Non-existence of some natural operators on connections

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Abstract. Let $n, r, k$ be natural numbers such that $n \geq k + 1$. Non-existence of natural operators $C^r_0 \leadsto Q(\text{reg}
{T_k^r} \rightarrow K^r_k)$ and $C^r_0 \leadsto Q(\text{reg}
{T_k^r} \rightarrow K^r_k)$ over $n$-manifolds is proved. Some generalizations are obtained.

0. Introduction. Let $n, r$ and $k$ be natural numbers such that $n \geq k + 1$. In [1], C. Ehresmann constructed functorially the fiber bundle $K^r_k M = \text{reg}
{T_k^r} M / L_k^r$ of contact $(k, r)$-elements over an $n$-dimensional manifold $M$ and obtained the bundle functor $K^r_k : \mathcal{M}f_n \rightarrow \mathcal{F}M$ from the category $\mathcal{M}f_n$ of $n$-dimensional manifolds and their embeddings into the category $\mathcal{F}M$ of fibered manifolds and their fibered maps. In [5], I. Kolár, P. W. Michor and J. Slovák studied the problem of how a vector field $X$ on $M$ induces a vector field $A(X)$ on $K^r_k M$ and proved that for sufficiently large $n$ every natural operator $A : T|\mathcal{M}f_n \leadsto TK^r_k$ is a constant multiple of the complete lifting $K^r_k$. In [6], I. Kolár and the author investigated the naturality problem for bundle mappings $B : K^r_k M \rightarrow K^r_k M$ and deduced the so called rigidity theorem for $K^r_k$ saying that the only natural transformation $B : K^r_k \rightarrow K^r_k$ over $n$-manifolds is the identity. They also studied the naturality problem for affinors (i.e. tensor fields of type $(1, 1)$) $C : TK^r_k M \rightarrow TK^r_k M$ on $K^r_k M$ and derived that for sufficiently large $n$ every natural affinor $C : TK^r_k \rightarrow TK^r_k$ on $K^r_k$ over $n$-manifolds is a constant multiple of the identity. Moreover they analysed how a 1-form $\omega$ on $M$ can induce a 1-form $D(\omega)$ on $K^r_k M$ and showed that for sufficiently large $n$ every natural operator $D : T^*|\mathcal{M}f_n \leadsto T^*K^r_k$ is a constant multiple of the vertical lifting. Some generalizations of the above results can be found in [8].

Similarly to $(k, r)$-elements, C. Ehresmann introduced the fiber bundle $K^{r\ast}k M = \text{reg}
{T_k^{r\ast}} M / L^{r\ast}_k$ of contact $(k, r)$-coelements over an $n$-dimensional manifold $M$. So, we have the bundle functor $K^{r\ast}_k : \mathcal{M}f_n \rightarrow \mathcal{F}M$. In [9], we

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studied for $K^r_kM$ the same naturality problems as for $K^r_kM$ and proved the same results.

In the present paper we continue the investigation of $K^r_kM$ and $K^r_kM$.

In the first part of the paper, we study the problem of whether a torsion free linear $r$-order connection $\lambda : TM \to J^rTM$ on an $n$-manifold $M$ induces a connection $A(\lambda) : T(\text{reg } T^r_kM) \to l^r_k = \text{Lie}(L^r_k)$ on the principal fiber bundle $\text{reg } T^r_kM \to K^r_kM$ with structure group $L^r_k$. This problem is reflected in the concept of natural operators $A : C_0^r \leadsto Q(\text{reg } T^r_k \to K^r_k)$ in the sense of [5]. We prove that if $n \geq k + 1$, then there are no natural operators $C_0^r \leadsto Q(\text{reg } T^r_k \to K^r_k)$. We give an example of a bundle functor $F : Mf_n \to FM$ (n ≥ k + 1) and a natural operator $A : F \leadsto Q(\text{reg } T^r_k \to K^r_k)$.

