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Positive periodic solutions of functional differential equations with infinite delay

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Abstract. The author applies a generalized Leggett–Williams fixed point theorem to the study of the nonlinear functional differential equation

$$x'(t) = -a(t, x(t))x(t) + f(t, x_t).$$

Sufficient conditions are established for the existence of multiple positive periodic solutions.

1. Introduction. In this paper, we are concerned with the functional differential equation

$$x'(t) = -a(t, x(t))x(t) + f(t, x_t), \tag{1.1}$$

where

- $a \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ with $a(t + \omega, x) \equiv a(t, x)$;
- $\forall t \in \mathbb{R}$, $x_t = x_t(\theta) = x(t + \theta)$, $-\infty < \theta \le 0$; we assume $x_t \in C$, where $C = C((-\infty, 0], \mathbb{R})$ is a Banach space with the norm $\|\varphi\|_C = \max_{\theta \in (-\infty,0]} |\varphi(\theta)|$;
- $f \in C(\mathbb{R} \times C^+, \mathbb{R})$ with $f(t + \omega, \varphi) \equiv f(t, \varphi)$, $\varphi \in C^+$, where $C^+ = \{\varphi \in C : \varphi(\theta) \ge 0, \theta \in (-\infty, 0]\}$;
- $\omega > 0$.

We make the following assumptions:

(H_1) there exist ω-periodic functions $a_1, a_2 \in C(\mathbb{R}, \mathbb{R})$ satisfying

$$a_1(t) \le a(t, x) \le a_2(t), \qquad \int_0^\omega a_1(t)\, dt > 0;$$

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(H$_2$) $f(t,\varphi)$ maps bounded sets into bounded sets and is a nonnegative continuous function defined on $\mathbb{R}\times C^+$.

Our purpose is to prove that (1.1) has multiple positive periodic solutions by using a generalized Leggett–Williams fixed point theorem. For the biological senses of (1.1), we refer to [2, 3, 6].

From (1.1) we obtain

$$\Big[x(t)\exp\Big(\int_0^t a(\tau,x(\tau))\,d\tau\Big)\Big]' = \exp\Big(\int_0^t a(\tau,x(\tau))\,d\tau\Big) f(t,x_t). \tag{1.2}$$

After integration from t to $t+\omega$, we obtain

$$x(t) = \int_t^{t+\omega} G(t,s) f(s,x_s)\,ds, \tag{1.3}$$

where

$$G(t,s) = \frac{\exp(\int_t^s a(\tau,x(\tau))\,d\tau)}{\exp(\int_0^\omega a(\tau,x(\tau))\,d\tau)-1}. \tag{1.4}$$

Let

$$M_1 = \inf_{0\le t\le s\le\omega} \exp\Big(\int_t^s a_1(\tau)\,d\tau\Big),\qquad M_2 = \sup_{0\le t\le s\le\omega} \exp\Big(\int_t^s a_2(\tau)\,d\tau\Big),$$

$$k_1 = \exp\Big(\int_0^\omega a_1(\tau)\,d\tau\Big),\qquad k_2 = \exp\Big(\int_0^\omega a_2(\tau)\,d\tau\Big),\qquad \delta = \frac{M_1(k_1-1)}{M_2(k_2-1)};$$

we know that $0<\delta<1$. Furthermore, we have from (H$_1$) and (1.4)

$$\frac{M_1}{k_2-1} \le G(t,s) \le \frac{M_2}{k_1-1}.$$

Now, let X be the set of all real ω-periodic continuous functions, endowed with the usual linear structure as well as the norm $\|x\| = \sup_{t\in[0,\omega]}|x(t)|$. It is a Banach space with a cone

$$P = \Big\{x\in X : x(t)\ge\delta\|x\|,\ x(t)\exp\Big(\int_0^t a_1(\tau)\,d\tau\Big) \text{ is nondecreasing on } [0,\omega]\Big\}.$$

Furthermore, for all $x\in X$, we have

$$\|x\| = \|x_t\|_C \quad \text{for each } t\in[0,\omega],$$

and for all $x\in P$, we deduce that $x(t)\exp(\int_0^t a_2(\tau)\,d\tau)$ is nondecreasing on $[0,\omega]$.

