ANNALES
POLONICI MATHEMATICI
81.2 (2003)

Unicity theorems for meromorphic functions that share three values

by Wei-Ran Lü and Hong-Xun Yi (Jinan)

Abstract. We deal with the problem of uniqueness of meromorphic functions sharing three values, and obtain several results which improve and extend some theorems of M. Ozawa, H. Ueda, H. X. Yi and other authors. We provide examples to show that results are sharp.

1. Introduction and main results. In this paper, a meromorphic function means meromorphic in the complex plane. We denote by E (resp. I) a set of finite (resp. infinite) linear measure, not necessarily the same at each occurrence. It is assumed that the reader is familiar with the standard notations of Nevanlinna's theory such as $T(r,f)$, $m(r,f)$, $N(r,f)$, $\overline{N}(r,f)$, and so on, which can be found in [2]. In particular, $S(r,f)$ denotes any quantity that satisfies $S(r,f)=o(T(r,f))$ $(r\to\infty,\ r\notin E)$. For a complex number a, we say that two non-constant meromorphic functions f and g *share the value* a *CM* provided $f-a$ and $g-a$ have the same zeros counting multiplicities (see [9]). We say that two non-constant meromorphic functions f and g *share* ∞ *CM* provided that $1/f$ and $1/g$ share 0 CM.

Let f be a non-constant meromorphic function, and let k be a positive integer. We denote by $N_{k)}(r,f)$ the counting function of poles of f with multiplicity $\le k$, and by $N_{(k}(r,f)$ the counting function of poles of f with multiplicity $\ge k$ (see [9]). Let

$$\delta_{k)}(a,f)=1-\limsup_{r\to\infty}\frac{N_{k)}\left(r,\frac{1}{f-a}\right)}{T(r,f)}.$$

In 1976, M. Ozawa [3] proved the following result.

Theorem A. *Let f and g be two non-constant entire functions of finite order such that f and g share $0,1$ CM. If $\delta(0,f)>1/2$, then $f\cdot g\equiv 1$ or $f\equiv g$.*

2000 *Mathematics Subject Classification*: Primary 30D35.

Key words and phrases: meromorphic functions, shared values, uniqueness theorem.
Project supported by the NSFC (No. 19871050) and the RFDP (No. 1998042209).

In 1983, H. Ueda [4] obtained the following theorem.

THEOREM B. *Let f and g be two non-constant meromorphic functions sharing $0, 1$ and ∞ CM. If*

$$\limsup_{r\to\infty} \frac{N(r,f)+N(r,1/f)}{T(r,f)} < \frac{1}{2},$$

then either $f \equiv g$ or $f \cdot g \equiv 1$.

In 1990, H. Yi [6] proved the following further result.

THEOREM C. *Let f and g be two non-constant meromorphic functions sharing $0, 1$ and ∞ CM. If*

$$N_{1)}(r,f) + N_{1)}(r,1/f) < (\lambda + o(1))T(r) \quad (r \in I),$$

where $\lambda < 1/2$, $T(r) = \max\{T(r,f), T(r,g)\}$, then either $f \equiv g$ or $f \cdot g \equiv 1$.

EXAMPLE 1. Let $f(z) = -e^{\gamma(z)} - e^{2\gamma(z)}$, $g = -e^{-\gamma(z)} - e^{-2\gamma(z)}$, where $\gamma(z)$ is a non-constant entire function.

EXAMPLE 2. Let $f(z) = -e^{2z}/(e^z + 1)$, $g(z) = -e^{-z}/(e^z + 1)$.

It is easy to see that Examples 1 and 2 show that the number $1/2$ in the above theorems is sharp.

In this paper, we improve and generalize the above theorems, and obtain the following results:

THEOREM 1. *Let f and g be two distinct non-constant meromorphic functions sharing $0, 1$ and ∞ CM. If*

$$\limsup_{\substack{r\to\infty\\ r\in I}} \frac{N_{1)}(r,f)+N_{1)}(r,1/f)}{T(r,f)} < 1, \tag{1}$$

then

$$f = \frac{e^{s\gamma}-1}{e^{-(k+1-s)\gamma}-1}, \qquad g = \frac{e^{-s\gamma}-1}{e^{(k+1-s)\gamma}-1}, \tag{2}$$

where s and k are positive integers $(1 \le s \le k)$ such that $(s, k+1) = 1$, and γ is a non-constant entire function.