In the second part we study the problem of whether a torsion free linear $r$-order connection $\lambda : TM \to J^rTM$ on an $n$-manifold $M$ induces a connection $A(\lambda) : T(\text{reg } T^r_kM) \to l^r_k$ on the principal fiber bundle $\text{reg } T^r_kM \to K^{r*}_kM$ with structure group $L^r_k$. We prove that if $n \geq k + 1$ then there are no natural operators $C_0^r \leadsto Q(\text{reg } T^{r*}_k \to K^{r*}_k)$ over $n$-manifolds. We find an assumption on a bundle functor $F : Mf_n \to FM$ under which there are no natural operators $F \leadsto Q(\text{reg } T^{r*}_k \to K^{r*}_k)$. We observe that there is a bundle functor $F : Mf_n \to FM$ with $n \geq k + 1$ and a natural operator $A : F \leadsto Q(\text{reg } T^{r*}_k \to K^{r*}_k)$.

Natural operations with connections have been studied by many authors; see e.g. [2], [4], [5], [7], etc.

From now on $x^1, \ldots, x^n$ and $t^1, \ldots, t^k$ are the usual coordinates on $\mathbb{R}^n$ and $\mathbb{R}^k$ respectively.

All manifolds are assumed to be finite-dimensional and smooth, i.e. of class $C^\infty$. Maps between manifolds are assumed to be smooth.

I. Torsion free linear connections of order $r$ and the bundle of contact $(k, r)$-elements

1. The bundle functor $K^r_k$ of contact $(k, r)$-elements. Let $n$ and $k$ be natural numbers. For every $n$-manifold $M$ we have the bundle $T^r_kM = J^0_0(\mathbb{R}^k, M)$ over $M$ of so-called $(k, r)$-velocities on $M$. Every embedding $\varphi : M \to N$ of $n$-manifolds induces a bundle map $T^r_k\varphi : T^r_kM \to T^r_kN$ by $T^r_k\varphi(j^0_0\gamma) = j^0_0(\varphi \circ \gamma)$ for $\gamma : \mathbb{R}^k \to M$. The correspondence $T^r_k : Mf_n \to FM$ is a bundle functor of order $r$ from the category $Mf_n$ of $n$-dimensional manifolds and their embeddings into the category $FM$ of fibered manifolds and their fibered maps.

Every $\xi = j^0_0\psi \in L^r_k = \text{inv } J^0_0(\mathbb{R}^k, \mathbb{R}^k)_0$, the Lie group of invertible $r$-jets $\mathbb{R}^k \to \mathbb{R}^k$ with source and target $0 \in \mathbb{R}^k$, induces a natural automorphism
$\xi : T^r_k \to T^r_k, \bar{\xi} : T^r_k M \to T^r_k M$ by $\bar{\xi}(j^r_k \gamma) = j^r_0(\gamma \circ \psi^{-1})$ for $\gamma : \mathbb{R}^k \to M$. This defines a group homomorphism $L^r_k \to \text{Aut}(T^r_k)$.

It is well known that if $n \geq k + 1$ then the above homomorphism is an isomorphism, i.e. $\text{Aut}(T^r_k) \cong L^r_k$.

Assume $n \geq k + 1$. For every $n$-manifold $M$, $\tilde{T}^r_k M = \text{reg } J^r_0(\mathbb{R}^k, M) = \{ j^r_0 \gamma \mid \gamma : \mathbb{R}^k \to M, \text{rank}(d_0 \gamma) = k \}$ is an open subbundle of $T^r_k M$. Elements of $\tilde{T}^r_k M$ are called regular $(k, r)$-velocities on $M$. For every embedding $\varphi : M \to N$ of $n$-manifolds, $T^r_k \varphi(\tilde{T}^r_k M) \subset \tilde{T}^r_k N$, and we let $\tilde{T}^r_k \varphi : \tilde{T}^r_k M \to \tilde{T}^r_k N$ be the restriction of $T^r_k \varphi$. The correspondence $\tilde{T}^r_k : \mathcal{M} f_n \to \mathcal{F}M$ is a bundle functor of order $r$.

