma_{*,A}$ is not maximal, so we have $\dim \ker \sigma_{*,A} > n^2 - n$; by [6, Corollary, p. 83], C_A is an analytic set with $\dim C_A = \dim L_A = n^2 - n$, so we have $C_A \subsetneq \ker \sigma_{*,A}$.

The following proposition characterizes the tangent cone C_A as the set of "generalized" tangent vectors at A to an entire curve in Ω_n through A (therefore contained in L_A).

Proposition 1. *Let $A \in \Omega_n$ and $B \in \mathcal{M}_n$. Then there are $m \in \mathbb{N} = \{m \in \mathbb{Z} : m > 0\}$, $m \le n!$, and $\varphi \in \mathcal{O}(\mathbb{C}, \Omega_n)$ with $\varphi(0) = A$, $\varphi'(0) = \cdots = \varphi^{(m-1)}(0) = 0$, $\varphi^{(m)}(0) = B$ if and only if $B \in C_A$.*

Proposition 1 implies that C_A is contained in the zero set of the *singular Kobayashi pseudometric* (cf. [14])

$$\widehat{k}_{\Omega_n}(A;B) = \inf\{|\alpha| : \exists m \in \mathbb{N},\ \varphi \in \mathcal{O}(\mathbb{D}, \Omega_n) : \operatorname{ord}_0(\varphi - z) \ge m,\ \alpha\varphi^{(m)}(0) = m!X\}.$$

A consequence of the proof of Proposition 1 is the following.

Corollary 2. *Let $A \in \Omega_n$ and $B \in C_A$. Then the following conditions are equivalent*:

(a) *There is $\varphi \in \mathcal{O}(\mathbb{C}, \Omega_n)$ with $\varphi(0) = A$ and $\varphi'(0) = B$.*

(b) *There are* $r_j \to \infty$ *and* $\varphi_j \in \mathcal{O}(r_j\mathbb{D}, \Omega_n)$, $j \in \mathbb{N}$, *uniformly bounded near* 0, *such that* $\varphi_j(0) = A$ *and* $\varphi_j'(0) = B$.

(c) *There are* $r > 0$ *and* $\varphi \in \mathcal{O}(r\mathbb{D}, L_A)$ *with* $\varphi(0) = A$ *and* $\varphi'(0) = B$.

Before stating the next proposition, we shall define an algebraic cone $C_A' \subset \mathcal{M}_n$, $A \in \Omega_n$.

For a function g holomorphic near A, and X in a neighborhood of A, let $g(X) - g(A) = g_A^*(X - A) + \cdots$, where g_A^* stands for the homogeneous polynomial of lowest nonzero degree in the expansion of g near A (the omitted terms are thus of higher order).

Set

$$C_A^* := \{B \in \mathcal{M}_n : (\sigma_1)_A^*(B) = 0, \dots, (\sigma_n)_A^*(B) = 0\},$$
$$C_A' := \bigcap_{\lambda \in \mathrm{sp}(A)} (\Phi_\lambda)_{*,A}^{-1}(C_{\Phi_\lambda(A)}^*),$$

where

$$\Phi_\lambda(A) := (A - \lambda I)(I - \overline{\lambda} A)^{-1}. \tag{1}$$

Note that

$$C_A \subset C_A^* \subset \ker \sigma_{*,A}.$$

For the first inclusion, see [6, p. 86, lines 4–6]), and for the second one, use the fact that

$$\ker \sigma_{*,A} = \{(\sigma_j)_A^* = 0 \text{ for all } j \text{ such that } \deg(\sigma_j)_A^* = 1\}.$$

Since C_A and $\ker \sigma_{*,A}$ are invariant under automorphisms of Ω_n, it follows that

$$C_A \subset C_A' \subset \ker \sigma_{*,A}.$$

Moreover, if $\dim C_A = \dim C_A^*$, that is, $\dim C_A^* = n^2 - n$, then $C_A = C_A^* = C_A'$ (cf. [6, p. 112, Corollary 2]).

Proposition 3. *Let* $A \in \Omega_n \setminus \mathcal{C}_n$.

(i) *If* $\widehat{k}_{\Omega_n}(A; B) = 0$, *then* $B \in C_A'$.

(ii) $C_A' \neq \ker \sigma_{*,A}$.

Remark. The cone C_A^* may coincide with $\ker \sigma_{*,A}$ for some $A \in \Omega_n \setminus \mathcal{C}_n$, $n \geq 3$. For example, if $A := \mathrm{diag}(t, \dots, t, 0)$, $t \in \mathbb{D}_*$, then

$$C_A^* = \ker \sigma_{*,A} = \{B \in \mathcal{M}_n : \mathrm{tr}\, B = b_{nn} = 0\}.$$

The main consequence of Proposition 3 is that for $A \in \Omega_n \setminus \mathcal{C}_n$ and $B \in \ker \sigma_{*,A} \setminus C_A'$, a lifting for the corresponding Carathéodory–Fejér problem is not possible and $k_{\Omega_n}(\cdot; B)$ is not a continuous function at A. This generalizes previous discontinuity results (see [13] and references therein).

Note also that the cone C_A' may coincide with C_A in some cases, for example, for any $A \in \Omega_2$ (then also $C_A^* = C_A$) and any $A \in \Omega_3$ (see

Proposition 6 and the discussion before it). We do not know whether this holds in general. On the other hand, it is not hard to find cases where $C_A \subsetneq C^*_A$.

PROPOSITION 4. *For any $n \geq 3$ there is $A \in \Omega_n$ such that $C_A \subsetneq C^*_A$.*

Now, we state a conjecture about the zero set of k_{Ω_n}.

