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On oscillation of solutions of forced nonlinear neutral differential equations of higher order II

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Abstract. Sufficient conditions are obtained so that every solution of

$$[y(t)-p(t)y(t-\tau)]^{(n)}+Q(t)G(y(t-\sigma))=f(t)$$

where $n\geq 2$, $p,f\in C([0,\infty),\mathbb{R})$, $Q\in C([0,\infty),[0,\infty))$, $G\in C(\mathbb{R},\mathbb{R}),\tau>0$ and $\sigma\geq 0$, oscillates or tends to zero as $t\to\infty$. Various ranges of $p(t)$ are considered. In order to accommodate sublinear cases, it is assumed that $\int_0^\infty Q(t)\,dt=\infty$. Through examples it is shown that if the condition on Q is weakened, then there are sublinear equations whose solutions tend to $\pm\infty$ as $t\to\infty$.

1. Introduction. In this paper, we study oscillatory and asymptotic behaviour of solutions of

$$[y(t)-p(t)y(t-\tau)]^{(n)}+Q(t)G(y(t-\sigma))=f(t), \tag{1}$$

where $n\geq 2$, $p,f\in C([0,\infty),\mathbb{R})$, $Q\in C([0,\infty),[0,\infty))$, $G\in C(\mathbb{R},\mathbb{R}),\tau>0$ and $\sigma\geq 0$. In [5], most of the results hold for G satisfying

$$\liminf_{|u|\to\infty} G(u)/u>\lambda>0.$$

We remove this restriction on G in the present work. In most of our results here we assume

$$\int_0^\infty Q(t)\,dt=\infty. \tag{H_1}$$

The technique employed in this paper is motivated by Lemma 2.1 (see Section 2). The results hold true for homogeneous equations associated with (1).

By a solution of (1) we mean a real-valued continuous function y on $[T_y-\varrho,\infty)$ for some $T_y\geq 0$, where $\varrho=\max\{\tau,\sigma\}$, such that $y(t)-p(t)y(t-\tau)$

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is n-times continuously differentiable and (1) is satisfied for $t \in [T_y, \infty)$. A solution of (1) is said to be *oscillatory* if it has arbitrarily large zeros; otherwise, it is called *nonoscillatory*.

2. Main results. The following assumptions are needed below:

(H_2) G is nondecreasing and $uG(u) > 0$ for $u \neq 0$.
(H_3) There exists $F \in C^n([0,\infty),\mathbb{R})$ such that $\lim_{t\to\infty} F(t) = 0$ and $F^{(n)}(t) = f(t)$.
(H_4) For $u > 0$ and $v > 0$, there exists $\alpha > 0$ such that $G(u) + G(v) \geq \alpha G(u+v)$.
($\widetilde{H}_4$) For $x < 0$ and $y < 0$ there exists $\beta > 0$ such that $G(x) + G(y) \leq \beta G(x+y)$.
(H_5) $G(uv) = G(u)G(v)$.
(H_6) $\int_\tau^\infty Q^*(t)\,dt = \infty$, where $Q^*(t) = \min\{Q(t), Q(t-\tau)\}$.

Further, $p(t)$ satisfies one of the following conditions:

(A_1) $0 \leq p(t) \leq p_1 < 1$,
(A_2) $-1 < -p_2 \leq p(t) \leq 0$,
(A_3) $-p_3 \leq p(t) \leq 0$,
(A_4) $p(t)$ changes sign with $-1 < -p_4 \leq p(t) \leq p_5 < 1$ such that $p_4 + p_5 < 1$, where $p_4 > 0$ and $p_5 > 0$,
(A_5) $1 \leq p(t) \leq p_6$,
(A_6) $0 \leq p(t) \leq p_7$.

REMARK 1. The prototype of a function G satisfying (H_1)–($\widetilde{H}_4$) is $G(u) = |u|^\gamma \operatorname{sgn} u$, $\gamma > 0$ (see [4, p. 292]). Note that (H_2) and (H_5) imply that $G(-u) = -G(u)$. Further, (H_4) and $G(-u) = -G(u)$ imply ($\widetilde{H}_4$).