For every $n$-manifold $M$, $\tilde{T}^r_k M$ is invariant with respect to the action of $L^r_k = \text{Aut}(T^r_k)$ on $T^r_k M$. So, we have (by restriction) the left action of $L^r_k$ on $\tilde{T}^r_k M$, and the quotient bundle $K^r_k M = \tilde{T}^r_k M/L^r_k$ over $M$ of so-called contact $(k, r)$-elements. Let $\pi : K^r_k M \to M$ be the bundle projection. This bundle was introduced by C. Ehresmann [1]. The quotient projection $\kappa : \tilde{T}^r_k M \to K^r_k M$ is a principal fiber bundle with structure group $L^r_k$. The right principal bundle action of $L^r_k$ on $\tilde{T}^r_k M$ is given by $v \cdot \xi = \bar{\xi}^{-1}(v)$ for $\xi \in L^r_k, v \in \tilde{T}^r_k M$. For every embedding $\varphi : M \to N$ of $n$-manifolds, $\tilde{T}^r_k \varphi$ commutes with the left action of $L^r_k$ on $\tilde{T}^r_k M$ and we have the quotient map $K^r_k \varphi : K^r_k M \to K^r_k N$. Then $\tilde{T}^r_k \varphi$ is a principal bundle morphism covering $K^r_k \varphi$. The correspondence $K^r_k : \mathcal{M} f_n \to \mathcal{F}M$ is a bundle functor of order $r$.

2. Principal fiber bundle $\kappa^0 : P^0 \to Q^0$ and non-existence of $\text{GL}(\mathbb{R}^n)$-invariant connections on $P^0$. Let $n \geq k + 1$. Let $\kappa^0 : P^0 \to Q^0$ be the restriction of $\kappa : \tilde{T}^r_k \mathbb{R}^n \to K^r_k \mathbb{R}^n$ to the fibers over $0 \in \mathbb{R}^n$. Then $\kappa^0$ is a principal fiber bundle (a principal subbundle of $\kappa$) with structure group $L^r_k$. The right action of $L^r_k$ on $P^0$ is the restriction of the right action of $L^r_k$ on $\tilde{T}^r_k \mathbb{R}^n$.

There is a left action $\alpha^0 : \text{GL}(\mathbb{R}^n) \times P^0 \to P^0$ given by $\alpha^0(\eta, p) = \tilde{T}^r_k \eta(p)$ for $\eta \in \text{GL}(\mathbb{R}^n)$, $p \in P^0$. This action covers the left action $\beta^0 : \text{GL}(\mathbb{R}^n) \times Q^0 \to Q^0$ defined by $\beta^0(\eta, q) = K^r_k \eta(q)$ for $\eta \in \text{GL}(\mathbb{R}^n), q \in Q^0$. For every $\eta \in \text{GL}(\mathbb{R}^n)$ the mapping $\alpha^0_\eta = \alpha^0(\eta, \cdot) : P^0 \to P^0$ is a principal fiber bundle isomorphism covering $\beta^0_\eta = \beta^0(\eta, \cdot) : Q^0 \to Q^0$.

A connection $\omega : TP^0 \to L^r_k$ is $\text{Lie}(L^r_k)$ on $P^0$ is called $\text{GL}(\mathbb{R}^n)$-invariant if $(\alpha^0_\eta)^* \omega = \omega$ for every $\eta \in \text{GL}(\mathbb{R}^n)$.

**Proposition 1.** There are no $\text{GL}(\mathbb{R}^n)$-invariant connections $\omega : TP^0 \to L^r_k$ on $P^0$.

**Proof.** Suppose $\omega : TP^0 \to L^r_k$ is a $\text{GL}(\mathbb{R}^n)$-invariant connection. According to the general theory of invariant connections (see Kobayashi and
Nomizu [3]), there is a linear map $A : \mathfrak{gl}(\mathbb{R}^n) \to L'_k$ satisfying the following two conditions:

(1) $A(X) = \lambda(X)$ for $X \in \text{Lie}(J)$;
(2) $A(\text{ad}(j)(X)) = \text{ad}(\lambda(j))A(X)$ for $j \in J$ and $X \in \mathfrak{gl}(\mathbb{R}^n)$.

Here $J \subset GL(\mathbb{R}^n)$ is the stabilizer of $\kappa^0(\sigma_0) \in Q^0$, $\sigma_0 = j^0_0(t^1, \ldots, t^k, 0, \ldots, 0) \in P^0$, and $\lambda : \text{Lie}(J) \to L'_k$ is the Lie algebra homomorphism corresponding to the group homomorphism $\lambda : J \to L'_k$, that is, $j_0.\sigma_0 = \sigma_0.\lambda(j)$ for $j \in J$.

We recall that $A(X) = \omega_{\sigma_0}(\tilde{X})$ for $X \in \mathfrak{gl}(\mathbb{R}^n)$, where $\tilde{X}$ is the vector field on $P^0$ induced by $X$.

