On the Noether exponent

by Anna Stasica (Kraków and Le Bourget du Lac)

Abstract. We obtain, in a simple way, an estimate for the Noether exponent of an ideal $I$ without embedded components (i.e. we estimate the smallest number $\mu$ such that $(\text{rad} \ I)^\mu \subset I$).

1. Introduction. Let $k[X]$ be the polynomial ring of $n$ variables over an algebraically closed field. The Nullstellensatz guarantees that for a given ideal $I \subset k[X]$ there is a number $\mu$ such that $(\text{rad} \ I)^\mu \subset I$. The smallest such $\mu$ is called the Noether exponent of the ideal $I$ and will be denoted by $\mu_I$. An estimate of this exponent was obtained by Kollár in [K]. Subsequently several authors contributed to this problem, especially in the easier case when the ideal $I$ has only isolated components in its primary decomposition (see e.g. [CP], [FPT], [JOW], [STV], [AM]).

In this note we also consider the case without embedded components. We give a very simple method to obtain an estimate for the Noether exponent (Theorem 7) which is sharper than the results obtained in [FPT] and [JOW]. More precisely for the ideal $I = (f_1, \ldots, f_k)$, where $\deg f_2 \geq \ldots \geq \deg f_k \geq \deg f_1$ we show that

$$\mu_I \leq \max_{i \in \{r_1, \ldots, r_m\}} \left\{ \frac{\deg f_1 \cdot \ldots \cdot \deg f_i}{d_i} \right\},$$

where $r_1, \ldots, r_m$ are all possible codimensions of irreducible components of the zero set of the ideal $I$, and $d_i$ is the minimal degree of irreducible components of codimension $i$ of the variety given by the ideal $I$.

We conjecture that this estimate is also valid in the general case.

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2. Preliminaries. We denote by \( \mathbb{A}^n \) the affine space of dimension \( n \) and by \( k[X] = k[x_1, \ldots, x_n] \) the polynomial ring over the algebraically closed field \( k \). The zero set of an ideal \( I \) is denoted by \( V(I) \). For an algebraic set \( Z \subset \mathbb{A}^n \) we consider the ideal \( I(Z) = \{ f \in k[X] \mid f|_Z = 0 \} \).

Let \( Z_1, \ldots, Z_r \subset \mathbb{A}^n \) be hypersurfaces and let \( V \) be an irreducible component of \( Z_1 \cap \ldots \cap Z_r \). We say that \( Z_1, \ldots, Z_r \) meet properly along \( V \) or that this intersection is proper along \( V \) if \( \dim V = n - r \). Recall

**Definition 1.** Let \( Z_1 = \{ F_1 = 0 \}, \ldots, Z_r = \{ F_r = 0 \} \subset \mathbb{A}^n \) be hypersurfaces meeting properly along the variety \( V \). Then we define the index of intersection of \( Z_1, \ldots, Z_r \) along \( V \) to be the number

\[
i(Z_1 \cdot \ldots \cdot Z_r; V) := e_{k[X]}((F_1, \ldots, F_r)_p),
\]

where

\[
e_{k[X]}((F_1, \ldots, F_r)_p) t^n + \ldots
\]

is the Hilbert–Samuel polynomial of \( k[X]/(F_1, \ldots, F_r)_p \) and \( p = I(V) \) is the ideal of the variety \( V \).

**Remark 2.** Since \( k[X]_p \) is a Cohen–Macaulay ring we have

\[
i(Z_1 \cdot \ldots \cdot Z_r; V) = \text{length}(k[X]/(F_1, \ldots, F_r)_p).
\]

See e.g. [F, Example 7.1.10, p. 123].

We will need the following facts:

**Theorem 3** (Associativity formula from [Na, (24.7)]). Let \( (R, \mathfrak{m}) \) be a local ring and let \( I \) be an \( \mathfrak{m} \)-primary ideal generated by a system of parameters \( f_1, \ldots, f_n \in \mathfrak{m} \). Let \( b \) be the ideal generated by \( f_1, \ldots, f_r \) for some \( r \leq n \), and let \( p_i \) be the minimal prime ideals of \( b \) such that \( \text{length}(p_i) = k \) and \( \dim R/p_i = n - r \). Then

\[
e(I) = \sum_i e((I + p_i)p_i) \cdot e(bR_{p_i}).
\]

**Theorem 4.** Let \( \Phi : k^n \to k^n \) be a generically finite polynomial mapping of geometric degree \( \text{gdeg} \Phi \) (by geometric degree we mean the number of points in a generic fiber). Then for each \( y \in k^n \) the number of isolated points in the fiber \( \Phi^{-1}(y) \) is not greater than \( \text{gdeg} \Phi \).