CONJECTURE 5. *$k_{\Omega_n}(A;B) = 0$ if and only if there is $\varphi \in \mathcal{O}(\mathbb{C}, \Omega_n)$ with $\varphi(0) = A$ and $\varphi'(0) = B$. In particular, if $k_{\Omega_n}(A;B) = 0$, then $B \in C_A$.*

Conversely, however, there are matrices $B \in C_A$ such that $k_{\Omega_n}(A;B) \neq 0$ (see Proposition 6(ii) and Corollary 7).

There are some cases where our conjecture can be checked.

For example, since Ω_n is a balanced domain, $l_{\Omega_n}(0,\cdot)$ and $k_{\Omega_n}(0;\cdot)$ coincide with the Minkowski function, that is, with the spectral radius. Thus the zeros of $k_{\Omega_n}(0;\cdot)$ are exactly the zero-spectrum matrices, and the set of those matrices is a union of complex lines through 0.

Also, if A is a scalar matrix, that is, $A = \lambda I$, $\lambda \in \mathbb{C}$, then $B \in C_A$ if and only if there is $\varphi \in \mathcal{O}(\mathbb{C}, \Omega_n)$ with $\varphi(0) = A$ and $\varphi'(0) = B$. To see this, use an automorphism of Ω_n of the form (1) to reduce to the case $A = 0$.

Since the derogatory matrices in Ω_2 are exactly the scalar matrices, we may choose $m = 1$ in Proposition 1 if $n = 2$, and C_A coincides both with the zeros of $k_{\Omega_2}(A;\cdot)$ and with the matrices $B = \varphi'(0)$ for some entire curve φ in Ω_2 (on the other hand, $\ker \sigma_{*,A} = \{B \in \mathcal{M}_2 : \operatorname{tr} B = 0\}$).

Now we shall study the zero set of $k_{\Omega_3}(A;\cdot)$, when A is a nonscalar derogatory matrix. Using first an automorphism of the form (1) and then an automorphism of the form $C \mapsto D^{-1}CD$ reduces the problem to the following two cases:

$$A = A_t := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & t \end{pmatrix}, \quad t \in \mathbb{D}_*, \qquad A = \widetilde{A} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that

$$C_{A_t} \subset C^*_{A_t} = C'_{A_t} = \{B \in \mathcal{M}_3 : b_{33} = b_{11} + b_{22} = b_{11}^2 + b_{12}b_{21} = 0\},$$
$$C_{\widetilde{A}} \subset C'_{\widetilde{A}} = \{B \in \mathcal{M}_3 : b_{11} + b_{22} + b_{33} = b_{32} = b_{12}b_{31} = 0\}$$

(to prove, for example, the second inclusion, use the fact that if $B_\varepsilon = A + \varepsilon B + o(\varepsilon)$, then $\operatorname{tr} B_\varepsilon = \varepsilon \operatorname{tr} B + o(\varepsilon)$, $\sigma_2(B_\varepsilon) = -\varepsilon b_{32} + o(\varepsilon)$ and $\det B_\varepsilon = \varepsilon^2(b_{12}b_{31} - b_{11}b_{32}) + o(\varepsilon^2)$). The next proposition implies, in particular, that $C_{A_\lambda} = C'_{A_\lambda}$ and $C_{\widetilde{A}} = C'_{\widetilde{A}}$ (use the fact that the tangent cones are closed or the dimensional reasoning mentioned above).

Proposition 6.

(i) *For any* $B \in C'_{A_t}$ $(t \neq 0)$ *there is* $\varphi \in \mathcal{O}(\mathbb{C}, \Omega_3)$ *with* $\varphi(0) = A_t$ *and* $\varphi'(0) = B$.

(ii) *Let* $B \in C'_{\widetilde{A}}$. *Then there is* $\varphi \in \mathcal{O}(\mathbb{C}, \Omega_n)$ *with* $\varphi(0) = \widetilde{A}$ *and* $\varphi'(0) = B$ *if and only if* $b_{11} = 0$ *or* $b_{12} \neq b_{31}$. *Otherwise*, $k_{\Omega_3}(\widetilde{A}; B) > 0$.

Corollary 7. *For any* $n \geq 3$ *there are* $A \in \Omega_n$ *and* $B \in C_A$ *such that* $k_{\Omega_n}(A; B) > 0$.

Since $k_{\Omega_3}(A; B) > 0$ if $B \notin C'_A$, Proposition 6 and the discussion before give a complete description of the zero set of k_{Ω_3}.

Note that the situation is much easier for the *Carathéodory–Reiffen pseudometric*

$$\gamma_{\Omega_n}(A; B) = \sup\{|f'(A)B| : f \in \mathcal{O}(D, \mathbb{D})\}.$$

Here $\gamma_{\Omega_n}(A; B) = 0$ if and only if $\sigma_{*,A}(B) = 0$. Indeed, if $\sigma_{*,A}(B) \neq 0$, then

$$\gamma_{\Omega_n}(A; B) \geq \gamma_{\mathbb{G}_n}(A; \sigma_{*,A}(B)) > 0.$$

On the other hand, if $A \in \mathcal{C}_n$ and $\sigma_{*,A}(B) = 0$, then

$$0 = k_{\Omega_n}(A; B) \geq \gamma_{\Omega_n}(A; B) \geq 0.$$

It remains to use the density of $\mathcal{C}_n$ in Ω_n and the continuity of the Carathéodory–Reiffen pseudometric.

The rest of the paper is organized as follows. The proofs of Propositions 3 and 4 are given in Section 2, the proofs of Proposition 6 and Corollary 7 in Section 3, and the proofs of Proposition 1 and Corollary 2 in Section 4.