We need the following lemma.

LEMMA 2.1 ([3, p. 19]). *Let $F, G, p \in C([t_0,\infty),\mathbb{R})$, $t_0 \geq 0$, be such that $F(t) = G(t) - p(t)G(t-\tau)$, $t \geq t_0 + \varrho$, $\tau \geq 0$, $G(t) \neq 0$ for $t \geq t_0$, $\liminf_{t\to\infty} G(t) = 0$ if $G(t) > 0$ or $\limsup_{t\to\infty} G(t) = 0$ if $G(t) < 0$, and $\lim_{t\to\infty} F(t) = L$ exists. If $p(t)$ satisfies (A_i), $i = 1, 2, 5, 6$, then $L = 0$.*

THEOREM 2.2. *Let (H_1), (H_2) and (H_3) hold. If $p(t)$ satisfies either (A_1) or (A_2), then every solution of (1) oscillates or tends to zero as $t \to \infty$.*

Proof. Let $y(t)$ be a nonoscillatory solution of (1). Then $y(t) > 0$ or $y(t) < 0$ for $t \geq T_0 > T_y \geq 0$. Let $y(t) > 0$ for $t \geq T_0$. For $t \geq T_1 > T_0 + \varrho$, we set

$$z(t) = y(t) - p(t)y(t-\tau), \qquad w(t) = z(t) - F(t). \tag{2}$$

From (1) it follows that

$$w^{(n)}(t) = -Q(t)G(y(t-\sigma)) \leq 0 \tag{3}$$

for $t \geq T_1$. Hence, for large t, each of $w, w', w'', \ldots, w^{(n-1)}$ is monotonic and of constant sign.

We claim that $y(t)$ is bounded. If not, then there exists a strictly increasing sequence $\{t_n\} \subset [T_4, \infty)$ satisfying

$$(4) \qquad \lim_{n\to\infty} t_n = \infty, \qquad \lim_{n\to\infty} y(t_n) = \infty, \qquad y(t_n) = \max\{y(t) : t \in [T_1, t_n]\}.$$

From (2) we obtain

$$(5) \qquad w(t_n) = y(t_n) - p(t_n)y(t_n - \tau) - F(t_n).$$

Assume (A_1) holds. Then from (5) we get $w(t_n) > (1-p_1)y(t_n) - F(t_n)$ for n large enough such that $t_n - \tau > T_1$. Thus $\lim_{n\to\infty} w(t_n) = \infty$. Consequently, $w(t) > 0$ and $w'(t) > 0$, $t \geq T_2 > T_1$. From (3) it follows that $w^{(n-1)}(t) > 0, t \geq T_3 > T_2$. On the other hand, for $0 < \varepsilon < w(T_3)$, $|F(t)| < \varepsilon$ for $t \geq T_4 > T_3$. Hence, for $t \geq T_4$, $w(t) < y(t) - F(t) < y(t) + \varepsilon$, which implies that $0 < w(T_3) - \varepsilon < w(t) - \varepsilon < y(t)$. Setting $u(t) = w(t) - \varepsilon$, $t \geq T_4$, we obtain $0 < u(t) < y(t)$, $u'(t) = w'(t) > 0$, $u^{(n-1)}(t) = w^{(n-1)}(t) > 0$ and $u^{(n)}(t) = w^{(n)}(t) = -Q(t)G(y(t-\sigma))$. Integrating from T_5 to t ($T_4 + \sigma < T_5 < t$) and using (H_2) we obtain

$$\begin{aligned} u^{(n-1)}(t) &= u^{(n-1)}(T_5) - \int_{T_5}^{t} Q(s)G(y(s-\sigma))\,ds \\ &< u^{(n-1)}(T_5) - G(u(T_5 - \sigma)) \int_{T_5}^{t} Q(s)\,ds. \end{aligned}$$