**Proof.** The statement is obvious for a quasi-finite mapping. The general case follows from the Stein factorization applied to the compactification of \( \Phi \). Indeed, let \( \Gamma = \text{graph} \Phi \) and let \( \overline{\Gamma} \subset \mathbb{P}^n \times k^n \) be its closure. Consider the projection \( \bar{f} : \overline{\Gamma} \to k^n \). Due to the Stein factorization theorem there exist a normal variety \( W \) and two morphisms, \( q : \overline{\Gamma} \to W \) which has connected fibers and \( u : W \to k^n \) which is finite, such that \( \bar{f} = u \circ q \). Moreover, \( \text{gdeg} u = \text{gdeg} \Phi \). Consequently, every fiber \( \bar{f}^{-1}(y) \) has no more than \( \text{gdeg} u = \text{gdeg} \Phi \).
connected components. This implies that the number of isolated points in a fiber $\Phi^{-1}(y)$ is not greater than $\text{gdeg} \Phi$.

We have a useful characterization of the index of proper intersection:

**Proposition 5.** Let $Z_1 = \{F_1 = 0\}, \ldots, Z_r = \{F_r = 0\} \subset \mathbb{A}^n$ be hypersurfaces given by polynomials $F_i$ which meet along $V$ properly. Let $H_j = V(\alpha_j)$ be the hyperplane given by a linear form $\alpha_j$ for $j \in \{1, \ldots, n - r\}$. Define $\Phi := (F_1, \ldots, F_r, \alpha_1, \ldots, \alpha_{n-r})$. If the intersection $Z_1 \cap \ldots \cap Z_r \cap H_1 \cap \ldots \cap H_{n-r}$ is proper at a point $Q$, then

$$i(Z_1 \cdot \ldots \cdot Z_r; V) = i(Z_1 \cdot \ldots \cdot Z_r \cdot L; Q) = \mu_Q(\Phi)$$

for every linear subspace $L$ of dimension $n - r$ which meets $\bigcap_{i=1}^r Z_i$ transversely at $Q$. Here $\mu_Q(\Phi)$ denotes the multiplicity index of $\Phi$ at $Q$.

**Proof.** We follow [No]. Set $F = \{F_1, \ldots, F_r\}$ and $H = \{\alpha_1, \ldots, \alpha_{n-r}\}$. Clearly $F_1, \ldots, F_r$ form a system of parameters of the localization $k[X]_p$, where $p$ is the ideal of the variety $V$. Since the hyperplanes $H_j$ for $j \in \{1, \ldots, n - r\}$ meet $V$ transversely at $Q$, it follows that $\{F, H\}$ is a system of local parameters in the ring $k[X]_{m_Q}$ and hence from the associativity formula we get

$$e((F, H)) = e(((F, H) + p)/p) \cdot e(k[X]_p/(F)_p).$$

Since $(F, H)$ generate the maximal ideal in the local ring $k[X]_{m_Q}$, we get $e(((F, H) + p)/p) = 1$ and

$$e(F, H) = e(k[X]_p/(F)_p),$$

which proves that $i(Z_1 \cdot \ldots \cdot Z_r; V) = i(Z_1 \cdot \ldots \cdot Z_r \cdot L; Q)$.

We will also need the following version of the Bézout Theorem.

**Theorem 6** (Bézout Theorem, an affine version). Let $Z_1 = \{F_1 = 0\}, \ldots, Z_r = \{F_r = 0\} \subset \mathbb{A}^n$ be hypersurfaces given by polynomials $F_i$ which meet along $V_1, \ldots, V_s$ properly. Then

$$\sum_{i=1}^s i(Z_1 \cdot \ldots \cdot Z_r; V_i) \deg V_i \leq \prod_{i=1}^r \deg F_i.$$

**Proof.** According to the previous proposition the index of intersection is independent of the choice of a generic point $P$. Take generic hyperplanes $H_j$ given by linear forms $\alpha_j$ $(j = 1, \ldots, n - r)$ and set $d_i := \deg V_i$ for $i = 1, \ldots, s$.

Clearly the intersection $S := (\bigcup_{i=1}^s V_i) \cap \bigcap_{j=1}^{n-r} H_j$ is a finite set. Let $\{P_1^i, \ldots, P_{d_i}^i\} := V_i \cap S$ for $i \in \{1, \ldots, s\}$. Consider the map $\Phi := (F_1, \ldots, F_r, \alpha_1, \ldots, \alpha_{n-r})$. Note that it is generically finite and hence for a generic fiber $\Phi^{-1}(y)$ we have, by Theorem 4 and by the inequality of Rusek–Winiarski
where $\mu_P(\Phi)$ is the multiplicity of $\Phi$ at $P$. Since there are exactly $d_i$ points in $V_i \cap S$ we have
$$ \sum_{P \in \Phi^{-1}(y)} \mu_P(\Phi) = \sum_{i=1}^{s} \mu_{P_i}(\Phi) \cdot d_i, $$
and finally due to Proposition 5 we get
$$ \sum_{i=1}^{s} i(Z_1 \cdot \ldots \cdot Z_r; V_i) \deg V_i \leq \prod_{i=1}^{r} \deg F_i. \quad \blacksquare $$