EXAMPLE 3. Let

$$f(z) = \frac{e^{-z}-1}{2e^z-1}, \qquad g(z) = \frac{e^z-1}{\frac{1}{2}e^{-z}-1}.$$

This example shows that condition (1) in Theorem 1 is best possible.

From Theorem 1, we obtain the following corollaries.

COROLLARY 1. *Let f and g be two distinct non-constant meromorphic functions sharing $0, 1$ and ∞ CM. If $\delta_{1)}(\infty, f) + \delta_{1)}(0, f) > 1$, then the conclusion of Theorem 1 is valid.*

COROLLARY 2. *Let f and g be two distinct non-constant meromorphic functions sharing $0, 1$ and ∞ CM. If $N_{1)}(r, f) = S(r, f)$ and $\delta_{1)}(0, f) > 0$, then*

$$f = -e^{k\gamma} - e^{(k-1)\gamma} - \ldots - e^{\gamma}, \qquad g = -e^{-k\gamma} - e^{-(k-1)\gamma} - \ldots - e^{-\gamma},$$

where k is a positive integer and γ is a non-constant entire function.

REMARK 1. Clearly, from Theorem 1, we can get

$$\limsup_{\substack{r\to\infty \\ r\notin E}} \frac{N_{1)}(r, f) + N_{1)}(r, 1/f)}{T(r, f)} = 1 - \frac{1}{k};$$

from this we know that $fg \equiv 1$ when $k = 1$, hence we see that Theorem 1 contains Theorems A, B and C.

REMARK 2. From Remark 1, we can get the following result: Let f and g be two distinct non-constant meromorphic functions sharing $0, 1$ and ∞ CM. If $N_{1)}(r, f) = S(r, f)$ and $1/3 < \delta_{1)}(0, f) \le 1/2$, then $f = -e^{2\gamma} - e^{\gamma}$, $g = -e^{-2\gamma} - e^{-\gamma}$, where γ is a non-constant entire function. This shows that Example 1 is the only exceptional example for Theorem A when $1/3 < \delta(0, f) \le 1/2$.

THEOREM 2. *Let f and g be two distinct non-constant meromorphic functions sharing $0, 1$ and ∞ CM. If $N_{1)}(r, f) = S(r, f)$ and*

$$\limsup_{\substack{r\to\infty \\ r\in I}} \frac{N_{1)}\left(r, \frac{1}{f-1}\right) + N_{1)}\left(r, \frac{1}{f}\right)}{T(r, f)} < 2, \tag{3}$$

then f and g assume one of the following forms:

(a) $f = e^{k\gamma} + e^{(k-1)\gamma} + \ldots + e^{\gamma} + 1$, $g = e^{-k\gamma} + e^{-(k-1)\gamma} + \ldots + e^{-\gamma} + 1$,
(b) $f = -e^{k\gamma} - e^{(k-1)\gamma} - \ldots - e^{\gamma}$, $g = -e^{-k\gamma} - e^{-(k-1)\gamma} - \ldots - e^{-\gamma}$,

where k is a positive integer and γ is a non-constant entire function.

From Theorem 2, we obtain the following corollary.

COROLLARY 3. *Let f and g be two distinct non-constant meromorphic functions sharing $0, 1$ and ∞ CM. If $N_{1)}(r, f) = S(r, f)$ and $\delta_{1)}(1, f) + \delta_{1)}(0, f) > 0$, then the conclusion of Theorem 2 is true.*

In 1980, H. Ueda [5] proved the following theorem.

THEOREM D. *Let f and g be two non-constant entire functions of finite order such that f and g share $0, 1$ CM and $0 < \delta(0, f) \le 1/2$. If there exists at least one zero z_0 of f such that*

$$f^{(j)}(z_0) = g^{(j)}(z_0) = 0 \quad (j = 0, 1, \ldots, n-1), \qquad f^{(n)}(z_0) = g^{(n)}(z_0) \ne 0,$$

where n is a positive integer, then $f \equiv g$.

In this paper, we improve Theorem D and obtain the following result.