Define $T: P\to X$ as

$$Tx(t) = \int_t^{t+\omega} G(t,s) f(s,x_s)\,ds. \tag{1.5}$$

Similar to the proofs of Lemmas 2.2 and 2.4 in [4], one can easily show

LEMMA 1.1. *$T : P \to P$ is well defined and completely continuous.*

One can easily see that x is a positive solution of (1.1) if and only if x is a fixed point of T on P.

For convenience, we present a generalized Leggett–Williams fixed point theorem due to Avery and Henderson [1]. Let

$$P(\delta, e) = \{x \in P : \delta(x) < e\},$$
$$\partial P(\delta, e) = \{x \in P : \delta(x) = e\},$$
$$\overline{P(\delta, e)} = \{x \in P : \delta(x) \le e\}.$$

LEMMA 1.2. *Let X be a real Banach space, P a cone of X, γ and α two nonnegative increasing continuous function on P, and θ a nonnegative continuous function on P with $\theta(0) = 0$ such that there are positive numbers c and M satisfying*

$$\gamma(x) \le \theta(x) \le \alpha(x), \quad \|x\| \le M\gamma(x) \quad \text{for } x \in \overline{P(\gamma, c)}.$$

Moreover, assume that $T : \overline{P(\gamma, c)} \to P$ is completely continuous and there are positive numbers $0 < a < b < c$ such that

$$\theta(\lambda x) \le \lambda\theta(x) \quad \text{for all } \lambda \in [0, 1] \text{ and } x \in \partial P(\theta, b),$$

and

(i) $\gamma(Tx) > c$ *for* $x \in \partial P(\gamma, c)$;
(ii) $\theta(Tx) < b$ *for* $x \in \partial P(\theta, b)$;
(iii) $\alpha(Tx) > a$ *and* $P(\alpha, a) \ne \emptyset$ *for* $x \in \partial P(\alpha, a)$.

Then T has at least two fixed points $x_1, x_2 \in \overline{P(\gamma, c)}$ satisfying

$$a < \alpha(x_1), \quad \theta(x_1) < b, \quad b < \theta(x_2), \quad \gamma(x_2) < c.$$

The following lemma is similar to Lemma 1.2.

LEMMA 1.3. *The conclusion of Lemma* 1.2 *still holds if we replace* (i)–(iii) *there by*

(i) $\gamma(Tx) < c$ *for* $x \in \partial P(\gamma, c)$;
(ii) $\theta(Tx) > b$ *for* $x \in \partial P(\theta, b)$;
(iii) $\alpha(Tx) < a$ *and* $P(\alpha, a) \ne \emptyset$ *for* $x \in \partial P(\alpha, a)$.

LEMMA 1.4 ([5]). *Let P be a cone of a real Banach space X, Ω a bounded open subset of X, and $0 \in \Omega$. Moreover, assume that $T : \overline{P \cap \Omega} \to P$ is completely continuous and satisfies*

$$Tx = \lambda x \text{ for some } x \in P \cap \partial\Omega \ \Rightarrow\ \lambda < 1.$$

Then

$$i(T, P \cap \Omega, P) = 1.$$

REMARK 1.1. We know that $\Omega = P(\alpha, a)$ in Lemma 1.3 is a bounded open subset of X, and by (iii) of Lemma 1.3, we have

$$Tx < x \quad \text{for all } x \in P \cap \partial\Omega = \partial P(\alpha, a).$$

From Lemma 1.3, Lemma 1.4 and Remark 1.1, we have the following result.

LEMMA 1.5. *Let the conditions of Lemma* 1.3 *hold. Furthermore, assume* $\theta \in P(\alpha, a)$. *Then* T *has at least three fixed points* $x_1, x_2, x_3 \in \overline{P(\gamma, c)}$ *satisfying*

$$\alpha(x_1) < a, \quad a < \alpha(x_2), \quad \omega(x_2) < b, \quad b < \omega(x_3), \quad \gamma(x_3) < c.$$

