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Bundle functors on all foliated manifold morphisms have locally finite order

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Abstract. We prove that any bundle functor $F:\mathcal{F}ol\to\mathcal{FM}$ on the category $\mathcal{F}ol$ of all foliated manifolds without singularities and all leaf respecting maps is of locally finite order.

Let $\mathcal{M}f_m$ be the category of m-dimensional manifolds and their embeddings and $\mathcal{FM}$ be the category of all fibred manifolds and their fibred maps. In [9], R. Palais and C. Terng showed that any natural bundle in the sense of A. Nijenhuis [8] (bundle functor) $F:\mathcal{M}f_m\to\mathcal{FM}$ has finite order $\operatorname{ord}(F)\le 2^f+1$, where $f=\dim(F_0\mathbb{R}^m)$. (We remark that a bundle functor $F:\mathcal{M}f_m\to\mathcal{FM}$ is of order r if for any $\mathcal{M}f_m$-maps $\varphi,\psi:M\to N$ and any $x\in M$, from $j^r_x\varphi=j^r_x\psi$ it follows that $F\varphi=F\psi$ on the fiber of FM over x.) In [1], D. Epstein and W. Thurston showed that $\operatorname{ord}(F)\le 2f+1$. In [11], A. Zajtz presented the best inequality

$$\operatorname{ord}(F)\le\max\left(\frac{f}{m-1},\frac{f}{m}+1\right)$$

if $m>1$. In [2], I. Kolář, P. Michor and J. Slovák extended the result from [11] to bundle functors $F:\mathcal{FM}_{m,n}\to\mathcal{FM}$, where $\mathcal{FM}_{m,n}$ is the category of fibred manifolds with m-dimensional bases and n-dimensional fibers and their fibred embeddings, and obtained the estimate $\operatorname{ord}(F)\le 2f+1$ for all m,n, and

$$\operatorname{ord}(F)\le\max\left(\frac{f}{m-1},\frac{f}{m}+1,\frac{f}{n-1},\frac{f}{n}+1\right)$$

if $m>1$ and $n>1$, where $f=\dim(F_{(0,0)}(\mathbb{R}^m\times\mathbb{R}^n))$ (the definition of the order of bundle functors on $\mathcal{FM}_{m,n}$ is a direct generalization of the one for bundle functors on $\mathcal{M}f_m$). From [2] it follows that every product preserving bundle functor $F:\mathcal{M}f\to\mathcal{FM}$, where $\mathcal{M}f$ is the category of all

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manifolds and all maps, is of finite order $\mathrm{ord}(F) = \mathrm{ord}(F|\mathcal{M}f_1)$. In [6], the second author presented an example of a vector bundle functor $\mathcal{M}f \to \mathcal{VB}$ of strictly infinite order.

EXAMPLE 1 ([6]). We recall that $T^{(r)}M = (J^r(M,\mathbb{R})_0)^*$ denotes the rth order vector tangent bundle of a manifold M. Let $d_r = \dim(T_0^{(r)}\mathbb{R}^r)$. We set $GM = \bigoplus_{k=1}^{\infty} \bigwedge^{d_k} T^{(k)}M$. Then GM is a finite-dimensional vector bundle for every manifold M because for $k > \dim(M)$ the vector bundle $\bigwedge^{d_k} T^{(k)}M$ is the zero-bundle. Hence the direct sum in the definition of GM is in fact a finite sum. For a mapping $f : M \to N$ the induced mapping $Gf : GM \to GN$ is defined in the natural way from $T^{(k)}f : T^{(k)}M \to T^{(k)}N$. The vector bundle functor G is of strictly infinite order because its restriction to the category $\mathcal{M}f_k$ is of order at least k.

In [7], the second author proved that every bundle functor $F : \mathcal{M}f \to \mathcal{FM}$ has locally finite order in the following sense.