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2. Proofs of Propositions 3 and 4

Proof of Proposition 4. Set $A := \operatorname{diag}(0, \ldots, 0, t, t)$, $t \in \mathbb{D}_*$. It is easy to see that

$$(\sigma_1)^*_A(B) = \sum_{j=1}^{n} b_{jj},$$

$$(\sigma_2)^*_A(B) = 2t \sum_{j=1}^{n-2} b_{jj} + t(b_{n-1,n-1} + b_{nn}),$$

$$(\sigma_3)^*_A(B) = t^2 \sum_{j=1}^{n-2} b_{jj}.$$

Therefore, $(\sigma_3)^*_A = t(\sigma_2)^*_A - t^2(\sigma_1)^*_A$ and $\dim C^*_A > n^2 - n = \dim C_A$.

Proof of Proposition 3. (i) Let $\widehat{\gamma}_{\Omega_n}(A;B)$ be the *singular Carathéodory metric* (cf. [12])

$$\widehat{\gamma}_{\Omega_n}(A;B) := \sup\left\{\left|\frac{f^{(k)}(A)B}{k!}\right|^{1/k} : k\in\mathbb{N},\ f\in\mathcal{O}(\Omega_n,\mathbb{D}),\ \mathrm{ord}_A f\ge k\right\},$$

where $\left|\frac{f^{(k)}(A)B}{k!}\right| = \sum_{|\alpha|=k} D^\alpha f(A)B^\alpha$. Since

$$\widehat{k}_{\Omega_n}(A;B) \ge \widehat{\gamma}_{\Omega_n}(A;B),$$

it is enough to show that $\widehat{\gamma}_{\Omega_n}(A;B)>0$ if $B\notin C'_A$. Then $B\in C^*_{\Phi_\lambda(A)}$ for some $\lambda\in\mathrm{sp}(A)$. Replacing A and B by $\Phi_\lambda(A)$ and $(\Phi_\lambda)_{*,A}(B)$, respectively, we may assume that $B\notin C^*_A$. Then there is σ_j such that $(\sigma_j)^*_A(B)\ne 0$. Denoting by k the degree of $(\sigma_j)^*_A$, it follows that

$$\widehat{\gamma}_{\Omega_n}(A;B) \ge \left|\frac{(\sigma_j)^*_A(B)}{\binom{n}{j}}\right|^{1/k} > 0.$$

(ii) Since $A\in\Omega_n\setminus\mathcal{C}_n$, at least two of the eigenvalues of A are equal, say to λ. Applying the automorphism Φ_λ of Ω_n, we may assume that $\lambda=0$. Since the map $A\mapsto P^{-1}AP$ is a linear automorphism of Ω_n for any $P\in\mathcal{M}_n^{-1}$, we may also assume that A is in Jordan form. In particular,

$$A=\begin{pmatrix} A_0 & 0\\ 0 & A_1\end{pmatrix},$$

where $A_0\in\mathcal{M}_m$, $2\le m\le n$, $\mathrm{sp}(A_0)=\{0\}$, $A_1\in\mathcal{M}_{n-m}$, $0\notin\mathrm{sp}(A_1)$. Furthermore, there is a set $J\subsetneq\{2,\dots,m\}$, possibly empty, such that $a_{j-1,j}=1$ for $j\in J$, and all other coefficients a_{ij} are zero for $1\le i,j\le m$. Define $0\le r:=\#J=\mathrm{rank}\,A_0\le m-2$.

We set

$$B:=\begin{pmatrix} B_0 & 0\\ 0 & 0\end{pmatrix}\in\mathcal{M}_n,$$

where $B_0=(b_{ij})_{1\le i,j\le m}$ is such that $b_{j-1,j}=-1$ for $j\in\{2,\dots,m\}\setminus J$, $b_{m1}=1$, and $b_{ij}=0$ otherwise. To complete the proof, it is enough to show the following.

Lemma 8. $(\sigma_m)^*_A(B)=1$, *but* $\sigma_{*,A}(B)=0$.

Proof. We begin by computing $\sigma_j(A_0+hB_0)$, $1\le j\le m$, $h\in\mathbb{C}$. Expanding with respect to the first column, we see that

$$\det(tI-(A_0+hB_0)) = t^m+(-1)^{m-1}h^{m-r}.$$

Comparing the corresponding coefficients of both sides, it follows that

$$\sigma_j(A_0+hB_0)=\begin{cases} 0, & 1\le j\le m-1,\\ h^{m-r}, & j=m.\end{cases} \tag{2}$$

Next, we need a general formula for the functions σ_j. Given a matrix $M = (m_{ij})_{1\le i,j\le n}$ and a set $E \subset \{1,\dots,n\}$, we write $\delta_E(M)$ for the determinant of the matrix $(m_{ij})_{i,j\in E} \in \mathcal{M}_{\#E}$. By convention, $\delta_\emptyset(M) = \sigma_0(M) := 1$. Then

$$\sigma_j(M) = \sum_{E\subset\{1,\dots,n\},\,\#E=j} \delta_E(M). \tag{3}$$

Because of the block structure of our matrices,

$$\delta_E(A+hB) = \delta_{E\cap\{1,\dots,m\}}(A_0+hB_0)\delta_{E\cap\{m+1,\dots,n\}}(A_1).$$

Therefore

$$\begin{aligned}\sigma_j(A+hB) &= \sum_{\max(0,j-n+m)\le k\le\min(m,j)} \Big(\sum_{E'\subset\{1,\dots,m\},\,\#E'=k} \delta_{E'}(A_0+hB_0)\Big) \\ &\qquad\times\Big(\sum_{E''\subset\{m+1,\dots,n\},\,\#E''=j-k} \delta_{E''}(A_1)\Big) \\ &= \sum_{\max(0,j-n+m)\le k\le\min(m,j)} \sigma_k(A_0+hB_0)\sigma_{j-k}(A_1).\end{aligned}$$