2. Main results. Fix $0 \le \eta < l \le \omega$ and define nonnegative, increasing, continuous functions γ, θ, and α on P by

$$\gamma(x) = \min_{\eta \le t \le l} e^{\int_0^t a_1(\tau)\,d\tau} x(t) = x(\eta) e^{\int_0^\eta a_1(\tau)\,d\tau},$$

$$\theta(x) = \max_{0 \le t \le \eta} e^{\int_0^t a_1(\tau)\,d\tau} x(t) = x(\eta) e^{\int_0^\eta a_1(\tau)\,d\tau},$$

$$\alpha(x) = \min_{l \le t \le \omega} e^{\int_0^t a_2(\tau)\,d\tau} x(t) = x(l) e^{\int_0^l a_2(\tau)\,d\tau}.$$

We have

$$\gamma(x) = \theta(x) \le \alpha(x), \quad x \in P,$$

and

$$\gamma(x) = x(\eta) e^{\int_0^\eta a_1(\tau)\,d\tau} \ge e^{\int_0^\eta a_1(\tau)\,d\tau} \delta \|x\| \quad \text{for each } x \in P, \tag{1.6}$$

$$\alpha(x) = x(l) e^{\int_0^l a_2(\tau)\,d\tau} \ge e^{\int_0^l a_2(\tau)\,d\tau} \delta \|x\| \quad \text{for each } x \in P. \tag{1.7}$$

Then

$$\|x\| \le e^{-\int_0^\eta a_1(\tau)\,d\tau} \frac{1}{\delta} \gamma(x) = e^{-\int_0^\eta a_1(\tau)\,d\tau} \frac{1}{\delta} \theta(x) \quad \text{for each } x \in P, \tag{1.8}$$

$$\|x\| \le e^{-\int_0^l a_2(\tau)\,d\tau} \frac{1}{\delta} \alpha(x) \quad \text{for each } x \in P, \tag{1.9}$$

$$\theta(\lambda x) = \lambda \theta(x) \quad \text{for all } \lambda \in [0,1] \text{ and } x \in P.$$

For the notational convenience, we set

$$\sigma_1 = \frac{M_1(\omega - \eta)}{k_2 - 1} e^{\int_0^\eta a_1(\tau)\,d\tau}, \quad \varrho_1 = \frac{M_2 \omega}{k_1 - 1} e^{\int_0^\eta a_1(\tau)\,d\tau},$$

$$\sigma_2 = \frac{M_1(\omega - l)}{k_2 - 1} e^{\int_0^l a_2(\tau)\,d\tau}, \quad \varrho_2 = \frac{M_2 \omega}{k_1 - 1} e^{\int_0^l a_2(\tau)\,d\tau}.$$

THEOREM 2.1. *Suppose that there are positive numbers* $a < b < c$ *such that*

$$0 < a < \frac{\sigma_2}{\varrho_1} b < \frac{\sigma_2 \delta}{\varrho_1} c.$$

Assume $f(t,\varphi)$ *satisfies the following conditions*:

(A) $f(t,\varphi) > c/\sigma_1$ *for* $(t,\varphi) \in [\eta,\omega] \times K_1$,
(B) $f(t,\varphi) < b/\varrho_1$ *for* $(t,\varphi) \in [0,\omega] \times K_2$,
(C) $f(t,\varphi) > a/\sigma_2$ *for* $(t,\varphi) \in [l,\omega] \times K_3$,

where

$$K_1 = \{\varphi \in C^+ : ce^{-\int_0^\eta a_1(\tau)\,d\tau} \le \|\varphi\|_C \le (c/\delta)e^{-\int_0^\eta a_1(\tau)\,d\tau}\},$$
$$K_2 = \{\varphi \in C^+ : 0 \le \|\varphi\|_C \le (b/\delta)e^{-\int_0^\eta a_1(\tau)\,d\tau}\},$$
$$K_3 = \{\varphi \in C^+ : ae^{-\int_0^l a_2(\tau)\,d\tau} \le \|\varphi\|_C \le (a/\delta)e^{-\int_0^l a_2(\tau)\,d\tau}\}.$$

Then (1.1) *has at least two positive periodic solutions* x_1 *and* x_2 *satisfying*

$$a < \alpha(x_1), \quad \theta(x_1) < b, \quad b < \theta(x_2), \quad \gamma(x_2) < c.$$

Proof. By Lemma 1.1, $T : \overline{P(\gamma,c)} \to P$ and T is completely continuous. Now, we show that (i)–(iii) of Lemma 1.2 are satisfied.