Hence $u^{(n-1)}(t) < 0$ for large t by (H_1), contradicting $u^{(n-1)}(t) > 0$ for $t \geq T_4$. If (A_2) is satisfied, then from (5) we get $w(t_n) \geq y(t_n) - F(t_n)$. Hence $\lim_{n\to\infty} w(t_n) = \infty$. Thus $w(t) > 0$ and $w'(t) > 0$ for $t \geq T_2 > T_1$. Consequently, from (3) it follows that $w^{(n-1)}(t) > 0$ for $t \geq T_3 > T_2$. Since $w(t)$ is increasing, we have, for $t > T_3 + \tau$,

$$\begin{aligned} (6) \quad (1-p_2)w(t) &\leq w(t) + p(t)w(t-\tau) \\ &= y(t) - F(t) - p(t)p(t-\tau)y(t-2\tau) - p(t)F(t-\tau) \\ &\leq y(t) - F(t) - p(t)F(t-\tau). \end{aligned}$$

For $0 < \varepsilon < (1-p_2)w(T_3)$, there exists $T_4 > T_3$ such that $|F(t)| < \varepsilon/2$, $t \geq T_4$. From (6) it follows that

$$(1-p_2)w(T_3) < (1-p_2)w(t) \leq y(t) + \varepsilon/2 - p(t)\varepsilon/2 < y(t) + \varepsilon$$

for $t > T_4 + \tau$ because $w(t)$ is increasing and $-p(t) < 1$. Setting $u(t) = (1-p_2)w(t) - \varepsilon$ for $t > T_4 + \tau$, we obtain $0 < (1-p_2)w(T_3) - \varepsilon < u(t) < y(t)$, $u'(t) = (1-p_2)w'(t) > 0$, $u^{(n-1)}(t) = (1-p_2)w^{(n-1)}(t) > 0$ and

$$(7) \qquad u^{(n)}(t) = (1-p_2)w^{(n)}(t) = -(1-p_2)Q(t)G(y(t-\sigma)).$$

Integrating (7) from T_5 to t ($T_4 + 2\varrho < T_5 < t$) we obtain

$$u^{(n-1)}(t) < u^{(n-1)}(T_5) - (1-p_2)\int_{T_5}^{t} Q(s)G(u(s-\sigma))\,ds$$
$$< u^{(n-1)}(T_5) - (1-p_2)G(u(T_5-\sigma))\int_{T_5}^{t} Q(s)\,ds,$$

because u is increasing. Hence $u^{(n-1)}(t) < 0$ for large t, due to (H_1), a contradiction.

Thus $y(t)$ is bounded. Consequently, whether $p(t)$ satisfies (A_1) or (A_2), $w(t)$ is bounded and hence

$$(8) \qquad (-1)^{n+k}w^{(k)}(t) < 0, \qquad k = 1,\dots,n-1,$$

for $t \ge T_6 > T_1$. If $\liminf_{t\to\infty} y(t) = \alpha > 0$, then $y(t) > \beta > 0$ for $t \ge T_7 > T_6$. Hence from (3) we get

$$w^{(n-1)}(t) < w^{(n-1)}(T_7+\sigma) - G(\beta)\int_{T_7+\sigma}^{t} Q(s)\,ds.$$

Thus, for large t, $w^{(n-1)}(t) < 0$ by (H_1), contrary to (8). We conclude that $\liminf_{t\to\infty} y(t) = 0$.

Since $w(t)$ is bounded, $\lim_{t\to\infty} w(t)$ exists due to (8) whether n is odd or even. Hence $\lim_{t\to\infty} z(t)$ exists. Lemma 2.1 applied to (2) yields $\lim_{t\to\infty} z(t) = 0$.

If (A_1) holds then for $t \ge T_1$, $z(t) \ge y(t) - p_1 y(t-\tau)$ implies $y(t) \le z(t) + p_1 y(t-\tau)$. Hence $\limsup_{t\to\infty} y(t) \le \lim_{t\to\infty} z(t) + p_1 \limsup_{t\to\infty} y(t-\tau)$, that is, $(1-p_1)\limsup_{t\to\infty} y(t) \le 0$. Consequently, $\lim_{t\to\infty} y(t) = 0$.