We have the Lie algebra isomorphism $l'_n = J^r_0(T\mathbb{R}^n)_0 := \{j_0^rX \mid X \in \mathcal{X}(\mathbb{R}^n), X_0 = 0\}$, where the bracket in $J^r_0(T\mathbb{R}^n)_0$ is given by $[j_0^rX, j_0^rY] = j_0^r([Y, X])$ (see [5]).

Consider the elements

$$X^1 = j_0^r(x^1 \frac{\partial}{\partial x^{k+1}}), \quad X^3 = j_0^r(x^{k+1} \frac{\partial}{\partial x^{k+1}}),$$

$$X^2 = j_0^r(x^1 \frac{\partial}{\partial x^1}), \quad X^4 = j_0^r(x^{k+1} \frac{\partial}{\partial x^1})$$

of $\mathfrak{gl}(\mathbb{R}^n) \subset L'_n$. Their one-parameter subgroups in $GL(\mathbb{R}^n) \subset L'_n$ are

$$a^1_t = j_0^r(x^1, \ldots, x^k, x^{k+1} + tx^1, x^{k+2}, \ldots, x^n),$$

$$a^2_t = j_0^r(e^tx^1, x^2, \ldots, x^n),$$

$$a^3_t = j_0^r(x^1, \ldots, x^k, e^tx^{k+1}, x^{k+2}, \ldots, x^n),$$

$$a^4_t = j_0^r(x^1 + tx^{k+1}, x^2, \ldots, x^n).$$

Of course $a^2_t, a^3_t, a^4_t \in J$, $\lambda(a^2_t) \neq \text{id}$, $\lambda(a^3_t) = \text{id}$, $\lambda(a^4_t) = \text{id}$ and $a^1_t \in GL(\mathbb{R}^n) \subset L'_n$. Hence $X^2, X^3, X^4 \in \text{Lie}(J)$, $X^1 \in \mathfrak{gl}(\mathbb{R}^n)$, $\lambda(X^3) = \lambda(X^4) = 0$ and $\lambda(X^2) \neq 0$.

On the other hand if $\varphi = (x^1 + x^{k+1}, x^2, \ldots, x^n)$ then $j = j_0^r\varphi \in J$ and $\lambda(j) = \text{id}$. So,

$$A(X^1) = \text{ad}(\lambda(j))A(X^1) = A(\text{ad}(j)X^1) = A\left(j_0^r\left(\varphi_*\left(x^1 \frac{\partial}{\partial x^{k+1}}\right)\right)\right)$$

$$= A(X^1 + X^2 - X^3 - X^4) = A(X^1) + A(X^2),$$

i.e. $A(X^2) = 0$.

This contradiction ends the proof of Proposition 1. ■

3. Linear connections of order $r$. A linear $r$-order connection on an $n$-manifold $M$ is a vector bundle morphism $\lambda : TM \to J^rTM$ such that $\pi^r_0 \circ \lambda = \text{id}_{TM}$, where $\pi^r_0 : J^rTM \to TM$ is the target projection (see [10]).
Natural operators on connections

Remark 1. If $M$ is an $n$-manifold we have the principal fiber bundle $P^r M = \text{inv} J^0_0(\mathbb{R}^n, M)$ over $M$ with standard group $L^r_n$. The right action of $L^r_n$ on $P^r M$ is given by the composition of jets. If $\varphi : M \to N$ is an embedding of $n$-manifolds, then we define $P^r(\varphi) : P(M) \to P(N)$ by composition of jets. There is a canonical bijection between connections $\omega : TP^r M \to l'_n$ on $P^r M$ and linear $r$-order connections $\lambda : TM \to J^r TM$ on $M$ by $H^\omega = P^r \circ (\lambda \times_M \text{id}_{P^r M})$, where $H^\omega : TM \times_M P^r M \to TP^r M$ is the horizontal lifting morphism of $\omega$, and $P^r : J^r TM \times_M P^r M \to TP^r M$ is the flow morphism of $P^r$ (see [10]).