It follows by (2) that $\sigma_j(A+hB) = S_1 + S_2$, where

$$S_1 = \begin{cases} \sigma_j(A_1), & j \le n-m, \\ 0, & \text{otherwise}, \end{cases} \qquad S_2 = \begin{cases} h^{m-r}\sigma_{j-m}(A_1), & j \ge m, \\ 0, & \text{otherwise}. \end{cases}$$

In particular,

$$\sigma_j(A) = \begin{cases} \sigma_j(A_1), & j \le n-m, \\ 0, & \text{otherwise}. \end{cases}$$

Then

$$\sigma_j(A+hB) - \sigma_j(A) = \begin{cases} h^{m-r}\sigma_{j-m}(A_1), & j \ge m, \\ 0, & \text{otherwise}. \end{cases}$$

Since $m - r \ge 2$ we conclude that $\sigma_{*,A}(B) = 0$, but $(\sigma_m)^*_A(B) = 1$. ∎

3. Proofs of Proposition 6 and Corollary 7

Proof of Proposition 6. (i) Let first $B \in C'_{A_t}$. We shall write B in the form $B = X + [Y, A_t]$, where X is such that $\psi(\zeta) = A_t + \zeta X \in L_{A_t}$ for any $\zeta \in \mathbb{C}$. Then $\varphi(\zeta) = e^{\zeta Y}\psi(\zeta)e^{-\zeta Y}$ has the required properties.

It is easy to compute that $\psi(\mathbb{C}) \subset L_{A_t}$ if and only if $\operatorname{sp}(X) = \{0\}$ and $x_{11} + x_{22} = x_{11}^2 + x_{12}x_{21} = 0$. On the other hand,

$$[Y, A_t] = t\begin{pmatrix} 0 & 0 & y_{13} \\ 0 & 0 & y_{23} \\ -y_{31} & -y_{32} & 0 \end{pmatrix}.$$

So we may take

$$X = \begin{pmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = t^{-1} \begin{pmatrix} 0 & 0 & b_{13} \\ 0 & 0 & b_{23} \\ -b_{31} & -b_{32} & 0 \end{pmatrix}.$$

(ii) Let first $B \in \mathbb{C}'_{\widetilde{A}}$. If $b_{11} = 0$ or $b_{12} \neq b_{31}$, it is enough to find (as above) X and Y such that $B = X + [Y, \widetilde{A}]$ and $\widetilde{A} + \zeta X \in L_{\widetilde{A}}$ for any $\zeta \in \mathbb{C}$. The last condition means that $\mathrm{sp}(X) = \{0\}$ and $x_{32} = x_{12}x_{31} = 0$. On the other hand,

$$[Y, \widetilde{A}] = \begin{pmatrix} 0 & 0 & y_{12} \\ -y_{31} & -y_{33} & y_{22} - y_{33} \\ 0 & 0 & y_{32} \end{pmatrix}.$$

Assume that $b_{31} = 0$ (the computations are similar in the case $b_{12} = 0$). Then we have to choose X of the form

$$X = \begin{pmatrix} b_{11} & b_{12} & b_{13} - y_{12} \\ b_{21} + y_{31} & b_{22} + y_{32} & b_{23} - y_{22} + y_{33} \\ 0 & 0 & -b_{11} - b_{22} - y_{32} \end{pmatrix}$$

such that $\det X = 0$ and $\sigma_2(X) = 0$, that is, $DT = 0$, $D = T^2$, where we write

$$D := \begin{vmatrix} b_{11} & b_{12} \\ b_{21} + y_{31} & b_{22} + y_{32} \end{vmatrix}, \quad T := b_{11} + b_{22} + y_{32}.$$

These conditions are satisfied if and only if

$$y_{32} = -b_{11} - b_{22}, \qquad y_{31} = \begin{cases} -b_{21}, & b_{11} = 0, \\ -b_{21} - b_{11}^2/b_{12}, & b_{12} \neq 0. \end{cases}$$

It remains to show that if $b_{11} \neq 0$ and $b_{12} = b_{31} = 0$, then $k_{\Omega_3}(\widetilde{A}; B) > 0$. We may assume that $b_{11} = 1$. Set

$$\widetilde{X} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Choosing X and Y as above yields $B = \widetilde{X} + [Y, A_t]$. Let $\alpha > 0$ and $\varphi \in \mathcal{O}(\alpha\mathbb{D}, \Omega_3)$ be such that $\varphi(0) = A_t$ and $\varphi'(0) = B$. Setting $\widetilde{\varphi}(\zeta) = e^{-\zeta Y}\varphi(\zeta)e^{\zeta Y}$, we have $\widetilde{\varphi} \in \mathcal{O}(\alpha\mathbb{D}, \Omega_3)$, $\widetilde{\varphi}(0) = \widetilde{A}$ and $\widetilde{\varphi}'(0) = X$. It follows that $k_{\Omega_3}(A_t; B) \geq k_{\Omega_3}(A_t; X)$. The opposite inequality follows in the same way.