THEOREM 3. *Let f and g be two non-constant entire functions sharing $0, 1$ CM and such that $\delta_{1)}(1, f) + \delta_{1)}(0, f) > 0$. If there exists at least one zero z_0 of f such that*

$$f^{(j)}(z_0) = g^{(j)}(z_0) = 0 \quad (j = 0, 1, \ldots, n-1), \quad f^{(n)}(z_0) = g^{(n)}(z_0) \neq 0,$$

where n is a positive integer, then $f \equiv g$.

2. Some lemmas. Let f and g share $0, 1$ and ∞ CM. We denote by $N_0(r)$ the counting function of the zeros of $f - g$ that are not zeros of f, $1/f$ or $f - 1$ (see [7] or [10]). In this section, we present some lemmas which are necessary for the proofs of our result.

LEMMA 1 ([7, Lemma 4]). *Let f and g be two non-constant meromorphic functions sharing $0, 1, \infty$ CM. If $f \not\equiv g$, then*

$$N_{(2}\left(r, \frac{1}{f}\right) + N_{(2}\left(r, \frac{1}{f-1}\right) + N_{(2}(r, f) = S(r, f).$$

LEMMA 2 ([1, Lemma 3] or [10, Lemma 7]). *Let f and g be two distinct non-constant meromorphic functions sharing $0, 1$ and ∞ CM. If f is a Möbius transformation of g, then f and g satisfy one of the following relations*:

(i) $f \cdot g \equiv 1$,
(ii) $(f-1)(g-1) \equiv 1$,
(iii) $f + g \equiv 1$,
(iv) $f \equiv cg$,
(v) $f - 1 \equiv c(g-1)$,
(vi) $[(c-1)f + 1] \cdot [(c-1)g - c] \equiv -c$,

where c $(\neq 0, 1)$ is a constant.

LEMMA 3 ([10, Theorem 1]). *Let f and g be two distinct non-constant meromorphic functions sharing $0, 1$ and ∞ CM. If*

$$\limsup_{\substack{r\to\infty \\ r\notin E}} \frac{N_0(r)}{T(r, f)} > \frac{1}{2},$$

then f is a Möbius transformation of g.

LEMMA 4 ([10, Theorem 2]). *Let f and g be two non-constant meromorphic functions sharing $0, 1$ and ∞ CM. If*

$$0 < \limsup_{\substack{r\to\infty \\ r\notin E}} \frac{N_0(r)}{T(r, f)} \leq \frac{1}{2},$$

then f is not a Möbius transformation of g, and f and g satisfy one of the following relations:

$$\text{(I)}\ f \equiv \frac{e^{s\gamma}-1}{e^{(k+1)\gamma}-1},\ g \equiv \frac{e^{-s\gamma}-1}{e^{-(k+1)\gamma}-1},$$
$$\text{(II)}\ f \equiv \frac{e^{(k+1)\gamma}-1}{e^{s\gamma}-1},\ g \equiv \frac{e^{-(k+1)\gamma}-1}{e^{-s\gamma}-1},$$
$$\text{(III)}\ f \equiv \frac{e^{s\gamma}-1}{e^{-(k+1-s)\gamma}-1},\ g \equiv \frac{e^{-s\gamma}-1}{e^{(k+1-s)\gamma}-1},$$

where s and k (≥ 2) are positive integers such that $1 \leq s \leq k$, s and $k+1$ are relatively prime, and γ is a non-constant entire function.

LEMMA 5 ([9, Theorem 5.13]). *Let f and g be two non-constant meromorphic functions sharing $0, 1$ and ∞ CM. If f is not a Möbius transformation of g, then*

$$T(r,f)+T(r,g) = N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f-1}\right)+N(r,f)+N_0(r)+S(r,f).$$

3. Proofs of the main results

Proof of Theorem 1. Since $f \not\equiv g$, by Lemma 1, it follows that

$$N\left(r,\frac{1}{f-a}\right) = N_{1)}\left(r,\frac{1}{f-a}\right)+S(r,f), \qquad a = 0,1,\infty.$$

Now we discuss the following two cases.

CASE 1. Suppose that f is a Möbius transformation of g. By Lemma 2, f and g satisfy one of the six relations (i)–(vi) of Lemma 2. We discuss them separately.