Given an $n$-manifold $M$ we define $C^r(M) = (\text{id}_{T^* M} \otimes \alpha^0_0)^{-1}(\text{id}_{TM}) \subset T^* M \otimes J^r TM$ to be the subbundle in $T^* M \otimes J^r TM$, where $\alpha^0_0 : J^r TM \to TM$ is the target projection. It is called the bundle of linear $r$-order connections on $M$. The sections of $C^r(M)$ are exactly the linear $r$-order connections on $M$. For every embedding $\varphi : M \to N$ the mapping $T^* \varphi \otimes J^r T \varphi : T^* M \otimes J^r TM \to T^* N \otimes J^r TN$ sends $C^r(M)$ into $C^r(N)$ and we have (by restriction) a fiber bundle map $C^r(\varphi) : C^r(M) \to C^r(N)$. The correspondence $C^r : \mathcal{M}f_n \to \mathcal{F}M$ is a bundle functor of order $r + 1$.

An $r$-order linear connection $\lambda : TM \to J^r TM$ is called torsion free if $\{\lambda(u), \lambda(v)\} = 0$ for any $u, v \in T_x M$, $x \in M$. Here $\{\cdot, \cdot\} : J^r TM \times_M J^r TM \to J^{r - 1} TM$ is the algebraic bracket given by

$$
\{j^r_x X, j^r_x Y\} = j^{r - 1}_x ([X, Y])
$$

(see [10]).

By the Frobenius theorem, an $r$-order connection $\lambda$ on an $n$-manifold $M$ is torsion free iff for each $x \in M$ there is a chart $\Phi$ near $x$ such that $\Phi(x) = 0$ and $C^r(\Phi)(\lambda_x) = \lambda^0$, where $\lambda^0 : T_0 \mathbb{R}^n \to J^0_0 \mathbb{R}^n$ is defined by $\lambda^0(u) = j_0^u \tilde{u}$ for $u \in T_0 \mathbb{R}^n$ and $\tilde{u}$ is the constant vector field on $\mathbb{R}^n$ such that $\tilde{u}_0 = u$. The coordinates $\Phi$ corresponding to $\Phi$ are called normal coordinates of $\lambda$ at $x$ (see [10]). Clearly, if $\tilde{\Phi}$ are another normal coordinates of $\lambda$ at $x$ then $\tilde{\Phi} = A \circ \Phi$ for some $A \in \text{GL}(\mathbb{R}^n)$.

Given an $n$-manifold $M$ we define $C^0_0(M)$ to be the orbit in $T^* M \otimes J^r TM$ of $\lambda^0$ with respect to embeddings $\psi : \mathbb{R}^n \to M$, where $\lambda^0 \in (T^*_0 \mathbb{R}^n) \otimes (J^0_0 \mathbb{R}^n)$ is as above. It is a subbundle of $T^* M \otimes J^r TM$ and it is called the bundle of torsion free linear $r$-order connections on $M$. The sections of $C^0_0(M)$ are exactly the torsion free linear $r$-order connections on $M$. For every embedding $\varphi : M \to N$ the mapping $T^* \varphi \otimes J^r T \varphi : T^* M \otimes J^r TM \to T^* N \otimes J^r TN$ sends $C^r_0(M)$ into $C^r_0(N)$ and we have (by restriction) a fiber bundle map $C^r_0(\varphi) : C^r_0(M) \to C^r_0(N)$. The correspondence $C^r_0 : \mathcal{M}f_n \to \mathcal{F}M$ is a bundle functor of order $r + 1$.

4. Non-existence of natural operators $C^r_0 \sim Q(T^r_k \to K^r_k)$. Let $P \to P$ be a principal fiber bundle with structure group $L^r_k$ and $m$-dimensio-
nal base. According to the general theory, connections $TP \rightarrow l^r_k$ on $P$ can be identified with sections of some fiber bundle $QP = Q(P \rightarrow P)$ over $P$, called the bundle of connections on $P$. It is known that every local principal bundle isomorphism $\Psi : (P \rightarrow P) \rightarrow (H \rightarrow H)$ covering $\Psi : P \rightarrow H$ induces functorially a fiber bundle map $Q\Psi : QP \rightarrow QH$ over $\Psi$. The correspondence $Q : \mathcal{P}_m(L^r_k) \rightarrow \mathcal{F}M$ is a gauge bundle functor from the category $\mathcal{P}_m(L^r_k)$ of principal fiber bundles with structure group $L^r_k$ and $m$-dimensional bases and their local principal bundle isomorphisms (see [5]).