First, we verify that $x \in \partial P(\gamma,c)$ implies $\gamma(Tx) > c$. Since $\gamma(x) = x(\eta)e^{\int_0^\eta a_1(\tau)\,d\tau} = c$, one gets

$$x(t)e^{\int_0^t a_1(\tau)\,d\tau} \ge c \quad \text{for } t \in [\eta,\omega].$$

From (1.8), we have

$$ce^{-\int_0^\eta a_1(\tau)\,d\tau} \le \|x_t\|_C \le e^{-\int_0^\eta a_1(\tau)\,d\tau}\,\frac{c}{\delta} \quad \text{for } t \in [\eta,\omega].$$

Then we get

$$\begin{aligned}\gamma(Tx) &= (Tx)(\eta)e^{\int_0^\eta a_1(\tau)\,d\tau} = e^{\int_0^\eta a_1(\tau)\,d\tau}\int_\eta^{\eta+\omega} G(\eta,s)f(s,x_s)\,ds\\ &\ge e^{\int_0^\eta a_1(\tau)\,d\tau}\int_\eta^{\omega} G(\eta,s)f(s,x_s)\,ds\\ &> e^{\int_0^\eta a_1(\tau)\,d\tau}\int_\eta^{\omega} \frac{M_1}{k_2-1}\frac{c}{\sigma_1}\,ds = c.\end{aligned}$$

Secondly, we prove that $x \in \partial P(\theta,b)$ implies $\theta(Tx) < b$. Since $\theta(x) = b$ implies $x(\eta)e^{\int_0^\eta a_1(\tau)\,d\tau} = b$, we have

$$0 \le x(t)e^{\int_0^t a_1(\tau)\,d\tau} \le b \quad \text{for } t \in [0,\eta].$$

From (1.8), we have

$$0 \le \|x_t\|_C \le \frac{b}{\delta}\,e^{-\int_0^\eta a_1(\tau)\,d\tau} \quad \text{for } t \in [0,\omega] \text{ or } t \in [\eta,\eta+\omega].$$

Then

$$\theta(Tx) = (Tx)(\eta)e^{\int_0^\eta a_1(\tau)\,d\tau} = e^{\int_0^\eta a_1(\tau)\,d\tau} \int_\eta^{\eta+\omega} G(\eta,s)f(s,x_s)\,ds$$
$$\le e^{\int_0^\eta a_1(\tau)\,d\tau} \int_\eta^{\eta+\omega} \frac{M_2}{k_1-1}\frac{b}{\varrho_1}\,ds = b.$$

Finally, we show that

$$P(\alpha,a) \ne \emptyset, \quad \alpha(Tx) > a \quad \text{for all } x \in \partial P(\alpha,a).$$

The nonemptiness is obvious. On the other hand, $\alpha(x) = x(l)e^{\int_0^l a_2(\tau)\,d\tau} = a$ implies $a \le x(t)e^{\int_0^t a_2(\tau)\,d\tau}$ for $t \in [l,\omega]$. Recalling (1.9), we know

$$ae^{-\int_0^l a_2(\tau)\,d\tau} \le \|x_t\|_C \le e^{-\int_0^l a_2(\tau)\,d\tau}\frac{a}{\delta} \quad \text{for } t \in [l,\omega].$$

Thus

$$\alpha(Tx) = (Tx)(l)e^{\int_0^l a_2(\tau)\,d\tau} = e^{\int_0^l a_2(\tau)\,d\tau} \int_l^{l+\omega} G(l,s)f(s,x_s)\,ds$$
$$\ge e^{\int_0^l a_2(\tau)\,d\tau} \int_l^{\omega} G(l,s)f(s,x_s)\,ds$$
$$> (\omega - l)e^{\int_0^l a_2(\tau)\,d\tau}\frac{M_1}{k_2-1}\frac{a}{\sigma_2} = a.$$

Thus by Lemma 1.2, T has at least two different fixed points x_1 and x_2, which are positive periodic solutions of (1.1). The proof is complete.

Similarly, by Lemma 1.5, we have the following result.