3. Main result. Our main result is the following estimate.

Theorem 7. Let $I = (f_1, \ldots, f_k)$ be an ideal generated by polynomials $f_j \in k[X]$, where $\deg f_2 \geq \ldots \geq \deg f_k \geq \deg f_1$. Assume that there is a primary decomposition $I = \bigcap_{i=1}^{m} q_i$ without embedded components, where $q_i$ are $p_i$-primary ideals. Set $r_i := \text{codim } V(q_i)$, and define $d_t := \min\{\deg V(q_i) \mid \text{codim } V(q_i) = t\}$ for $t \in \{r_1, \ldots, r_m\}$. Then
$$ \mu_I \leq \max_{t \in \{r_1, \ldots, r_m\}} \left\{ \frac{\deg f_1 \cdot \ldots \cdot \deg f_t}{d_t} \right\}. $$

To prove this theorem we will proceed by reduction to the case where the intersection along components of $V(I)$ is proper. The proof will be given in the next section.

Observe that for an ideal $I$ without embedded components we are able to find in this ideal a finite set $F$ of polynomials such that each component of $V(I)$ can be represented as a proper intersection of some hypersurfaces given by polynomials which lie in the set $F$. In fact we have the following

Lemma 8. Let $I = (f_1, \ldots, f_k)$ be an ideal with only isolated $p_i$-prime components $q_i$, say $(f_1, \ldots, f_k) = \bigcap_{i=1}^{m} q_i$, and define $r_i := \text{codim } V(q_i)$. Then there exists a family of polynomials
$$ \begin{align*}
F_1 &:= f_1, \\
F_u &:= a_u^1 f_u + \ldots + a_u^k f_k \quad \text{for } u \geq 2,
\end{align*} $$
where $a_u^j \in k$, such that for each $i \in \{1, \ldots, m\}$ the intersection $V(F_1) \cap \ldots \cap V(F_{r_i})$ along $V(q_i)$ is proper.

Proof. We will construct such a family inductively. Obviously for $i$ such that codim $V(q_i) = 1$ the statement is true. Assume that for all $s < l \leq \max\{r_1, \ldots, r_m\}$ we have polynomials $F_s$ such that the intersection $W_s := V(F_1) \cap \ldots \cap V(F_s)$ is proper along each component which does not lie
in the set \( V(I) \). Moreover, suppose that if there is \( i \in \{1, \ldots, m\} \) such that \( r_i = s \) then the hypersurfaces \( V(F_1), \ldots, V(F_s) \) meet properly along \( V(q_i) \). Consider a decomposition \( W_{l-1} = W_{l-1}^1 \cup \ldots \cup W_{l-1}^{s_{l-1}} \) into irreducible components. Take points \( x_{l-1}^p \in W_{l-1}^p \setminus V(I) \) (for non-empty \( W_{l-1}^p \setminus V(I) \)). Since not all \( f_j \) for \( j = l, \ldots, k \) vanish at \( x_{l-1}^p \), for generic \( a_j \), the intersection \( W_{l-1} \cap V(F_i) \) along all components of codimension \( l \) is proper. Thus if there is \( V(q_j) \) such that \( r_j = l \) then clearly it must be contained in \( W_{l-1} \cap V(F_i) \) and hence \( V(F_1), \ldots, V(F_s) \) meet properly along this \( V(q_j) \). Continuing this process, in a finite number of steps we obtain a family of polynomials \( F_u \) such that for each \( i \) the intersection \( V(F_1) \cap \ldots \cap V(F_{r_i}) \) is proper along \( V(q_i) \).

4. Estimates. In this section we give the proof of the main result, hence we work throughout under the assumptions and notation of Theorem 7. First, consider the rings \( k[X]_{p_i}/I_{p_i} \) for \( i \in \{1, \ldots, m\} \). Since the ideal \( I \) has only isolated components we obtain for each \( i \) an isomorphism

\[
k[X]_{p_i}/I_{p_i} \cong k[X]_{p_i}(q_i)_{p_i} =: R_i.
\]

Consider in the ring \( R_i \) an increasing family of modules

\[
M^i_s(h) := ((q_i)_{p_i} : h^s) = \{ g \in R_i \mid gh^s \in (q_i)_{p_i}\},
\]

for some \( h \in k[X] \) with \( h|_{V(I)} = 0 \). We have

**Lemma 9.** If \( s \) is such that

\[
M^i_0(h) \subsetneq \ldots \subsetneq M^i_s(h) = M^i_{s+1}(h) = \ldots
\]

then \( h^s \in q_i \).