According to [5] a natural operator $A : C^r_0 \rightsquigarrow Q(\tilde{T}^r_k \rightarrow K^r_k)$ over $n$-manifolds is a family of regular operators

$$A : C^\infty_M(C^r_0(M)) \rightarrow C^\infty_{K^r_k}(Q(\tilde{T}^r_k M \rightarrow K^r_k M))$$

from the set of torsion free $r$-order linear connections on $M$ (sections of $C^r_0(M)$) into the set of connections on $\kappa : \tilde{T}^r_k M \rightarrow K^r_k M$ (sections of $Q(\tilde{T}^r_k M \rightarrow K^r_k M)$) for every $n$-manifold $M$ such that the naturality condition with respect to $\mathcal{MF}_n$-morphisms is satisfied. This means that for every $\lambda \in C^\infty_M(C^r_0(M))$ and $\bar{\lambda} \in C^\infty_N(C^r_0(N))$ and every $\mathcal{MF}_n$-morphism $\varphi : M \rightarrow N$, if $\bar{\lambda}$ and $\lambda$ are $\varphi$-related, then $A(\bar{\lambda})$ and $A(\lambda)$ are $Q\tilde{T}^r_k \varphi$-related.

The first main result of this paper is the following theorem.

**Theorem 1.** Let $n$, $k$ and $r$ be natural numbers such that $n \geq k + 1$. There are no natural operators $C^r_0 \rightsquigarrow Q(\tilde{T}^r_k \rightarrow K^r_k)$ over $n$-manifolds.

**Proof.** Suppose that $A : C^r_0 \rightsquigarrow Q(\tilde{T}^r_k \rightarrow K^r_k)$ is a natural operator over $n$-manifolds. We define $H^\omega : TQ^0 \times Q^0 P^0 \rightarrow TP^0$ by

$$H^\omega(u, v) = H^A(\tilde{\lambda}^0)(u, v)$$

for $u \in T_\sigma Q^0$, $\sigma \in Q^0$, $v \in P^0$, $\kappa^0(v) = \sigma$, where $H^A(\tilde{\lambda}^0) : TK^r_k \mathbb{R}^n \times K^r_k \mathbb{R}^n \rightarrow TT^r_k \mathbb{R}^n$ is the horizontal lifting morphism of $A(\tilde{\lambda}^0)$ and where $\tilde{\lambda}^0$ is the translation invariant section of $C^r_0(\mathbb{R}^n)$ such that $\tilde{\lambda}^0_0 = \lambda^0$. Here $\lambda_0$ is the element from $C^r_0(\mathbb{R}^n)$ as in Section 3.

Clearly, $H^\omega(u, v) \in T_{\tilde{\lambda}^0} T\tilde{T}^r_k \mathbb{R}^n$ and it projects on $u$ under $T\kappa$. Hence $H^\omega(u, v)$ is $(\tilde{T}^r_k \mathbb{R}^n \rightarrow \mathbb{R}^n)$-vertical because $u$ is $\pi$-vertical. Therefore, $H^\omega(u, v) \in TP^0$. Since $H^A(\tilde{\lambda})$ is a horizontal lifting, so is $H^\omega$.

Let $\omega : TP^0 \rightarrow l^r_k$ be the corresponding connection on $P^0$. Since $\tilde{\lambda}^0$ is $\text{GL}(\mathbb{R}^n)$-invariant, so is $A(\tilde{\lambda}^0)$ because of the naturality of $A$. Hence $\omega$ is $\text{GL}(\mathbb{R}^n)$-invariant.

This is a contradiction by Proposition 1. □

We have the following obvious corollaries of Theorem 1.

**Corollary 1.** Let $n$, $k$ and $r$ be natural numbers such that $n \geq k + 1$. There are no natural operators $C^r \rightsquigarrow Q(\tilde{T}^r_k \rightarrow K^r_k)$ over $n$-manifolds.
COROLLARY 2. Let $n$, $k$ and $r$ be natural numbers such that $n \geq k + 1$. There are no canonical connections $\omega : T\tilde{T}_k^r M \to l_k^r$ on $T\tilde{T}_k^r M$ over $n$-manifolds.

5. A generalization. Using the same method as in the proof of Theorem 1 we obtain the following general fact.

THEOREM 2. Let $n$, $k$ and $r$ be natural numbers such that $n \geq k + 1$. Let $F : \mathcal{M}f_n \to \mathcal{F}M$ be a bundle functor such that there is a $GL(\mathbb{R}^n)$-invariant element $\mu^0 \in F_0\mathbb{R}^n$. There are no natural operators $A : F \sim Q(T^r_k \to K^r_k)$.