Write $\widetilde{\varphi}$ in the form

$$\widetilde{\varphi}(\zeta) = \widetilde{A} + \zeta\widetilde{X} + \zeta^2\widehat{X} + o(\zeta^2).$$

Then we compute that

$$\sigma_2(\widetilde{\varphi}(\zeta)) = \zeta^2(1-\widehat{x}_{32}) + o(\zeta^2), \qquad \det\widetilde{\varphi}(\zeta) = -\zeta^3\widehat{x}_{32} + o(\zeta^3).$$

Since $|\sigma_2\circ\varphi| < 3$ and $|\det\varphi| < 1$, by the Cauchy inequalities we get

$$|\widehat{x}_{32} - 1| \le 3\alpha^{-2}, \qquad |\widehat{x}_{32}| \le \alpha^{-3}.$$

So

$$k_{\Omega_3}(A_t; B) = k_{\Omega_3}(\widetilde{A}; \widetilde{X}) \ge \min_{t\in\mathbb{C}} \max\{\sqrt{|t-1|/3}, \sqrt[3]{|t|}\} > 0.$$

Proof of Corollary 7. Set

$$\widetilde{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \widetilde{B}_\varepsilon = \begin{pmatrix} 1 & \varepsilon & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$A = \begin{pmatrix} \widetilde{A} & O \\ O & O \end{pmatrix}, \qquad B_\varepsilon = \begin{pmatrix} \widetilde{B}_\varepsilon & O \\ O & O \end{pmatrix}.$$

It follows as in the proof of Proposition 6(ii) that

- $k_{\Omega_n}(A; B_0) > 0$;
- for $\varepsilon \ne 0$, there is $\varphi_\varepsilon \in \mathcal{O}(\mathbb{C}, \Omega_n)$ with $\varphi_\varepsilon(0) = A$ and $\varphi'_\varepsilon(0) = B_\varepsilon$.

Then $B_\varepsilon \in C_A$ for $\varepsilon \ne 0$, and hence $B_0 \in C_A$.

4. Proofs of Proposition 1 and Corollary 2

Proof of Corollary 2. The implication (a)⇒(b) is trivial and the main implication (c)⇒(a) is a particular case of Proposition 9 below.

It remains to prove that (b)⇒(c). Let $\psi(\zeta) = \sum_{k=0}^\infty A_k\zeta^k \in \mathcal{O}(s\mathbb{D}, \Omega_n)$ for some $s > 0$. Let $a_{k,\psi} \in \mathbb{C}^{(k+1)\times n^2}$ be the vector with components the entries of $A_0, \dots, A_k$ (taken in some order). Note that

$$\sigma_l(\psi(\zeta)) = \sum_{k=0}^\infty p_{l,k}(a_{k,\psi})\zeta^k, \qquad 1 \le l \le n,$$

where the $p_{l,k}$ are polynomials.

Let now $r_j \to \infty$ and $\varphi_j \in \mathcal{O}(r_j\mathbb{D}, \Omega_n)$, $j \in \mathbb{N}$, uniformly bounded near 0, be such that $\varphi_j(0) = A$ and $\varphi'_j(0) = B$. Then we may assume that $\varphi_j \to \varphi \in \mathcal{O}(r\mathbb{D}, \Omega_n)$ for some $r > 0$. Hence $p_{l,k}(a_{k,\varphi_j}) \to p_{l,k}(a_{k,\varphi})$. On the other hand, $|p_{l,k}(a_{k,\varphi_j})| \le \binom{n}{l}/r_j^k \to 0$, $k > 0$, by the Cauchy inequalities. Hence $p_{l,k}(a_{k,\varphi}) = 0$, $k > 0$, that is, $\sigma_l(\varphi(\zeta)) = \sigma_l(\varphi(0)) = \sigma_l(A)$. This means that $\varphi(\zeta) \in L_A$.

Proof of Proposition 1. It is clear that if such a φ exists, then $B \in C_A$.

Conversely, let $B \in C_A$. Then, by [6, p. 86, Proposition 1], there exists a one-dimensional irreducible analytic variety $L_{A,B} \subset L_A$, tangent to B at A. Now, by [6, p. 80, Proposition], there exist $m \in \mathbb{N}$, $r > 0$ and $\psi \in \mathcal{O}(r\mathbb{D}, L_{A,B})$ such that $\psi(\zeta) = A + \zeta^m B + o(\zeta^m)$.

The integer m is the number of sheets in the (local) branched covering provided by the orthogonal projection from L_A to a suitable linear subspace of dimension $n^2 - n$ (see [6]). This number of sheets corresponds to the cardinality of the solution set, in each generic fiber of the projection, of the equations $\sigma_j(M) = \sigma_j(A)$, $1 \le j \le n$. Bézout's theorem shows that this cardinality is less than or equal to the product of the degrees of the polynomials, so here $m \le n!$.

The above considerations will prove the estimate in Proposition 1, provided that we can replace ψ by an entire map $\widetilde{\psi}$ with the same expansion up to order m near 0. So the proof of Proposition 1 reduces to the following.

Proposition 9. *If $A \in \Omega_n$, $m \in \mathbb{N}$ and $\psi \in \mathcal{O}(r\mathbb{D}, L_A)$ for some $r > 0$, then there is $\widetilde{\psi} \in \mathcal{O}(\mathbb{C}, L_A)$ with $\widetilde{\psi}(\zeta) = \psi(\zeta) + o(\zeta^m)$.*

Proof. We want to reduce the problem by replacing each matrix $\psi(\zeta)$ by a conjugate matrix $\varphi(\zeta)$ (in particular, they will have the same spectrum, so we remain inside L_A and inside Ω_n). If we can manage this so that $\varphi(\zeta)$ is upper triangular, then an entire map with the same spectrum matching φ up to order m can be obtained by taking the Taylor polynomial of degree m of each coefficient of φ.

To proceed with this program, first we need to show that conjugation (with a holomorphic change of basis) does not change the problem.

Let $\mathcal{M}_n^{-1}$ stand for the group of all invertible $n \times n$ matrices.