**Proof.** Take the smallest \( n \in \mathbb{N} \) such that \( h^n \in (q_i)_{p_i} \) and assume that \( n > s \). Clearly \( h^{n-s-1} \in M^i_{s+1}(h) \) and hence also \( h^{n-s-1} \in M^i_s(h) \), but this means that \( h^{n-1} \in (q_i)_{p_i} \), contrary to the minimality of \( n \). Thus \( h^s = a^b \) for some \( a \in q_i \) and \( b \not\in \text{rad } q_i \). This means that \( bh^s \in q_i \) and since the ideal \( q_i \) is primary, \( h^s \in q_i \).

Define

\[
s_i(h) := \min \{ s \mid M^i_s(h) = M^i_{s+1}(h) \}, \quad s_i := \max \{ s_i(h) \mid h|_{V(I)} = 0 \}.
\]

This is well defined since \( R_i \) is the Artin ring. Finally define \( s \) as the maximum of \( s_i \) for \( i \in \{1, \ldots, m\} \). Then we have the following inequalities:

**Lemma 10.** \( \mu_I \leq s \leq \text{length}(R_i) \) for \( i \) such that \( s_i = s \). Consequently, \( \mu_I \leq \max_{i \in \{1, \ldots, m\}} \{ \text{length}(R_i) \} \).

**Proof.** Take \( h \) such that \( h|_{V(I)} = 0 \). Then \( h = 0 \) on each \( V_i \) and hence \( h^s \in q_i \) for each \( i \in \{1, \ldots, m\} \) by Lemma 9. This proves the first inequality. The second one is a consequence of the definition of the length.
Proof of Theorem 7. We choose polynomials \( F_u \) as in Lemma 8. Denote by \( Z_u \) the hypersurface \( V(F_u) \). Since for each \( i = 1, \ldots, m \) the intersection \( Z_1 \cap \ldots \cap Z_{r_i} \) is proper along \( V(q_i) \), Remark 2 yields
\[
\text{length}(k[X]_{p_i}/(F_1, \ldots, F_{r_i})_{p_i}) = \text{length}(Z_1 \cap \ldots \cap Z_{r_i} \cap V_i).
\]
Using the affine version of the Bézout Theorem we get
\[
i(Z_1 \cap \ldots \cap Z_{r_i} \cap V_i) \leq \deg F_1 \cdots \deg F_{r_i} \leq \deg f_1 \cdots \deg f_{r_i}.
\]
Clearly
\[
\text{length}(R_i) \leq \text{length}(k[X]_{p_i}/(F_1, \ldots, F_{r_i})_{p_i}),
\]
hence finally due to Lemma 10 we obtain
\[
\mu_I \leq \max_{i \in \{r_1, \ldots, r_m\}} \left\{ \frac{\deg f_1 \cdots \deg f_i}{d_i} \right\}.
\]

Note that for a set-theoretic complete intersection we have at once (see [PT])

**Corollary 11.** If an ideal \( I = (f_1, \ldots, f_r) \) is a set-theoretic complete intersection, then
\[
\mu_I \leq \frac{\deg f_1 \cdots \deg f_r}{\min_{i \in \{1, \ldots, s\}} \{\deg X_i\}},
\]
where \( V(I) = \bigcup_{i=1}^{s} X_i \).

The next corollary is a generalization of a result from [CP].

**Corollary 12.** Let \( f = (f_1, \ldots, f_n) \) be a polynomial mapping such that \( f^{-1}(0) \) is a finite, non-empty set. Set \( \mu := \max \{ \mu_a(f) \mid a \in f^{-1}(0) \} \), where \( \mu_a(f) \) is the local multiplicity of the map \( f \) at the point \( a \). Then for each polynomial \( g \in \mathbb{C}[X] \) such that \( g|_{f^{-1}(0)} = 0 \), we have \( g^\mu \in (f_1, \ldots, f_n) \).

Finally let us state the following

**Conjecture.** Let \( I = (f_1, \ldots, f_k) \) be an ideal generated by polynomials \( f_j \in k[X] \), where \( \deg f_2 \geq \ldots \geq \deg f_k \geq \deg f_1 \). Let \( \bigcap_{i=1}^{m} q_i = I \) be a primary decomposition. Set \( r_i := \text{codim} V(q_i) \) and define \( d_t := \min \{ \deg V(q_j) \mid \text{codim} V(q_j) = t \} \) for \( t \in \{ r_1, \ldots, r_m \} \). Then
\[
\mu_I \leq \max_{t \in \{ r_1, \ldots, r_m \}} \left\{ \frac{\deg f_1 \cdots \deg f_t}{d_t} \right\}.
\]

**References**