Proof. In the proof of Theorem 1 we replace $\tilde{\chi}^0$ by the translation invariant section $\tilde{\mu}^0$ of $F\mathbb{R}^n$ such that $\tilde{\mu}^0|_0 = \mu^0$. ■

REMARK 2. There are many natural bundles $FM$ satisfying the assumption of Theorem 2. For example, $C^0(M)$ and $C^s(M)$ for all $s$, fiber products $C^s_0 (M) \times_M \ldots \times_M C^s_k (M)$, $T^A M$ for all Weil algebras $A$, all natural vector bundles $(\bigotimes^p T^r M \otimes \bigotimes^q TM, T^r_k M, T^r_k M, \ldots)$, etc.

COROLLARY 3. Let $n, k$ and $r$ be natural numbers such that $n \geq k + 1$. There are no natural operators $\Lambda$ sending a generalized connection $\Gamma$ on $\tilde{T}_k^r M \to M$ (or $K^r_k M \to M$) and a linear connection $\nabla$ on $M$ to a connection $\Lambda(\Gamma, \nabla) : T\tilde{T}_k^r M \to l_k^r$.

Proof. Every $r$-order linear connection $\lambda : TM \to J^r TM$ on $M$ gives rise to a generalized connection $\Gamma$ on $\tilde{T}_k^r M \to M$ (or $K^r_k M \to M$) because $\tilde{T}_k^r$ (or $K^r_k$) is of order $r$. Next we apply Theorem 2 and Remark 2. ■

The following example shows that Theorem 1 is not true for an arbitrary natural bundle $F$ instead of $C^r_0$.

EXAMPLE 1. Let $F = P^{r+1} : \mathcal{M}f_n \to \mathcal{F}M$ be the bundle functor from Remark 1. Consider a connection $\omega : T\tilde{T}_k^r \mathbb{R}^n \to l_k^r$ on $\kappa : T\tilde{T}_k^r \mathbb{R}^n \to K^r_k \mathbb{R}^n$. Given a section $\varphi$ of $FM$ define $A(\varphi) : T\tilde{T}_k^r M \to l_k^r$ by $A(\varphi)(v) = \omega(T\tilde{T}_k^r \varphi^{-1}(v)))$ for $j_0^{r+1} \varphi = \varphi(x)$, $v \in (T\tilde{T}_k^r M)_x$, $x \in M$. Then $A(\varphi)$ is a connection on $\kappa : \tilde{T}_k^r M \to K^r_k M$. In this way we obtain a natural operator $A : F \sim Q(\tilde{T}_k^r \to K^r_k)$.

II. Torsion free linear connections of order $r$ and the bundle functor of contact $(k,r)$-coelements

6. The bundle functor $K^r_k$ of contact $(k,r)$-coelements. Let $n$ and $k$ be natural numbers. For every $n$-manifold $M$ we have the bundle $T^r_k \mathbb{R}^k = J^r(M, \mathbb{R}^k)_0$ over $M$ of so-called $(k,r)$-covelocities on $M$. Every embedding $\varphi : M \to N$ of $n$-manifolds induces a bundle map $T^r_k \varphi : T^r_k M \to T^r_k N$
by $T^*_k \varphi(j^r_x \gamma) = j^r_x(\varphi \circ \varphi^{-1})$ for $\gamma : M \to \mathbb{R}^k$, $x \in M$, $\gamma(x) = 0$. The correspondence $T^*_k : \mathcal{M} f_n \to \mathcal{F} M$ is a bundle functor of order $r$.

Every $\xi = j^r_0 \psi \in L^r_k$ induces a natural automorphism $\tilde{\xi} : T^*_k \to T^*_k$, $\bar{\xi} : T^*_k M \to T^*_k M$, defined by $\tilde{\xi}(j^r_x \gamma) = j^r_x(\psi \circ \gamma)$ for $\gamma : M \to \mathbb{R}^k$, $x \in M$, $\gamma(x) = 0$. This defines a group homomorphism $L^r_k \to \text{Aut}(T^*_k)$.

If $n \geq k + 1$ then the above homomorphism is an isomorphism, i.e. $\text{Aut}(T^*_k) \cong L^r_k$.

