On the mean-value property of superharmonic functions

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Abstract. We complement a previous result concerning a converse of the mean-value property for smooth superharmonic functions. The case of harmonic functions was treated by Kuran and an improvement was given by Armitage and Goldstein.

Recall that a function $u$ is harmonic (resp. superharmonic) on an open set $U \subset \mathbb{R}^n$ ($n \geq 1$) if $u \in C^2(U)$ and $\Delta u = 0$ (resp. $\Delta u \leq 0$) on $U$. Denote by $H(U)$ the space of harmonic functions on $U$ and by $SH(U)$ the subset of $C^2(U)$ consisting of superharmonic functions on $U$. Notice that superharmonic functions are usually defined in a more general sense (see [5] and Remark 3).

If $A \subset \mathbb{R}^n$ is Lebesgue measurable, $L^1(A)$ denotes the space of Lebesgue integrable functions on $A$. If $A$ has finite measure we denote by $|A|$ the Lebesgue measure of $A$.

In [3] we proved the following theorem.

Theorem 1. Let $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) be a bounded open set. Suppose that there exists $x_0 \in \Omega$ such that

$$u(x_0) \geq \frac{1}{|\Omega|} \int_{\Omega} u(x) \, dx$$

for every $u \in SH(\Omega) \cap L^1(\Omega) \setminus H(\Omega)$. Then $\Omega$ is a ball with center $x_0$.

Theorem 1 extends a result obtained by Epstein and Schiffer [4] for harmonic functions. The final step concerning harmonic functions was achieved by Kuran [6] who proved the following theorem.

Theorem 2. Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a connected open set of finite measure. Suppose that there exists $x_0 \in \Omega$ such that

$$u(x_0) = \frac{1}{|\Omega|} \int_{\Omega} u(x) \, dx$$

(1)

for every $u \in H(\Omega) \cap L^1(\Omega)$. Then $\Omega$ is a ball with center $x_0$.

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Remark 1. The proof of Theorem 2 given in [2] for \( n = 2 \), which extends to \( n \geq 2 \), shows that the connectedness assumption is superfluous.

Remark 2. The hypothesis of Theorem 2 can be weakened to require only that (1) holds for all positive harmonic functions that are integrable over \( \Omega \): see [1].

We first give a proof of the following result.

**Theorem 3.** Let \( \Omega \subset \mathbb{R}^n (n \geq 2) \) be an open set of finite measure. Suppose that there exists \( x_0 \in \Omega \) such that

\[
 u(x_0) \geq \frac{1}{|\Omega|} \int_{\Omega} u(x) \, dx
\]

for every \( u \in SH(\Omega) \cap L^1(\Omega) \setminus H(\Omega) \). Then \( \Omega \) is a ball with center \( x_0 \).

**Proof.** We shall show that \( \Omega \) satisfies the assumptions of Theorem 2. Let \( h \in H(\Omega) \cap L^1(\Omega) \). Let \( y \in \mathbb{R}^n \setminus \Omega \) and \( a_n \in (-n,0) \). We define

\[
 v(x) = -\|x - y\|^{a_n}, \quad x \in \mathbb{R}^n \setminus \{y\}.
\]

We have \( \Delta v < 0 \) in \( \Omega \). Moreover, \( v \in L^1(\Omega) \). Indeed, let \( B(y,r) \) denote the open ball of fixed radius \( r > 0 \). Clearly \( v \in L^1(B(y, r)) \). Since \( \Omega \) has finite measure and \( v \) is bounded on \( \Omega \setminus B(y, r) \), \( v \in L^1(\Omega \setminus B(y, r)) \) and the result follows. Now for \( m \in \mathbb{N}^* = \{1, 2, 3, \ldots\} \) we set

\[
 u_m = h + \frac{1}{m} v.
\]

Then \( u_m \in SH(\Omega) \cap L^1(\Omega) \setminus H(\Omega) \) and we have

\[
 u_m(x_0) \geq \frac{1}{|\Omega|} \int_{\Omega} u_m(x) \, dx,
\]

that is,

\[
 h(x_0) + \frac{1}{m} v(x_0) \geq \frac{1}{|\Omega|} \int_{\Omega} h(x) \, dx + \frac{1}{m |\Omega|} \int_{\Omega} v(x) \, dx.
\]

Letting \( m \to +\infty \) we obtain

\[
 h(x_0) \geq \frac{1}{|\Omega|} \int_{\Omega} h(x) \, dx.
\]

As (3) holds for every \( h \in H(\Omega) \cap L^1(\Omega) \), replacing \( h \) by \(-h\) in (3) we conclude that

\[
 h(x_0) = \frac{1}{|\Omega|} \int_{\Omega} h(x) \, dx
\]

for all \( h \in H(\Omega) \cap L^1(\Omega) \). Then Theorem 2 implies that \( \Omega \) is a ball centered at \( x_0 \).
Now we have the following theorem.