PROPOSITION 1 ([7]). *Let $F : \mathcal{M}f \to \mathcal{FM}$ be a bundle functor. Let $r_m := \mathrm{ord}(F|\mathcal{M}f_m)$. For all maps $f_1, f_2 : M \to N$ and $x \in M$, from $j_x^{r_{\dim(M)+1}} f_1 = j_x^{r_{\dim(M)+1}} f_2$ it follows that $Ff_1 = Ff_2$ on the fiber over x.*

In [2], the above result is extended to bundle functors $F : \mathcal{FM}_m \to \mathcal{FM}$, where $\mathcal{FM}_m$ is the category of fibred manifolds with m-dimensional bases and their fibred maps covering embeddings. Namely, the following proposition is proved.

PROPOSITION 2 ([2]). *Let $F : \mathcal{FM}_m \to \mathcal{FM}$ be a bundle functor. Let $r_n = \mathrm{ord}(F|\mathcal{FM}_{m,n})$. For all $\mathcal{FM}_m$-maps $f_1, f_2 : Y \to Z$ and $x \in Y$, from $j_x^{r_{\dim(Y)-m+1}} f_1 = j_x^{r_{\dim(Y)-m+1}} f_2$ it follows that $Ff_1 = Ff_2$ on the fiber over x.*

From [4] it follows that any product preserving bundle functor $F : \mathcal{FM} \to \mathcal{FM}$ has finite order $\mathrm{ord}(F) = \mathrm{ord}(F|\mathcal{FM}_{1,1})$. In [3], a fiber-product preserving bundle functor $F : \mathcal{FM} \to \mathcal{FM}$ of strictly infinite order is given. From [3] it follows that any fiber-product preserving bundle functor $F : \mathcal{FM} \to \mathcal{FM}$ is of locally finite order in the following sense: for all $\mathcal{FM}$-maps $f_1, f_2 : Y \to Z$ and $x \in Y$ with $Y \in \mathcal{FM}_{m,n}$, from $j_x^{r_m} f_1 = j_x^{r_m} f_2$ it follows that $Ff_1 = Ff_2$ over x, where $r_m = \max(\mathrm{ord}(F|\mathcal{FM}_{m,0}), \mathrm{ord}(F|\mathcal{FM}_{m,1}))$. So, we have the following natural question.

QUESTION 1. Is any bundle functor $F : \mathcal{FM} \to \mathcal{FM}$ of locally finite order?

In this paper we give an affirmative answer to the above question. Since the category $\mathcal{FM}$ has the same skeleton as the category $\mathcal{F}ol$ of all foliated manifolds without singularities and all leaf respecting maps, it is sufficient to study the order of bundle functors on $\mathcal{F}ol$.

We recall (see [2]) that a bundle functor on $\mathcal{F}ol$ is a covariant functor $F : \mathcal{F}ol \to \mathcal{FM}$ satisfying:

(i) (*Base preservation*) $B_{\mathcal{FM}} \circ F = B_{\mathcal{F}ol}$, where $B_{\mathcal{FM}} : \mathcal{FM} \to \mathcal{M}f$ is the base functor and $B_{\mathcal{F}ol} : \mathcal{F}ol \to \mathcal{M}f$ is the functor $(M, \mathcal{F}) \to M$. Hence the induced projections form a natural transformation $\pi : F \to B_{\mathcal{F}ol}$.

(ii) (*Localization*) For every inclusion $i_{(U,\mathcal{F}|U)} : (U, \mathcal{F}|U) \to (M, \mathcal{F})$ of an open subset, $F(U, \mathcal{F}|U)$ is the restriction $\pi^{-1}(U)$ of $\pi : F(M, \mathcal{F}) \to M$ over U and $Fi_{(U,\mathcal{F}|U)}$ is the inclusion $\pi^{-1}(U) \to F(M, \mathcal{F})$.

(iii) (*Regularity*) F transforms smoothly parametrized families of $\mathcal{F}ol$-maps into smoothly parametrized families of fibred maps.