If (A_2) holds then $y(t) \le z(t)$ for $t \ge T_1$. Hence $\lim_{t\to\infty} y(t) = 0$. If $y(t) < 0$ for $t \ge t_0$, then we set $x(t) = -y(t)$ for $t \ge t_0$ to obtain

$$[x(t) - p(t)x(t-\tau)]^{(n)} + Q(t)\widetilde{G}(x(t-\sigma)) = \widetilde{f}(t),$$

where $\widetilde{f}(t) = -f(t)$ and $\widetilde{G}(u) = -G(-u)$. If $\widetilde{F}(t) = -F(t)$, then (H_2) and (H_3) are satisfied for $\widetilde{G}$ and $\widetilde{F}$ respectively. Hence $\lim_{t\to\infty} x(t) = 0$, that is, $\lim_{t\to\infty} y(t) = 0$. Thus the theorem is proved.

As a consequence of Theorem 2.2 we get the following.

Corollary 2.3. *Under the assumptions of Theorem* 2.2, *every nonoscillatory solution of* (1) *tends to zero as* $t\to\infty$ *and hence every unbounded solution of* (1) *oscillates.*

Remark 2. Theorem 2.2 remains true if $f(t) \equiv 0$.

For sublinear G, that is, G satisfying $\int_0^{\pm c} du/G(u) < \infty$, $c > 0$, or $\liminf_{|u|\to 0} G(u)/u > \mu > 0$, condition ($\mathrm{H}_1$) cannot be relaxed e.g. to

$\int_0^\infty t^{n-1}Q(t)\,dt = \infty$ used in [5]. This follows from the example given below.

EXAMPLE. Consider

$$[y(t)-py(t-1)]'' + \tfrac{3}{8}(t-1)^{-1/2}(t^{-3/2}-p(t-1)^{-3/2})y^{1/3}(t-1) = 0 \tag{9}$$

with either $0 < p < 1/2$ and $t \geq t_0 > \max\{2, (1-(2p)^{2/3})^{-1}\}$, or $-1 < p < 0$ and $t > t_0 > 2$. In the first case (H_1) fails because from the inequality

$$Q(t) = \tfrac{3}{8}[t^{-3/2}(t-1)^{-1/2} - p(t-1)^{-2}] < \tfrac{3}{8}[t^{-3/2} - p(t-1)^{-2}]$$

it follows that $\int_{t_0}^\infty Q(t)\,dt < \infty$. However,

$$Q(t) > \frac{3p}{8}[2(t-1)^{-2} - (t-1)^{-2}] = \frac{3p}{8}(t-1)^{-2}$$

implies that

$$\int_{t_0}^{t} sQ(s)\,ds > \frac{3p}{8}\int_{t_0}^{t} s(s-1)^{-2}\,ds > \ln[(t-1)/(t_0-1)].$$

Hence $\int_{t_0}^\infty Q(t)\,dt = \infty$. Similarly in the second case $\int_{t_0}^\infty Q(t)\,dt < \infty$ and $\int_{t_0}^t sQ(s)\,ds > \frac{3}{8}\int_{t_0}^t ds/s$ implies that $\int_{t_0}^\infty tQ(t)\,dt = \infty$. In both cases, (9) admits a solution $y(t) = t^{3/2}$ which is nonoscillatory and unbounded.

In the following theorem, $p(t)$ is allowed to change sign.

THEOREM 2.4. *Let* (H_1), (H_2) *and* (H_3) *hold. Let* $p(t)$ *be* τ*-periodic and* (A_4) *be satisfied. Then every solution of* (1) *oscillates or tends to zero as* $t \to \infty$.