**Theorem 4.** Let \( \Omega \subset \mathbb{R}^n \) \( (n \geq 2) \) be an open set of finite measure. Suppose that, for all \( y \in \partial \Omega \), there exists a sequence \( (y_j) \) in \( \mathbb{R}^n \setminus \Omega \) such that \( y_j \to y \) as \( j \to +\infty \). If (2) holds for every positive \( u \in SH(\Omega) \cap L^1(\Omega) \setminus H(\Omega) \), then \( \Omega \) is a ball with center \( x_0 \).

We shall need three lemmas.

**Lemma 1.** Let \( \Omega \subset \mathbb{R}^n \) \( (n \geq 1) \) be an open set of finite measure. Suppose that (2) holds for all positive \( u \in SH(\Omega) \cap L^1(\Omega) \setminus H(\Omega) \). Then (2) holds for all \( u \in SH(\Omega) \cap L^1(\Omega) \setminus H(\Omega) \) that are bounded from below.

**Proof.** Let \( u \in SH(\Omega) \cap L^1(\Omega) \setminus H(\Omega) \) be bounded from below. There exists \( c \in \mathbb{R} \) such that \( u > c \) on \( \Omega \). Then \( u - c \in SH(\Omega) \cap L^1(\Omega) \setminus H(\Omega) \) since \( \Omega \) has finite measure, and \( u - c \) is positive on \( \Omega \). By hypothesis, \( u - c \) has the mean value property (2). The constant function \( c \) has the mean value property (1). Hence \( u \) satisfies (2).

**Lemma 2.** Let \( \Omega \subset \mathbb{R}^n \) \( (n \geq 1) \) be an open set of finite measure and let \( \alpha \in (-n, 0) \).

(i) Suppose that there exists a sequence \( (y_j) \) in \( \mathbb{R}^n \setminus \overline{\Omega} \) such that \( y_j \to y \in \partial \Omega \) as \( j \to +\infty \).

\[ \lim_{j \to +\infty} \int_{\Omega} \|x - y_j\|^\alpha \, dx = \int_{\Omega} \|x - y\|^\alpha \, dx. \]

(ii) Suppose that there exists a sequence \( (z_j) \) in \( \mathbb{R}^n \setminus \overline{\Omega} \) such that \( \|z_j\| \to +\infty \) as \( j \to +\infty \). Then

\[ \lim_{j \to +\infty} \int_{\Omega} \|x - z_j\|^\alpha \, dx = 0. \]

**Proof.** (i) Since the arguments are similar, we only prove (ia). Since \( \Omega \) has finite measure the function \( x \mapsto \|x - y_j\|^\alpha \) is in \( L^1(\Omega) \) for every \( j \in \mathbb{N} \) and we have seen in the proof of Theorem 3 that \( x \mapsto \|x - y\|^\alpha \) is also in \( L^1(\Omega) \). Let \( r > 0 \) be fixed. There exists \( j(r) \in \mathbb{N} \) such that \( y_j \in B(y, r/3) \) for all \( j \geq j(r) \). For \( j \geq j(r) \) we can write