Assume that f and g satisfy (i). Let $f = -e^{\gamma}$, $g = -e^{-\gamma}$, where γ is a non-constant entire function. Hence the assertion of Theorem 1 is true with $k = 1$.

If f and g satisfy (ii), then 1 and ∞ are Picard values of f and $N_{1)}(r,1/f) = T(r,f)+S(r,f)$, which contradicts (1).

If (iii) holds, then 0 and 1 are Picard values of f and $N_{1)}(r,f) = T(r,f)+ S(r,f)$, which contradicts (1).

If (iv) holds, then 1 and c are Picard values of f. So, we have

$$T(r,f) = N_{1)}(r,f)+S(r,f), \quad T(r,f) = N_{1)}(r,1/f)+S(r,f),$$

which contradicts (1).

If (v) holds, then 0 and $1-c$ are Picard values of f. It follows from the second main theorem that $T(r,f) = N_{1)}(r,f)+S(r,f)$, and we have a contradiction.

If (vi) holds, then $1/(1-c)$ and ∞ are Picard values of f. Thus, we also get a contradiction for $T(r,f) = N_{1)}(r,1/f) + S(r,f)$.

CASE 2. Suppose that f is not a Möbius transformation of g. By Lemma 3, we consider the following two subcases.

SUBCASE 2.1. Assume that

$$0 < \limsup_{\substack{r\to\infty \\ r\notin E}} \frac{N_0(r)}{T(r,f)} \le \frac{1}{2}.$$

By Lemma 4, f and g satisfy one of the three relations (I)–(III) of Lemma 4.

If f and g satisfy (I), then noting that s and k are positive integers such that $(s, k+1) = 1$, we have

$$\begin{aligned} T(r,f) &= k\,T(r,e^{\gamma}) + S(r,f), \\ N_{1)}(r,f) &= k\,T(r,e^{\gamma}) + S(r,f), \\ N_{1)}(r,1/f) &= (s-1)\,T(r,e^{\gamma}) + S(r,f), \end{aligned}$$

which contradicts (1).

In a similar manner, we can prove that (II) is impossible.

If (III) is satisfied, then

$$\limsup_{r\to\infty} \frac{N_{1)}(r,f) + N_{1)}(r,1/f)}{T(r,f)} = 1 - \frac{1}{k} < 1.$$

Thus (2) holds.

SUBCASE 2.2. Assume that

$$\limsup_{\substack{r\to\infty \\ r\notin E}} \frac{N_0(r)}{T(r,f)} = 0.$$

Thus,

$$N_0(r) = S(r,f). \tag{4}$$

Noting that f and g share $0, 1$ and ∞ CM, by Lemma 1, Lemma 5 and (4) we get

$$T(r,f) \le N_{1)}(r,1/f) + N_{1)}(r,f) + S(r,f),$$

which is a contradiction.

Theorem 1 is thus completely proved.

Proof of Theorem 2. Since $f \not\equiv g$, by Lemma 1, it follows that

$$N\left(r, \frac{1}{f-a}\right) = N_{1)}\left(r, \frac{1}{f-a}\right) + S(r,f), \qquad a = 0, 1, \infty.$$

Now we discuss the following two cases.

CASE 1. Suppose that f is a Möbius transformation of g. By Lemma 2, f and g satisfy one of the six relations (i)–(vi) of Lemma 2.

If (i) holds, then $f = -e^{\gamma}$, $g = -e^{-\gamma}$, where γ is a non-constant entire function.

If (ii) holds, then $f = e^{\gamma} + 1$, $g = e^{-\gamma} + 1$, where γ is a non-constant entire function.

Hence the assertion of Theorem 2 is true with $k = 1$ in both subcases.

If (iii) (resp. (iv), (v)) holds, then 0 and 1 (resp. 1 and c, 0 and $1 - c$) are Picard values of f. By the second fundamental theorem, we obtain

$$N_{1)}(r, f) = T(r, f) + S(r, f),$$

which is a contradiction.

If (vi) holds, then $1/(1-c)$ and ∞ are Picard values of f. By the second fundamental theorem, we obtain

$$N_{1)}(r, 1/f) = T(r, f) + S(r, f), \qquad N_{1)}\left(r, \frac{1}{f-1}\right) = T(r, f) + S(r, f),$$

which contradicts (3).