THEOREM 2.2. *Suppose that there are positive numbers* $0 < a < b < c$ *such that*

$$0 < a < \delta b < \sigma_1 \delta c/\varrho_1.$$

Assume $f(t,\varphi)$ *satisfies the following conditions*:

(A′) $f(t,\varphi) < c/\varrho_1$ *for* $(t,\varphi) \in [0,\omega] \times K_1'$,
(B′) $f(t,\varphi) > b/\sigma_1$ *for* $(t,\varphi) \in [\eta,\omega] \times K_2'$,
(C′) $f(t,\varphi) < a/\varrho_2$ *for* $(t,\varphi) \in [0,\omega] \times K_3'$,

where

$$K_1' = \{\varphi \in C^+ : 0 \le \|\varphi\|_C \le (c/\delta)e^{-\int_0^\eta a_1(\tau)\,d\tau}\},$$
$$K_2' = \{\varphi \in C^+ : be^{-\int_0^\eta a_1(\tau)\,d\tau} \le \|\varphi\|_C \le (b/\delta)e^{-\int_0^\eta a_1(\tau)\,d\tau}\},$$
$$K_3' = \{\varphi \in C^+ : 0 \le \|\varphi\|_C \le (a/\delta)e^{-\int_0^l a_2(\tau)\,d\tau}\}.$$

Then (1.1) *has at least three positive solutions* x_1, x_2 *and* x_3 *satisfying*

$$a < \alpha(x_1), \quad \theta(x_1) < b, \quad b < \theta(x_2), \quad \gamma(x_2) < c.$$

The proof is omitted since it is similar to that of Theorem 2.1.

Now, we give theorems which may be considered as corollaries of Theorems 2.1 and 2.2.

Choose ε_1, ε_2, ε_3 such that

$$\varepsilon_1\sigma_2 e^{-\int_0^l a_2(\tau)\,d\tau} > 1, \qquad \varepsilon_2\sigma_1 e^{-\int_0^\eta a_1(\tau)\,d\tau} > 1,$$
$$0 < \varepsilon_3 \frac{\max\{\varrho_1, \varrho_2\}}{\delta} e^{-\int_0^\eta a_1(\tau)\,d\tau} < 1.$$

THEOREM 2.3. *Let the following conditions be satisfied*:

(D) $\displaystyle\lim_{\|\varphi\|_C\to 0}\min_{t\in[l,\omega]}\frac{f(t,\varphi)}{\|\varphi\|_C} > \varepsilon_1;\ \lim_{\|\varphi\|_C\to \infty}\min_{t\in[\eta,\omega]}\frac{f(t,\varphi)}{\|\varphi\|_C} > \varepsilon_2;$

(E) *there exists a* $p_1 > 0$ *such that for each* φ *with* $0 \le \|\varphi\|_C \le (p_1/\delta)e^{-\int_0^\eta a_1(\tau)\,d\tau}$,

$$f(t,\varphi) < p_1/\varrho_1 \quad \textit{for all } t \in [0,\omega].$$

Then (1.1) *has at least two positive periodic solutions.*

Proof. First, choose $b = p_1$; one gets

$$f(t,\varphi) < \frac{p_1}{\varrho_1} = \frac{b}{\varrho_1} \quad \text{for } t\in[0,\omega],\ 0 \le \|\varphi\|_C \le \frac{b}{\delta} e^{-\int_0^\eta a_1(\tau)\,d\tau}.$$

Secondly, since

$$\lim_{\|\varphi\|_C\to 0}\min_{t\in[l,\omega]}\frac{f(t,\varphi)}{\|\varphi\|_C} > \varepsilon_1,$$

there is $R_1 > 0$ sufficiently small such that

$$f(t,\varphi) > \varepsilon_1\|\varphi\|_C \quad \text{for } t\in[l,\omega],\ 0 \le \|\varphi\|_C \le R_1.$$

Without loss of generality, suppose

$$R_1 \le \frac{\sigma_2}{\varrho_1\delta} b e^{-\int_0^l a_2(\tau)\,d\tau}.$$

Choose $a > 0$ so that $a < \delta R_1 e^{\int_0^l a_2(\tau)\,d\tau}$. For

$$a e^{-\int_0^l a_2(\tau)\,d\tau} \le \|\varphi\|_C \le e^{-\int_0^l a_2(\tau)\,d\tau}\frac{a}{\delta},$$

we have $\|\varphi\|_C \le R_1$ and $a < (\sigma_2/\varrho_1)b$. Thus

$$f(t,\varphi) > \varepsilon_1\|\varphi\|_C \ge \varepsilon_1 a e^{-\int_0^l a_2(\tau)\,d\tau} > a/\sigma_2$$
$$\text{for } t\in[l,\omega], a e^{-\int_0^l a_2(\tau)\,d\tau} \le \|\varphi\|_C \le e^{-\int_0^l a_2(\tau)\,d\tau}\frac{a}{\delta}.$$