\[ \int_{\Omega \cap B(y,r)} \|x - y_j\|^\alpha \, dx = \int_{\Omega \cap B(y,r/3)} \|x - y_j\|^\alpha \, dx + \int_{\Omega \cap B(y,r) \setminus B(y,r/3)} \|x - y_j\|^\alpha \, dx. \]
We have
\[ \int_{\Omega \cap B(y_j, r/3)} \|x - y_j\|^\alpha \, dx \leq C_n r^{n+\alpha}, \quad \int_{\Omega \cap B(y, r)} \|x - y\|^\alpha \, dx \leq C_n r^{n+\alpha} \]
and
\[ \int_{\Omega \cap B(y, r) \setminus B(y_j, r/3)} \|x - y_j\|^\alpha \, dx \leq \frac{r}{3^n} \cdot \mu(\Omega \cap B(y, r) \setminus B(y_j, r/3)) \leq C_n r^{n+\alpha}, \]
where \( C_n > 0 \) is independent of \( j \) and \( r \). On the other hand, the Lebesgue dominated convergence theorem implies that
\[ \lim_{j \to +\infty} \int_{\Omega \setminus B(y, r)} \|x - y_j\|^\alpha \, dx = \int_{\Omega \setminus B(y, r)} \|x - y\|^\alpha \, dx. \] (4)

Now let \( f_j(x) = \|x - y_j\|^\alpha - \|x - y\|^\alpha, x \in \Omega \). For \( j \geq j(r) \) we write
\[ \left| \int_{\Omega} f_j(x) \, dx \right| \leq \left| \int_{\Omega \cap B(y, r)} f_j(x) \, dx \right| + \left| \int_{\Omega \setminus B(y, r)} f_j(x) \, dx \right| \]
\[ \leq 3C_n r^{n+\alpha} + \left| \int_{\Omega \setminus B(y, r)} f_j(x) \, dx \right|. \]

Let \( \varepsilon > 0 \). Take \( r > 0 \) such that \( 3C_n r^{n+\alpha} \leq \varepsilon / 2 \). By (4) there exists \( j_0 \geq j(r) \) such that
\[ \left| \int_{\Omega \setminus B(y, r)} f_j(x) \, dx \right| \leq \varepsilon / 2 \quad \forall j \geq j_0, \]
and (ia) follows.

(ii) For \( k \in \mathbb{N}^* \) define \( \Omega_k = \Omega \setminus B(0, k) \). Let \( k \in \mathbb{N}^* \) and \( r > 0 \) be fixed. We write
\[ \int_{\Omega} \|x - z_j\|^\alpha \, dx = \int_{\Omega \cap B(0, k)} \|x - z_j\|^\alpha \, dx + \int_{\Omega_k} \|x - z_j\|^\alpha \, dx. \]
We have
\[ \int_{\Omega_k \setminus B(z_j, r)} \|x - z_j\|^\alpha \, dx \leq r^\alpha \cdot \mu(\Omega_k) \]
and
\[ \int_{\Omega_k \cap B(z_j, r)} \|x - z_j\|^\alpha \, dx \leq C_n r^{n+\alpha}, \]
where \( C_n > 0 \) is independent of \( j, k \) and \( r \). Now let \( \varepsilon > 0 \). Take \( r > 0 \) such that \( C_n r^{n+\alpha} \leq \varepsilon / 3 \). Since \( \mu(\Omega_k) \to 0 \) as \( k \to +\infty \), there exists \( k \in \mathbb{N} \) such that \( r^\alpha \cdot \mu(\Omega_k) \leq \varepsilon / 3 \). By the Lebesgue dominated convergence theorem there exists \( j_0 \in \mathbb{N} \) such that
\[ \int_{\Omega \cap B(0, k)} \|x - z_j\|^\alpha \, dx \leq \varepsilon / 3 \quad \forall j \geq j_0, \]
and (ii) follows.
Lemma 3. In the setting of Theorem 4 there exists a sequence \((z_j)\) in \(\mathbb{R}^n \setminus \Omega\) such that \(\|z_j\| \to +\infty\) as \(j \to +\infty\).

Proof. Suppose first that \(\partial \Omega\) is bounded. Since \(\Omega\) has finite measure, we deduce that \(\mathbb{R}^n \setminus \overline{\Omega}\) is unbounded and the lemma follows. Now, if \(\partial \Omega\) is unbounded, there exists a sequence \((y_j)\) in \(\partial \Omega\) such that \(\|y_j\| \to +\infty\).

By hypothesis, for each \(j \in \mathbb{N}\) there exists \(z_j \in (\mathbb{R}^n \setminus \overline{\Omega}) \cap B(y_j, 1)\). Clearly \(\|z_j\| \to +\infty\).