Proof. Let $y(t)$ be a nonoscillatory solution of (1). Proceeding as in the proof of Theorem 2.2 we conclude that there is a sequence $\{t_n\}$ satisfying (4). Then (5) and (A_4) yield

$$w(t_n) \geq y(t_n) - p_5 y(t_n-\tau)F(t_n) \geq (1-p_5)y(t_n) - F(t_n).$$

Hence $\lim_{n\to\infty} w(t_n) = \infty$. Thus $w(t) > 0$ and $w'(t) > 0$ for $t \geq T_2 > T_1$. From (3) it follows that $w^{(n-1)}(t) > 0$ for $t \geq T_3 > T_2$. Since $w(t)$ is increasing and positive and $p(t) = p(t-\tau)$, we obtain

$$\begin{aligned}(1-p_4)w(t) &< w(t) - p_4 w(t-\tau) < w(t) + p(t)w(t-\tau)\\ &= y(t) - F(t) - p(t)p(t-\tau)y(t-2\tau) - p(t)F(t-\tau)\\ &= y(t) - F(t) - p^2(t)y(t-2\tau) - p(t)F(t-\tau)\\ &< y(t) - F(t) - p(t)F(t-\tau)\end{aligned}$$

for $t \geq T_4 > T_3 + \varrho$. Then proceeding as in the proof of Theorem 2.2 we arrive at a contradiction. Hence $y(t)$ is bounded. Consequently, $\lim_{t\to\infty} z(t)$ exists. Applying the argument of the proof of Theorem 2.2, we may show that $\liminf_{t\to\infty} y(t) = 0$. As Lemma 2.1 cannot be applied here to show that $\limsup_{t\to\infty} y(t) = 0$, we proceed as follows: observe that

$$\begin{aligned}\lim_{t\to\infty} z(t) &= \limsup_{t\to\infty}[y(t) - p(t)y(t-\tau)] \geq \limsup_{t\to\infty}[y(t) - p_5 y(t-\tau)] \\ &\geq \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}[-p_5 y(t-\tau)] \\ &= \limsup_{t\to\infty} y(t) - p_5 \limsup_{t\to\infty} y(t-\tau) \\ &= (1-p_5)\limsup_{t\to\infty} y(t)\end{aligned}$$

and

$$\begin{aligned}\lim_{t\to\infty} z(t) &= \liminf_{t\to\infty}[y(t) - p(t)y(t-\tau)] \leq \liminf_{t\to\infty}[y(t) + p_4 y(t-\tau)] \\ &\leq \liminf_{t\to\infty} y(t) + \limsup_{t\to\infty}[p_4 y(t-\tau)] = p_4 \limsup_{t\to\infty} y(t-\tau) \\ &= p_4 \limsup_{t\to\infty} y(t),\end{aligned}$$

since $\liminf_{t\to\infty} y(t) = 0$. Combining the above inequalities we get

$$0 \leq (1 - p_5 - p_4)\limsup_{t\to\infty} y(t) \leq 0,$$

because $p_4 + p_5 < 1$. Hence $\lim_{t\to\infty} y(t) = 0$. If $y(t) < 0$ for $t \geq T_0$, we obtain $\lim_{t\to\infty} y(t) = 0$ in a similar manner. Thus the theorem is proved.

THEOREM 2.5. *Suppose that* (H_1)–$(\widetilde{\mathrm{H}}_4)$ *hold and* Q *is decreasing. If* $p(t) \equiv -1$, *then every solution of* (1) *oscillates or tends to zero as* $t \to \infty$.

Proof. Proceeding as in the previous proofs we show that $w(t)$ is monotonic and hence $\lim_{t\to\infty} w(t) = \ell$, where $-\infty \leq \ell \leq \infty$. If $-\infty \leq \ell < 0$ then $z(t) < 0$ for large t, a contradiction. If $\ell = 0$, then $z(t) > y(t)$ implies that $\lim_{t\to\infty} y(t) = 0$. Suppose $0 < \ell \leq \infty$. Then $w^{(n-1)}(t) > 0$ for large t and hence $\lim_{t\to\infty} w^{(n-1)}(t)$ exists and is finite. Further, $z(t) > \lambda > 0$ for $t \geq T_2 > T_1$. Integrating (3) from T_2 to s ($s > T_2$) and then taking the limit as $s \to \infty$, we obtain