Lemma 10. *Let $r > 0$, $P \in \mathcal{O}(r\mathbb{D}, \mathcal{M}_n^{-1})$ and $\psi \in \mathcal{O}(r\mathbb{D}, \Omega_n)$. Write $\varphi(\zeta) := P(\zeta)^{-1}\psi(\zeta)P(\zeta)$, and assume that there exists $\widetilde{\varphi} \in \mathcal{O}(\mathbb{C}, \Omega_n)$ such that near 0, $\widetilde{\varphi}(\zeta) = \varphi(\zeta) + o(\zeta^m)$. Then there exists $\widetilde{\psi} \in \mathcal{O}(\mathbb{C}, \Omega_n)$ conjugate to $\widetilde{\varphi}$ (in particular having the same spectrum) such that near 0,*

$$\widetilde{\psi}(\zeta) = \psi(\zeta) + o(\zeta^m).$$

Note once again that Liouville's theorem implies that the entire maps $\widetilde{\varphi}, \widetilde{\psi}$ actually map to $L_{\widetilde{\varphi}(0)} = L_{\widetilde{\psi}(0)}$.

Proof. Note first that, because the exponential is locally Lipschitz, $\exp(A + M) = \exp A + O(M)$.

Denote by $L_m(x)$ the Taylor polynomial of degree m at 0 for the function $x \mapsto \ln(1 + x)$. Since $\exp(\ln(1 + x)) = 1 + x$ and $\ln(1 + x) = L_m(x) + o(x^m)$, we have $\exp(L_m(x)) = 1 + x + o(x^m)$. So $\exp(L_m(A)) = I + A + o(A^m)$.

Now write

$$P(\zeta) = P(0)\Big(I + \sum_{k=1}^{m} A_k\zeta^k + O(\zeta^{m+1})\Big) =: P(0)(I + M(\zeta)).$$

Define P_1 to be the unique matrix-valued polynomial of degree $\le m$ in ζ so that

$$L_m\Big(\sum_{k=1}^{m} A_k\zeta^k\Big) = P_1(\zeta) + o(\zeta^m).$$

Then, remarking that $M(\zeta) = o(1)$, we have

$$\exp(P_1(\zeta)) = \exp(L_m(M(\zeta)) + o(\zeta^m) = I + M(\zeta) + o(\zeta^m),$$

so that $P(0)\exp(P_1(\zeta)) = P(\zeta) + o(\zeta^m)$. Then it is easy to see that $\exp(-P_1(\zeta))P(0)^{-1} = P(\zeta)^{-1} + o(\zeta^m)$, and $\widetilde{P}(\zeta) := P(0)\exp(P_1(\zeta))$ defines an entire map. So $\widetilde{\psi} := \widetilde{P}\widetilde{\varphi}\widetilde{P}^{-1}$ satisfies the requirements. ∎

We now reduce the proof of Proposition 9 to the case of nilpotent matrices (that is, $\operatorname{sp}(A) = \{0\}$).

Lemma 11. *Suppose that the conclusion of Proposition* 9 *holds with the additional hypothesis that* $\operatorname{sp}(A) = \{0\}$*. Then it holds for an arbitrary matrix A.*

Proof. Write $\operatorname{sp}(\psi(\zeta)) = \operatorname{sp}(A) = \{\mu_j : 1 \le j \le k\}$ where the μ_j are distinct eigenvalues with respective algebraic multiplicities m_j. Let $S_j(\zeta) = \ker(\psi(\zeta) - \mu_j I)^{m_j}$ be the associated generalized eigenspace. Choose a basis $\{e_1, \ldots, e_n\}$ of $\mathbb{C}^n$ such that $S_j(0) = \operatorname{span}\{e_i : 1 \le i - \sum_{l=1}^{j-1} m_l \le m_j\}$. By Lemma 10, without loss of generality we may assume that the matrices are written in this basis, and therefore $\psi(0)$ is a block matrix.

By continuity of the various determinants involved, there exists some $r' > 0$ such that for $|\zeta| < r' \le r$, we still have, for each j, $\mathbb{C}^n = S_j(\zeta) \oplus \bigoplus_{l:l\ne j} S_l(0)$. Then there is a unique linear projection $\pi_{j,\zeta}$ defined on $\mathbb{C}^n$ such that $\pi_{j,\zeta}(\mathbb{C}^n) = S_j(\zeta)$ and $\operatorname{Ker}\pi_{j,\zeta} = \bigoplus_{l:l\ne j} S_l(0)$. This restricts to a linear isomorphism from $S_j(0)$ to $S_j(\zeta)$. Therefore the vectors $\{\pi_{j,\zeta}(e_i) : 1 \le i - \sum_{l=1}^{j-1} m_l \le m_j\}$, being obtained as solution of a Cramer system of linear equations with holomorphic coefficients, depend holomorphically on ζ in $D(0,r')$. Thus $\{\pi_{j,\zeta}(e_i) : 1 \le i - \sum_{l=1}^{j-1} m_l \le m_j, 1 \le j \le k\}$ form a basis of $\mathbb{C}^n$ adapted to the direct sum decomposition in the $S_j(\zeta)$. If we write $P(\zeta)$ for the matrix of the coordinates of the vectors of this new basis expressed in the standard basis, it depends holomorphically on ζ in $D(0,r')$, and the new matrix $\widehat{\psi}(\zeta) := P(\zeta)^{-1}\psi(\zeta)P(\zeta)$ has the same block structure as $\psi(0)$:

$$\widehat{\psi}(\zeta) = \begin{pmatrix} \widehat{\psi}_1(\zeta) & 0 & \cdots & 0 \\ 0 & \widehat{\psi}_2(\zeta) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \widehat{\psi}_k(\zeta) \end{pmatrix},$$