Proof of Theorem 4. Since \(\Omega\) has finite measure there exists a largest open ball \(B\) centered at \(x_0\) of radius \(r\) which lies in \(\Omega\). We will show that \(\Omega = B\). There exists \(y \in \partial \Omega \cap \partial B\) such that \(\|y - x_0\| = r\). Let \((y_j)\) in \(\mathbb{R}^n \setminus \overline{\Omega}\) be such that \(y_j \to y\) as \(j \to +\infty\) and let \((z_j)\) be as in Lemma 3. Define

\[
h(x) = r^{n-2}(\|x - x_0\|^2 - r^2)\|x - y\|^{-n}, \quad x \in \mathbb{R}^n \setminus \{y\},
\]

\[
h_j(x) = \|y_j - x_0\|^{-n}(\|x - x_0\|^2 - \|y_j - x_0\|^2)\|x - y_j\|^{-n}, \quad x \in \mathbb{R}^n \setminus \{y_j\},
\]

and

\[
v_j(x) = -\|x - z_j\|^a, \quad x \in \mathbb{R}^n \setminus \{z_j\},
\]

where \(a_n \in (-n, 0)\). Clearly \(h \in H(\mathbb{R}^n \setminus \{y\}), h_j \in H(\mathbb{R}^n \setminus \{y_j\})\) and \(\Delta v_j < 0\) in \(\Omega\). Moreover \(h, h_j\) and \(v_j\) are in \(L^1(\Omega)\), \(h(x_0) = -1\) and \(h > 0\) on \(\mathbb{R}^n \setminus B\).

Let \(u_j = 1 + h_j + v_j\). Then \(u_j \in SH(\Omega) \cap L^1(\Omega) \setminus H(\Omega)\) and \(u_j\) is bounded from below on \(\Omega\) for every \(j\). Therefore Lemma 1 implies that

\[
u_j(x_0) \geq \frac{1}{|\Omega|} \int_\Omega u_j(x)\, dx \quad \forall j \in \mathbb{N}.
\]

By Lemma 2 we can let \(j \to +\infty\) in (5) to obtain

\[
1 + h(x_0) \geq \frac{1}{|\Omega|} \int_\Omega (1 + h(x))\, dx.
\]

Since \(1 + h \in H(\Omega) \cap L^1(\Omega)\) we have

\[
0 = 1 + h(x_0) = \int_B (1 + h(x))\, dx.
\]

Now with the help of (6) and (7) we can write

\[
0 \geq \frac{1}{|\Omega|} \int_\Omega (1 + h(x))\, dx = \frac{1}{|\Omega|} \int_{\Omega \setminus B} (1 + h(x))\, dx + \frac{1}{|\Omega|} \int_B (1 + h(x))\, dx
\]

\[
= \frac{1}{|\Omega|} \int_{\Omega \setminus B} (1 + h(x))\, dx \geq \frac{|\Omega \setminus B|}{|\Omega|} \geq \frac{|\Omega \setminus \overline{B}|}{|\Omega|}.
\]
This implies that $|\Omega \setminus \overline{B}| = 0$. Then the open set $\Omega \setminus \overline{B}$ must be empty, hence $\Omega \subset \overline{B}$. Since $\Omega$ is open and $B \subset \Omega \subset \overline{B}$, we deduce that $\Omega = B$.

**Remark 3.** The assumption in Theorem 2 imposes a certain geometric restriction on the open set $\Omega$. We give an example that shows that this hypothesis cannot be omitted completely. Let $\Omega = B \setminus \{x\}$ where $B$ denotes an open ball centered at the origin in $\mathbb{R}^n$ ($n \geq 2$) and $x \in B \setminus \{0\}$. We claim that if $u \in L^1(\Omega)$ is a positive superharmonic function on $\Omega$, then (2) holds. Indeed such a function $u$ has a (unique) superharmonic extension $v$ on $B$ (see [5, Theorem 7.7, p. 130]). Then we have

$$u(0) = v(0) \geq \frac{1}{|B|} \int_B v(y) \, dy = \frac{1}{|\Omega|} \int_\Omega u(y) \, dy.$$ 

In fact, using Theorem 7.7 in [5], we can take $\Omega = B \setminus Z$ where $Z$ is a relatively closed polar subset of $B$ such that $0 \notin Z$. For instance in $\mathbb{R}^3$, $Z$ could be a line segment (see [5, Example 4, p. 127]).

**References**


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