$$\int_{T_2}^{\infty} Q(t)G(y(t-\sigma))\,dt < \infty. \tag{10}$$

On the other hand, for $T_3 > T_2 + \varrho$, $\int_{T_3}^{\infty} Q(t)G(z(t-\sigma))\,dt \geq G(\lambda)\int_{T_3}^{\infty} Q(t)\,dt$ implies $\int_{T_3}^{\infty} Q(t)G(z(t-\sigma))\,dt = \infty$ due to (H_1). Hence

$$\int_{T_3}^{\infty} Q(t)[G(y(t-\sigma)) + G(y(t-\sigma-\tau))]\,dt = \infty$$

by (H_4). Consequently, using (10), we obtain $\int_{T_3}^{\infty} Q(t)G(y(t-\sigma-\tau))\,dt = \infty$, that is,

$$\infty = \int_{T_3-\tau}^{\infty} Q(s+\tau)G(y(s-\sigma))\,ds < \int_{T_3-\tau}^{\infty} Q(s)G(y(s-\sigma))\,ds < \infty,$$

a contradiction. Hence $\ell = 0$ is the only possibility. The proof of the theorem is thus complete.

THEOREM 2.6. *Let* (H_2)–(H_6) *hold. If* $p(t)$ *satisfies* (A_3), *then every solution of* (1) *oscillates or tends to zero as* $t \to \infty$. (Observe that $(\mathrm{H}_6)\Rightarrow(\mathrm{H}_1)$ and it is not necessary to assume $Q(t)$ to be decreasing in Theorem 2.6. However, if $Q(t)$ is decreasing then (H_6) and (H_1) are equivalent.)

Proof. Using the notations of previous proofs, setting $\lim_{t\to\infty} w(t) = \ell$, we show that $-\infty \le \ell < 0$ is not possible and $\ell = 0$ implies $\lim_{t\to\infty} y(t) = 0$. Assume, if possible, that $0 < \ell \le \infty$. Hence $z(t) > \lambda > 0$ and $w^{(n-1)}(t) > 0$ for $t \ge T_2 > T_1 > T_0 + \varrho$. If $t \ge T_3 > T_2 + \varrho$, then using (H_2), (H_4), (H_5) and (H_6) we deduce from (3) that

$$\begin{aligned} 0 &= w^{(n)}(t) + Q(t)G(y(t-\sigma)) \\ &\quad + G(-p(t-\sigma))[w^{(n)}(t-\tau) + Q(t-\tau)G(y(t-\tau-\sigma))] \\ &\ge w^{(n)}(t) + G(p_3)w^{(n)}(t-\tau) + \alpha Q^*(t)G(z(t-\sigma)) \\ &\ge w^{(n)}(t) + G(p_3)w^{(n)}(t-\tau) + \alpha G(\lambda)Q^*(t). \end{aligned}$$

Integrating the above inequality and using (H_6) we get

$$w^{(n-1)}(t) + G(p_3)w^{(n-1)}(t-\tau) < 0$$

for large t, a contradiction. The case $y(t) < 0$ for $t \ge T_0$ may be dealt with similarly. Thus the theorem is proved.

The following example shows that condition (H_6) cannot be weakened for sublinear G.

EXAMPLE. Consider (9) with $p(t) = -3/2$ for $t \ge 2$. We have

$$Q(t) = (t-1)^{-1/2}\left[\tfrac{3}{8}t^{-3/2} + \tfrac{9}{16}(t-1)^{-3/2}\right]$$

and $Q(t)$ is decreasing. Further, $Q(t) < \frac{15}{16}(t-1)^{-2}$ for $t \ge 2$ implies that $\int_2^\infty Q(t)\,dt < \infty$. However, $tQ(t) > \frac{3}{8}t^{-1/2}(t-1)^{-1/2} > 3/(8t)$, so $\int_2^\infty tQ(t)\,dt = \infty$. Equation (9) admits a positive unbounded solution $y(t) = t^{3/2}$.