where $\widehat{\psi}_j \in \mathcal{O}(r'\mathbb{D}, \Omega_{m_j})$, $\mathrm{sp}(\widehat{\psi}_j(\zeta)) = \{\mu_j\}$. The map ω_j defined by

$$\omega_j(\zeta) := (\mu_j I_{m_j} - \widehat{\psi}_j(\zeta))(I_{m_j} - \bar{\mu}_j \widehat{\psi}_j(\zeta))^{-1}$$

is in $\mathcal{O}(r'\mathbb{D}, \Omega_{m_j})$, and its values are nilpotent matrices. By our hypothesis there are maps $\widetilde{\omega}_j \in \mathcal{O}(\mathbb{C}, \Omega_{m_j})$ such that $\widetilde{\omega}_j(\zeta) = \omega_j(\zeta) + o(\zeta^m)$ (and therefore with nilpotent values). Define

$$\widetilde{\psi}_j(\zeta) := (\mu_j I_{m_j} - \widetilde{\omega}_j(\zeta))(I_{m_j} - \overline{\mu}_j \widetilde{\omega}_j(\zeta))^{-1},$$

$$\widetilde{\psi}(\zeta) := \begin{pmatrix} \widetilde{\psi}_1(\zeta) & 0 & \cdots & 0 \\ 0 & \widetilde{\psi}_2(\zeta) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \widetilde{\psi}_k(\zeta) \end{pmatrix}.$$

It is easy to see that $\widetilde{\psi} \in \mathcal{O}(\mathbb{C}, \Omega_n)$ and $\widetilde{\psi}(\zeta) = \widehat{\psi}(\zeta) + o(\zeta^m)$. ∎

LEMMA 12. *If $m \in \mathbb{N}$ and $\psi \in \mathcal{O}(r\mathbb{D}, L_0)$ for some $r > 0$, then there are $r' \in (0, r)$ and $P \in \mathcal{O}(r'\mathbb{D}, \mathcal{M}_n^{-1})$ such that $\varphi(\zeta) := P(\zeta)^{-1}\psi(\zeta)P(\zeta)$ is a strictly upper triangular matrix for all $\zeta \in r'\mathbb{D}$.*

Proposition 9 follows from Lemma 12.

Indeed, by Lemma 11, we can make the additional hypothesis that ψ has nilpotent values, that is, $\psi \in \mathcal{O}(r\mathbb{D}, L_0)$. By Lemma 12, there is, for every ζ in a neighborhood of 0, a strictly upper triangular matrix $\varphi(\zeta) = P(\zeta)^{-1}\psi(\zeta)P(\zeta)$. If we replace each of the holomorphic coefficients $\varphi_{ij}(\zeta)$, $1 \le i < j \le n$, by its Taylor polynomial of order m, $\widetilde{\varphi}_{ij}(\zeta) := \sum_{k=0}^{m} \varphi_{ij}^{(k)}(0)\zeta^k/k!$, we obtain an approximation $\widetilde{\varphi}$ up to order m of our mapping which is entire, and still strictly upper triangular, therefore still with spectrum reduced to 0. Since $P(\zeta)$ depends holomorphically on ζ in a neighborhood of 0, we may apply Lemma 10 and obtain a matrix $\widetilde{\psi}(\zeta)$ still with spectrum reduced to 0, approximating ψ to order m, and entire in ζ.

Proof of Lemma 12. We are working with a matrix $\psi(\zeta)$ which satisfies $\psi(\zeta)^n = 0$ for all $\zeta \in r\mathbb{D}$. For $1 \le k \le n$, let $r_k(\zeta) := \mathrm{rank}(\psi(\zeta)^k)$. For a matrix M, $\mathrm{rank}(M) \le l < n$ if and only if all the minors of size $l + 1$ vanish; in our case they are holomorphic functions of ζ, therefore

$$r_k(\zeta) = \max_{\mathbb{D}} r_k =: \widetilde{r}_k$$

for all $\zeta \in r\mathbb{D}$ except on a discrete set. By replacing r by a smaller positive number if necessary, we may assume that $r_k(\zeta) = \widetilde{r}_k$ for all $\zeta \in r\mathbb{D}_* := r\mathbb{D} \setminus \{0\}$. Set

$$n_k := n - \widetilde{r}_k = \dim \ker \psi(\zeta)^k, \quad \zeta \in r\mathbb{D}_*.$$

It is a classical fact from linear algebra that, for a nilpotent matrix, if $p := \min\{k : n_k = n\}$, then $1 \le n_1 < \cdots < n_p = n_{p+1} = \cdots = n$.

For $1 \le k \le p$ and $\zeta \in r\mathbb{D}_*$, set $V_k(\zeta) := \ker \psi(\zeta)^k$. Since the Grassmannian $\mathcal{G}(n, n_k)$ is compact, we may find a sequence $\zeta_i \to 0$ and vector subspaces $V_k(0) \in \mathcal{G}(n, n_k)$ such that $\lim_{i\to\infty} V_k(\zeta_i) = V_k(0) \subset \ker \psi(0)^k$, $1 \le k \le p$.

Our problem will be solved if we find $\varepsilon > 0$ and holomorphic mappings $v_j \in \mathcal{O}(\varepsilon\mathbb{D}, \mathbb{C}^n)$ such that $\{v_j(\zeta) : 1 \le j \le n_k\}$ is a basis of $V_k(\zeta)$, $\zeta \in \varepsilon\mathbb{D}$, $1 \le k \le p$. (In particular, $\lim_{\zeta\to 0} V_k(\zeta) = V_k(0)$.)