Example 2. A well-known example of a bundle functor $F : \mathcal{F}ol \to \mathcal{FM}$ is the normal bundle functor $N : \mathcal{F}ol \to \mathcal{FM}$ transforming any foliated manifold $(M, \mathcal{F})$ into its normal bundle $N(M, \mathcal{F}) = TM/T\mathcal{F}$ and any $\mathcal{F}ol$-map $f : (M, \mathcal{F}) \to (M_1, \mathcal{F}_1)$ into the quotient map $Nf = [Tf] : N(M, \mathcal{F}) \to N(M_1, \mathcal{F}_1)$. This bundle functor N is product preserving. Another product preserving bundle functor $\mathcal{F}ol \to \mathcal{FM}$ can be found in [10]. (In [5], the second author described all product preserving bundle functors $F : \mathcal{F}ol \to F\mathcal{M}$ in terms of Weil algebra homomorphisms $\mu : A \to B$.)

Example 3. Let $F = T \otimes V : F\mathcal{M} \to \mathcal{VB}$ be the vector bundle functor sending any fibred manifold $p : Y \to M$ into the tensor product $FY = TM \otimes_Y VY$ of the tangent bundle TM with the vertical bundle $VY \to Y$ of $Y \to M$, and any $\mathcal{FM}$-map $f : Y \to Y_1$ covering $\underline{f} : M \to M_1$ into $Ff = T\underline{f} \otimes Vf : FY \to FY_1$. This bundle functor F is fibre product preserving but is not product preserving. Using the standard "gluing" argument one can uniquely extend F to $\widetilde{F} : \mathcal{F}ol \to \mathcal{FM}$. In this way we obtain a vector bundle functor which is not product preserving. (In [3], I. Kolář and the second author described all fibre product preserving vector bundle functors $F : \mathcal{FM} \to \mathcal{VB}$. The functors are of the form $F = G \otimes V : \mathcal{FM} \to \mathcal{VB}$ ($FY = GM \otimes_Y VY$, $Ff = G\underline{f} \otimes Vf$) for some vector bundle functor $G : \mathcal{M}f \to \mathcal{VB}$. Taking G of strictly infinite order (see Example 1), we produce $F : \mathcal{FM} \to \mathcal{FM}$ of strictly infinite order. Then using the standard "gluing" argument we produce $\widetilde{F} : \mathcal{F}ol \to \mathcal{FM}$ of strictly infinite order.)

Example 4. Let S be a manifold. We have a trivial bundle functor $F = \mathrm{id}_{\mathcal{F}ol} \times \mathrm{id}_S : \mathcal{F}ol \to \mathcal{FM}$, $F(M, \mathcal{F}) = M \times S$, $Ff = f \times \mathrm{id}_S$. This F is not a product preserving bundle functor if S is not one point. If S is not a vector bundle, then F is not a vector bundle functor.

We recall that a bundle functor $F : \mathcal{F}ol \to \mathcal{FM}$ is of *locally finite order* if for any m, n there exists a finite number $r_{m,n}$ such that for any foliated $(m+n)$-dimensional manifold M with n-dimensional foliation $\mathcal{F}$ and any $\mathcal{F}ol$-maps $f,g:(M,\mathcal{F})\to(N,\mathcal{F}_1)$ (into an arbitrary foliated manifold $(N,\mathcal{F}_1)$) and any $x\in M$, from $j^{r_{m,n}}_x f=j^{r_{m,n}}_x g$ it follows that $Ff=Fg$ on the fiber of $F(M,\mathcal{F})$ over x.

The purpose of the present note is to prove the following theorem which gives an affirmative answer to Question 1.