REMARK 3. In the literature there are a few papers dealing with the case $p(t) \ge 1$ (see [1, 2, 5, 6]). In the following some results in this direction are obtained.

Theorem 2.7. *Suppose that n is odd and* (H_1) *and* (H_2) *hold. If $p(t)$ satisfies* (A_3) *then every nonoscillatory solution of*

$$[y(t)-p(t)y(t-\tau)]^{(n)}+Q(t)G(y(t-\sigma))=0 \tag{11}$$

tends to $+\infty$ or $-\infty$ as $t\to\infty$.

Proof. Proceeding as in the proof of Theorem 2.2, we deduce that

$$\lim_{t\to\infty} z^{(n-1)}(t)=\lambda, \qquad -\infty\le\lambda<\infty.$$

Suppose that $-\infty<\lambda<\infty$.

If $\liminf_{t\to\infty} y(t)>0$, then $y(t)>\beta>0$ for $t\ge T_3>T_2$. Hence

$$\int_{T_3+\sigma}^{\infty} Q(t)G(y(t-\sigma))\,dt > G(\beta)\int_{T_3+\sigma}^{\infty} Q(t)\,dt$$

implies that $\int_{T_3+\sigma}^{\infty} Q(t)G(y(t-\sigma))\,dt=\infty$ by (H_1). On the other hand, integrating $z^{(n)}(t)+Q(t)G(y(t-\sigma))=0$, we obtain $\int_{T_3+\sigma}^{\infty} Q(t)G(y(t-\sigma))\,dt$ $=z^{(n-1)}(T_3+\sigma)-\lambda<\infty$, a contradiction. Hence $\liminf_{t\to\infty} y(t)=0$.

Thus there exists a sequence $\{t_n\}\subset[T_2,\infty)$ such that $t_n\to\infty$ and $y(t_n)\to 0$ as $n\to\infty$. As $z(t_n)=y(t_n)-p(t_n)y(t_n-\tau)<y(t_n)$, we have $\limsup_{n\to\infty} z(t_n)\le 0$. Similarly, $z(t_n+\tau)=y(t_n+\tau)-p(t_n+\tau)y(t_n)>-p_6y(t_n)$ implies that $\liminf_{n\to\infty} z(t_n+\tau)\ge 0$.

If $z(t)>0$ for $t\ge T_2$, then $\lim_{t\to\infty} z(t)=\mu$, where $0\le\mu\le\infty$. If $0<\mu\le\infty$ then $z(t)>a>0$ for $t\ge T_4>T_2$. Hence $0\ge\limsup_{n\to\infty} z(t_n)\ge a>0$, a contradiction. Thus $\mu=0$.

If $z(t)<0$ for $t\ge T_2$, then $\lim_{t\to\infty} z(t)=\mu$, where $-\infty\le\mu\le 0$. If $-\infty\le\mu<0$, then $z(t)<b<0$ for large t. Hence $0\le\liminf_{n\to\infty} z(t_n+\tau)\le b<0$, a contradiction. Thus $\mu=0$.

Consequently, $(-1)^{n+k}z^{(k)}(t)<0$, $k=0,1,\ldots,n-1$, for large t, and $\lim_{t\to\infty} z^{(k)}(t)=0$, $k=0,1,\ldots,n-1$. Since n is odd, $z'(t)<0$ for large t. Hence $z(t)>0$ for $t\ge T_2$. From (2) we obtain $y(t)>y(t-\tau)$. Hence $\liminf_{t\to\infty} y(t)>0$, a contradiction. Thus $\lambda=-\infty$. This implies that $\lim_{t\to\infty} y(t)=\infty$. Thus the proof of the theorem is complete.

Remark 4 (see [5]). The conclusion of Theorem 2.7 holds for G with $\liminf_{|u|\to\infty} G(u)/u>\lambda>0$ provided (H_1) is replaced by $\int_0^\infty t^{n-1}Q(t)\,dt$ $=\infty$.