We shall proceed by induction on k, and on j for each fixed k. The value of ε may be reduced at each step, but we keep the same notation.

By convention we will set $n_0 = 0$, and consider $\emptyset$ as a basis of $\{0\}$. Suppose that we have already determined $\{v_j : 1 \le j \le n_k\}$. Choose an $r_{k+1} \times r_{k+1}$ submatrix of ψ^{k+1} whose determinant, denoted δ_{k+1}, is holomorphic and does not vanish on $\varepsilon\mathbb{D}_*$ and eliminate the unknowns corresponding to the columns of this minor; the other unknowns are then expressed in terms of the former with coefficients which are rational in the coefficients of the matrix ψ^{k+1}, so that we obtain meromorphic vector-valued functions u_i on $\varepsilon\mathbb{D}$ so that $\{u_i(\zeta) : 1 \le j \le n_{k+1}\}$ is a basis of $V_{k+1}(\zeta)$ for $\zeta \in \varepsilon\mathbb{D}_*$. Those functions are of the form $u_i := f_i/\delta_{k+1}$, where $f_i \in \mathcal{O}(\varepsilon\mathbb{D}, \mathbb{C}^n)$.

By linear algebra, for each fixed ζ, there exists a set $I(\zeta)$ of $n_{k+1} - n_k$ indices i so that $\{v_j(\zeta), u_i(\zeta) : 1 \le j \le n_k, i \in I(\zeta)\}$ form a basis of $V_{k+1}(\zeta)$.

Using the fact that all determinants that we have to compute to determine the rank of a system of vectors are meromorphic in $\varepsilon\mathbb{D}$, and reducing ε if necessary, we may choose a fixed set I so that $\{v_j(\zeta), u_i(\zeta) : 1 \le j \le n_k, i \in I\}$ is a basis of $V_{k+1}(\zeta)$ for all $\zeta \in \varepsilon\mathbb{D}_*$. Re-index the functions $\{u_i : i \in I\}$ as w_j, $n_k + 1 \le j \le n_{k+1}$. Then $\{v_j(\zeta), w_l(\zeta) : 1 \le j \le n_k < l \le n_{k+1}\}$ is a basis of $V_{k+1}(\zeta)$ for $\zeta \in \varepsilon\mathbb{D}_*$.

Since δ_{k+1} vanishes at most at 0, we can multiply each vector-valued function w_l by ζ^{α_l}, with $\alpha_l \in \mathbb{Z}$ chosen such that $\zeta^{\alpha_l} w_l(\zeta)$ extends to a map $\widetilde{w}_l \in \mathcal{O}(\varepsilon\mathbb{D}, \mathbb{C}^n)$ with $\widetilde{w}_l(0) \neq 0$.

We need to modify the $\widetilde{w}_l$ to ensure that we still have a system of maximal rank at the origin (this is in the spirit of the Gram–Schmidt orthogonalization process). We proceed by induction on $j \ge n_k + 1$. Suppose we have determined v_j as in the statement of the lemma up to some $j_0 \ge n_k$ such that

$$\begin{aligned}&\operatorname{span}\{v_j(\zeta) : 1 \le j \le j_0\}\\&\qquad = \operatorname{span}\{v_j(\zeta), \widetilde{w}_l(\zeta) : 1 \le j \le n_k < l \le j_0\}, \quad \zeta \in \varepsilon\mathbb{D}_*,\\&\operatorname{rank}\{v_j(\zeta) : 1 \le j \le j_0\} = j_0, \quad \zeta \in \varepsilon\mathbb{D}.\end{aligned}$$

Let $W(\zeta) := \operatorname{span}\{v_1(\zeta), \ldots, v_{j_0}(\zeta), \widetilde{w}_{j_0+1}(\zeta)\}$, $\zeta \in \varepsilon\mathbb{D}_*$. Then $\dim W(\zeta) = j_0+1$. Again by using the compactness of the Grassmannian, we may choose a sequence $\zeta_i \to 0$ such that $\lim_{i\to\infty} W(\zeta_i) =: W(0)$ exists. Since all the v_j and $\widetilde{w}_{j_0+1}$ are continuous at 0, we easily deduce that $\operatorname{span}\{v_1(0), \ldots, v_{j_0}(0), \widetilde{w}_{j_0+1}(0)\} \subset W(0)$. Choose a vector w such that $\{v_1(0), \ldots, v_{j_0}(0), w\}$ is a basis of $W(0)$.

By reducing ε if necessary, we may assume that there exists a linear subspace Y such that $W(\zeta)$ and Y form a direct sum for any $\zeta \in \varepsilon\mathbb{D}$. Then define $v_{j_0+1}(\zeta)$ to be the projection of the fixed vector w onto $W(\zeta)$, parallel to Y, for $\zeta \ne 0$; in particular $v_{j_0+1}(0) = w$. Since $v_{j_0+1}(\zeta)$ is obtained by solving a system of linear equations with unique solution, it is meromorphic in a neighborhood of 0, but it is also clearly continuous near 0, hence holomorphic. Since the system $\{v_1(0), \ldots, v_{j_0}(0), v_{j_0+1}(0)\}$ is independent, $\{v_1(\zeta), \ldots, v_{j_0}(\zeta), v_{j_0+1}(\zeta)\}$ is also independent for ζ small enough, and since it is contained in $W(\zeta)$ by construction, it forms a basis of that subspace. Reducing ε yet again if necessary, the induction may proceed.

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Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev 8
1113 Sofia, Bulgaria
E-mail: nik@math.bas.bg

Laboratoire Émile Picard
UMR CNRS 5580
Université Paul Sabatier
118 Route de Narbonne
F-31062 Toulouse Cedex, France
E-mail: pthomas@cict.fr

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