THEOREM 1. *Any bundle functor $F:\mathcal{F}ol\to\mathcal{FM}$ has locally finite order in the following sense: Let m,n be positive integers, $(M,\mathcal{F})$ be an $(m+n)$-dimensional foliated manifold M with n-dimensional foliation $\mathcal{F}$, and $x\in M$ be a point. Then for all $\mathcal{F}ol$-maps $f_1,f_2:(M,\mathcal{F})\to(M_1,\mathcal{F}_1)$, from $j^{r(m,n)}_x f_1=j^{r(m,n)}_x f_2$ it follows that $Ff_1=Ff_2$ on the fibre over x, where $r(m,n)=\max(\mathrm{ord}(F|\mathcal{FM}_{m+1,n}),\mathrm{ord}(F|\mathcal{FM}_{m,n+1}))$.*

Proof. Let $f_1,f_2:(M,\mathcal{F})\to(M_1,\mathcal{F}_1)$ be $\mathcal{F}ol$-maps such that $j^{r(m,n)}_x f_1=j^{r(m,n)}_x f_2$ for some $x\in M$. We show that $Ff_1=Ff_2$ over x.

(I) First we assume that $p\ge m$ and $q\ge n$. Because of the regularity of F we can assume that d_xf_1 is of rank $m+n$. Then by the rank theorem we can assume $(M,\mathcal{F})=(\mathbb{R}^m\times\mathbb{R}^n,\{\{a\}\times\mathbb{R}^n\}_{a\in\mathbb{R}^m})$, $x=(0,0)$, $(M_1,\mathcal{F}_1)=(\mathbb{R}^p\times\mathbb{R}^q,\{\{c\}\times\mathbb{R}^q\}_{c\in\mathbb{R}^p})$, $f_1(0,0)=f_2(0,0)=(0,0)$ and

$$f_1(x,y)=((x,0),(y,0))$$

for any $(x,y)\in\mathbb{R}^m\times\mathbb{R}^n$. Let $f_i(x,y)=(\varphi_i(x),\psi_i(x,y))$ for any $(x,y)\in\mathbb{R}^m\times\mathbb{R}^n$, $i=1,2$. Define $\mathcal{F}ol$-maps $\Phi_i:\mathbb{R}^m\times\mathbb{R}^n\to\mathbb{R}^m\times\mathbb{R}^q$ by

$$\Phi_i(x,y)=(x,\psi_i(x,y)),\qquad (x,y)\in\mathbb{R}^m\times\mathbb{R}^n,$$

and $\Psi_i:\mathbb{R}^m\times\mathbb{R}^q\to\mathbb{R}^p\times\mathbb{R}^q$,

$$\Psi_i(x,z)=(\varphi_i(x),z),\qquad (x,z)\in\mathbb{R}^m\times\mathbb{R}^q.$$

Then $f_i=\Psi_i\circ\Phi_i$, $i=1,2$.

Define a bundle functor $G:\mathcal{FM}_m\to\mathcal{FM}$ by $G=F|\mathcal{FM}_m$. Of course, the Φ_i are $\mathcal{FM}_m$-maps and $j^{r(m,n)}_{(0,0)}\Phi_1=j^{r(m,n)}_{(0,0)}\Phi_2$. Then by Proposition 2 we have $G\Phi_1=G\Phi_2$ over $(0,0)\in\mathbb{R}^m\times\mathbb{R}^n$. So $F\Phi_1=F\Phi_2$ on the fibre $F_{(0,0)}$ of $F(\mathbb{R}^m\times\mathbb{R}^n,\{\{a\}\times\mathbb{R}^n\}_{a\in\mathbb{R}^m})$ over $(0,0)\in\mathbb{R}^m\times\mathbb{R}^n$.

Hence it remains to show that $F\Psi_1=F\Psi_2$ on $F\Phi_1(F_{(0,0)})$.