Corollary 2.8. *Let the conditions of Theorem* 2.7 *be satisfied. Then every bounded solution of* (11) *oscillates.*

Theorem 2.9. *Suppose that $p(t)$ satisfies* (A_6)*. Let* (H_2) *and* (H_3) *hold. If, for every sequence $\{\sigma_i\}\subset(0,\infty)$ such that $\lim_{i\to\infty}\sigma_i=\infty$ and for every*

$\gamma > 0$ *such that the intervals* $(\sigma_i - \gamma, \sigma_i + \gamma)$, $i = 1, 2, \ldots$, *are nonoverlapping,*

$$\sum_{i=1}^{\infty} \int_{\sigma_i-\gamma}^{\sigma_i+\gamma} Q(t)\,dt = \infty,$$

then every unbounded solution of (1) *oscillates or tends to* $\pm\infty$ *as* $t \to \infty$, *and every bounded solution of* (1) *oscillates or tends to zero as* $t \to \infty$.

Proof. Proceeding as in the proof of Theorem 2.7 we obtain

$$\lim_{t\to\infty} w^{(n-1)}(t) = -\infty \quad \text{or} \quad \lim_{t\to\infty} w^{(n-1)}(t) = \lambda.$$

If the former holds, then $\lim_{t\to\infty} w(t) = -\infty$. Since $w(t) > -p_7 y(t-\tau) - F(t)$, we have $\lim_{t\to\infty} y(t) = \infty$.

Suppose that the latter holds. Proceeding as in the proof of Theorem 2.7 and using (H_3), we have $\lim_{t\to\infty} w(t) = 0$. Hence $(-1)^{n+k} w^{(k)}(t) < 0$, $k = 0, 1, \ldots, n-1$, for $t \ge T_2 > T_1$ and $\lim_{t\to\infty} w^{(k)}(t) = 0$, $k = 0, 1, \ldots, n-1$.

If $y(t)$ is unbounded, then there exists a sequence $\{t_n\} \subset [T_2, \infty)$ such that $t_n \to \infty$ and $y(t_n) \to \infty$ as $n \to \infty$. Let $\mu > 0$. Then $y(t_n) > \mu$ for $n \ge N_1 > 0$. From the continuity of y it follows that there exists $\delta_n > 0$ with $\liminf_{n\to\infty} \delta_n > 0$ such that $y(t) > \mu$ for $t \in (t_n - \delta_n, t_n + \delta_n)$. Then choosing n large enough such that $\delta_n > \delta > 0$ for $n \ge N > N_1$, we obtain

$$\begin{aligned}\int_{T_2}^{\infty} Q(t)G(y(t-\sigma))\,dt &\ge \sum_{n=N}^{\infty} \int_{t_n-\delta_n+\sigma}^{t_n+\delta_n+\sigma} Q(t)G(y(t-\sigma))\,dt \\ &> G(\mu) \sum_{n=N}^{\infty} \int_{t_n-\delta_n+\sigma}^{t_n+\delta_n+\sigma} Q(t)\,dt \\ &> G(\mu) \sum_{n=N}^{\infty} \int_{t_n-\delta+\sigma}^{t_n+\delta+\sigma} Q(t)\,dt.\end{aligned}$$

From the hypothesis it follows that $\int_{T_2}^{\infty} Q(t)G(y(t-\sigma))\,dt = \infty$. Integrating (3) yields

$$\int_{T_2}^{\infty} Q(t)G(y(t-\sigma))\,dt = w^{(n-1)}(T_2) < \infty,$$

a contradiction.

If $y(t)$ is bounded, then we claim $\limsup_{t\to\infty} y(t) = 0$. If not, then $\limsup_{t\to\infty} y(t) = \alpha$, $\alpha > 0$. Then there exists a sequence $\{t_n\}$ such that $y(t_n) > \beta > 0$ for large n. Proceeding as above we arrive at a contradiction. Hence our claim holds. Consequently, $\lim_{t\to\infty} y(t) = 0$. The case $y(t) < 0$ for $t \ge T_0$ may be dealt with similarly. Thus the theorem is proved.

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