Thirdly, since

$$\lim_{\|\varphi\|_C\to \infty}\min_{t\in[\eta,\omega]}\frac{f(t,\varphi)}{\|\varphi\|_C} > \varepsilon_2,$$

there is $R_2 > 0$ sufficiently large such that

$$f(t,\varphi) > \varepsilon_2\|\varphi\|_C \quad \text{for } t\in[\eta,\omega],\ \|\varphi\|_C \ge R_2.$$

Without loss of generality, suppose $R_2 > (b/\delta)e^{-\int_0^\eta a_1(\tau)\,d\tau}$. Choose $c \geq R_2 e^{\int_0^\eta a_1(\tau)\,d\tau}$. Then

$$f(t,\varphi) > \varepsilon_2\|\varphi\|_C \geq \varepsilon_2 c e^{-\int_0^\eta a_1(\tau)\,d\tau} > c/\sigma_1$$

$$\text{for } t \in [\eta,\omega],\ c e^{-\int_0^\eta a_1(\tau)\,d\tau} \leq \|\varphi\|_C \leq e^{-\int_0^\eta a_1(\tau)\,d\tau}\,\frac{c}{\delta}.$$

We now get $0 < a < \sigma_2 b/\varrho_1 < \sigma_2\delta c/\varrho_1$, and then the conditions in Theorem 2.1 are all satisfied. By Theorem 2.1, (1.1) has at least two positive periodic solutions. The proof is complete.

THEOREM 2.4. *Let the following conditions be satisfied*:

(F) $\displaystyle\lim_{\|\varphi\|_C\to 0}\max_{t\in[0,\omega]}\frac{f(t,\varphi)}{\|\varphi\|_C} < \varepsilon_3$;

(G) *there exists a* $p_2 > 0$ *such that for each* φ *with* $p_2 e^{-\int_0^\eta a_1(\tau)\,d\tau} \leq \|\varphi\|_C \leq (p_2/\delta)e^{-\int_0^\eta a_1(\tau)\,d\tau}$,

$$f(t,\varphi) > p_2/\sigma_1 \quad \textit{for all } t \in [\eta,\omega].$$

Then (1.1) *has at least three positive periodic solutions.*

The following corollaries are obvious.

COROLLARY 2.1. *Let the following conditions be satisfied*:

(D′) $\displaystyle\lim_{\|\varphi\|_C\to 0}\min_{t\in[l,\omega]}\frac{f(t,\varphi)}{\|\varphi\|_C} = \infty;\ \lim_{\|\varphi\|_C\to \infty}\min_{t\in[\eta,\omega]}\frac{f(t,\varphi)}{\|\varphi\|_C} = \infty$;

(E) *there exists a* $p_1 > 0$ *such that for each* φ *with* $0 \leq \|\varphi\|_C \leq (p_1/\delta)e^{-\int_0^\eta a_1(\tau)\,d\tau}$,

$$f(t,\varphi) < p_1/\varrho_1 \quad \textit{for all } t \in [0,\omega].$$

Then (1.1) *has at least two positive periodic solutions.*

COROLLARY 2.2. *Let the following conditions be satisfied*:

(F)′ $\displaystyle\lim_{\|\varphi\|_C\to 0}\max_{t\in[0,\omega]}\frac{f(t,\varphi)}{\|\varphi\|_C} = 0$;

(G) *there exists a* $p_2 > 0$ *such that for each* φ *with* $p_2 e^{-\int_0^\eta a_1(\tau)\,d\tau} \leq \|\varphi\|_C \leq (p_2/\delta)e^{-\int_0^\eta a_1(\tau)\,d\tau}$,

$$f(t,\varphi) > p_2/\sigma_1 \quad \textit{for all } t \in [\eta,\omega].$$

Then (1.1) *has at least three positive periodic solutions.*

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