We define $\mathcal{F}ol$-maps $\widetilde{\Psi}_i:=\varphi_i\times\mathrm{id}_{\mathbb{R}^n}:\mathbb{R}^m\times\mathbb{R}^n\to\mathbb{R}^p\times\mathbb{R}^n$ and $I_s:\mathbb{R}^s\times\mathbb{R}^n\to\mathbb{R}^s\times\mathbb{R}^q$, $I_s(w,y)=(w,(y,0))$. Then $\Psi_i\circ I_m=I_p\circ\widetilde{\Psi}_i$ and $I_m=\Phi_1$. Clearly, FI_s is an embedding because I_s is (see [2]). Then it suffices to show that $F\widetilde{\Psi}_1=F\widetilde{\Psi}_2$ over $(0,0)\in\mathbb{R}^m\times\mathbb{R}^n$.

Define a bundle functor $H:\mathcal{M}f\to\mathcal{FM}$ by $HM=F(M\times\mathbb{R}^n,\{\{a\}\times\mathbb{R}^n\}_{a\in M})$, $H\varphi=F(\varphi\times\mathrm{id}_{\mathbb{R}^n})$. Clearly, $j^{r(m,n)}_0\varphi_1=j^{r(m,n)}_0\varphi_2$. Then by Proposition 1, $H\varphi_1=H\varphi_2$ over $0\in\mathbb{R}^m$. Therefore $F\widetilde{\Psi}_1=F\widetilde{\Psi}_2$ over $(0,0)\in\mathbb{R}^m\times\mathbb{R}^n$, as well, which implies $Ff_1=Ff_2$ over $x\in M$ under the assumption $p\ge m$ and $q\ge n$.

(II) Now let p and q be arbitrary. We may assume $(M,\mathcal{F}) = (\mathbb{R}^m \times \mathbb{R}^n, \{\{a\}\times\mathbb{R}^n\}_{a\in\mathbb{R}^m})$, $x = (0,0)$, $(M_1,\mathcal{F}_1) = (\mathbb{R}^p\times\mathbb{R}^q, \{\{c\}\times\mathbb{R}^q\}_{c\in\mathbb{R}^p})$, $f_1(0,0) = f_2(0,0) = (0,0)$. Let $\widetilde{p} \geq \max(m,p)$ and $\widetilde{q} \geq \max(n,q)$. Let $J : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^{\widetilde{p}} \times \mathbb{R}^{\widetilde{q}}$ be the $\mathcal{F}ol$-embedding given by $J(u,w) = ((u,0),(w,0))$. Then $j^{r(m,n)}_{(0,0)}(J\circ f_1) = j^{r(m,n)}_{(0,0)}(J\circ f_2)$. Hence, by (I) for $J\circ f_i$ instead of f_i and $(\widetilde{p},\widetilde{q})$ instead of (p,q), we have $F(J\circ f_1) = F(J\circ f_2)$ over $(0,0)\in\mathbb{R}^m\times\mathbb{R}^n$. But FJ is an embedding because J is. Then $Ff_1 = Ff_2$ over x as well. ∎

From Theorem 1 we immediately obtain the following corollary.

Corollary 1. *Any bundle functor $F : \mathcal{FM} \to \mathcal{FM}$ has locally finite order in the following sense: Let m,n be positive integers. Let $Y \to M$ be an $\mathcal{FM}_{m,n}$-object and $x \in Y$ be a point. Then for all $\mathcal{FM}$-morphisms $f_1, f_2 : Y \to Y_1$, from $j^{r(m,n)}_x f_1 = j^{r(m,n)}_x f_2$ it follows that $Ff_1 = Ff_2$ over x, where $r(m,n)$ is defined as in Theorem* 1.

Example 5. Let $G : \mathcal{M}f \to \mathcal{VB}$ be the vector bundle functor of strictly infinite order as in Example 1. We define a bundle functor $F = G : \mathcal{F}ol \to \mathcal{VB}$, $F(M,\mathcal{F}) = GM$, $Ff = Gf$. This bundle functor is of strictly infinite order. It is of locally finite order, but in this case we cannot replace $r(m,n)$ in Theorem 1 by an $r(m)$ depending only on m (in contrast to Example 3).

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