Case 2. Suppose that f is not a Möbius transformation of g. By Lemma 3, we consider the following two subcases.

Subcase 2.1. Assume that

$$0 < \limsup_{\substack{r\to\infty \\ r\notin E}} \frac{N_0(r)}{T(r, f)} \le \frac{1}{2}.$$

By Lemma 4, f and g satisfy one of the three relations (I)–(III) of Lemma 4.

If f and g satisfy (I), then $1 \le s \le k$ and $N_{1)}(r, f) = S(r, f)$, which is impossible.

If (II) holds, then noticing that $N_{1)}(r, f) = S(r, f)$, we have $s = 1$ and derive

$$f = e^{k\gamma} + e^{(k-1)\gamma} + \ldots + e^{\gamma} + 1, \quad g = e^{-k\gamma} + e^{-(k-1)\gamma} + \ldots + e^{-\gamma} + 1.$$

So we have proved the form (a).

If f and g satisfy (III), then we can easily obtain the form (b) with $k = s$.

Subcase 2.2. Assume that

$$\limsup_{\substack{r\to\infty \\ r\notin E}} \frac{N_0(r)}{T(r, f)} = 0.$$

Noting that f and g share $0, 1$ and ∞ CM, by Lemma 1, Lemma 5 and (4), we get

$$T(r, f) = N_{1)}\left(r, \frac{1}{f}\right) + S(r, f), \qquad T(r, f) = N_{1)}\left(r, \frac{1}{f-1}\right) + S(r, f),$$

which leads to a contradiction with (3).

This completes the proof of Theorem 2.

Proof of Theorem 3. Suppose that $f \not\equiv g$. According to Corollary 3, f and g assume one of the following forms:

(a) $f = e^{k\gamma} + e^{(k-1)\gamma} + \ldots + e^{\gamma} + 1,\ g = e^{-k\gamma} + e^{-(k-1)\gamma} + \ldots + e^{-\gamma} + 1,$
(b) $f = -e^{k\gamma} - e^{(k-1)\gamma} - \ldots - e^{\gamma},\ g = -e^{-k\gamma} - e^{-(k-1)\gamma} - \ldots - e^{-\gamma},$
where k is a positive integer and γ is a non-constant entire function.

If (a) holds, then

$$\frac{f-1}{g-1}\,\frac{g}{f} = e^{\gamma}.$$

Substituting z_0 into the above equation, we get $e^{\gamma(z_0)} = 1$. From this and (a) we have $f(z_0) = k+1$, which contradicts $f(z_0) = 0$. The same method leads to a contradiction in the second case, which proves Theorem 3.

Ackowledgements. The authors are grateful to the referee for his (or her) valuable comments.

References

[1] H. H. Al-Khaladi, *Unicity theorems for meromorphic functions*, Kodai Math. J. 23 (2000), 27–34.
[2] W. K. Hayman, *Meromorphic Functions*, Clarendon Press, Oxford, 1964.
[3] M. Ozawa, *Unicity theorems for entire functions*, J. Anal. Math. 30 (1976), 411–420.
[4] H. Ueda, *Unicity theorems for meromorphic or entire function II*, Kodai Math. J. 6 (1983), 26–36.
[5] —, *Unicity theorems for entire functions*, ibid. 3 (1980), 212–223.
[6] H. X. Yi, *Meromorphic functions that share two or three values*, ibid. 13 (1990), 363–372.
[7] —, *Unicity theorems for meromorphic functions that share three values*, ibid. 18 (1995), 300–314.
[8] —, *Meromorphic functions that share three values*, Chinese Ann. Math. Ser. A 9 (1988), 433–440.
[9] H. X. Yi and C. C. Yang, *Uniqueness Theory of Meromorphic Functions*, Pure Appl. Math. Monographs 32, Science Press, Beijing, 1995.
[10] Q. C. Zhang, *Meromorphic functions sharing three values*, Indian J. Pure Appl. Math. 30 (1999), 667–682.

Department of Mathematics
Shandong University
Jinan, Shandong 250100
People's Republic of China
E-mail: hxyi@sdu.edu.cn

Reçu par la Rédaction le 10.4.2002
Révisé le 4.11.2002 (1332)