Assume $n \geq k + 1$. For every $n$-manifold $M$, $\tilde{T}^*_k M = \text{reg} J^r(M, \mathbb{R}^k)_0 = \{ j^r_x \gamma | \gamma : M \to \mathbb{R}^k, x \in M, \gamma(x) = 0, \text{rank}(d_x \gamma) = k \}$ is an open subbundle of $T^*_k M$. Elements of $\tilde{T}^*_k M$ are called regular $(k, r)$-covelocities on $M$. For every embedding $\varphi : M \to N$ of $n$-manifolds, $T^*_k \varphi(\tilde{T}^*_k M) \subset \tilde{T}^*_k N$, and we let $\tilde{T}^*_k \varphi : \tilde{T}^*_k M \to \tilde{T}^*_k N$ be the restriction of $T^*_k \varphi$. The correspondence $\tilde{T}^*_k : \mathcal{M} f_n \to \mathcal{F} M$ is a bundle functor of order $r$.

For every $n$-manifold $M$, $\tilde{T}^*_k M$ is invariant with respect to the action of $L^r_k = \text{Aut}(T^*_k)$ on $T^*_k M$. So, we have (by restriction) the left action of $L^r_k$ on $\tilde{T}^*_k M$, and the quotient bundle $K^r_k M = \tilde{T}^*_k M / L^r_k$ over $M$ of so called contact $(k, r)$-coelements. This bundle was introduced in [9]. The quotient projection $\kappa^* : \tilde{T}^*_k M \to K^r_k M$ is a principal fiber bundle with structure group $L^r_k$. The right principal bundle action of $L^r_k$ on $\tilde{T}^*_k M$ is given by $v, \xi = \xi^{-1}(v)$ for $\xi \in L^r_k$, $v \in \tilde{T}^*_k M$. For every embedding $\varphi : M \to N$ of $n$-manifolds, $\tilde{T}^*_k \varphi$ commutes with the left action of $L^r_k$ on $\tilde{T}^*_k M$ and we have the quotient map $K^r_k \varphi : K^r_k M \to K^r_k N$. Then $\tilde{T}^*_k \varphi$ is a principal bundle morphism covering $K^r_k \varphi$. The correspondence $K^r_k : \mathcal{M} f_n \to \mathcal{F} M$ is a bundle functor of order $r$ (see [9]).

7. Principal fiber bundle $\kappa^0 : P^0 \to Q^0$ and non-existence of $\text{GL}(\mathbb{R}^n)$-invariant connections on $P^0$. Let $n \geq k + 1$. Let $\kappa^0 : P^0 \to Q^0$ be the restriction of $\kappa^* : \tilde{T}^*_k \mathbb{R}^n \to K^r_k \mathbb{R}^n$ to the fibers over $0 \in \mathbb{R}^n$. Then $\kappa^0$ is a principal fiber bundle (a principal subbundle of $\kappa^*$) with structure group $L^r_k$. Its right action of $L^r_k$ on $P^0$ is the restriction of the right action of $L^r_k$ on $\tilde{T}^*_k \mathbb{R}^n$.

There is a left action $\alpha^0 : \text{GL}(\mathbb{R}^n) \times P^0 \to P^0$ defined by $\alpha^0(\eta, p) = \tilde{T}^*_k \eta(p)$ for $\eta \in \text{GL}(\mathbb{R}^n)$, $p \in P^0$. This action covers the left action $\beta^0 : \text{GL}(\mathbb{R}^n) \times Q^0 \to Q^0$ given by $\beta^0(\eta, q) = K^r_k \eta(q)$ for $\eta \in \text{GL}(\mathbb{R}^n)$, $q \in Q^0$. For every $\eta \in \text{GL}(\mathbb{R}^n)$ the mapping $\alpha^0_\eta = \alpha^0(\eta, \cdot) : P^0 \to P^0$ is a principal fiber bundle isomorphism covering $\beta^0_\eta = \beta^0(\eta, \cdot) : Q^0 \to Q^0$.

A connection $\omega : TP^0 \to L^r_k = \mathcal{L}(L^r_k)$ on $P^0$ is called $\text{GL}(\mathbb{R}^n)$-invariant if $(\alpha^0_\eta)^* \omega = \omega$ for every $\eta \in \text{GL}(\mathbb{R}^n)$.

Proposition 2. There are no $\text{GL}(\mathbb{R}^n)$-invariant connections $\omega : TP^0 \to L^r_k$ on $P^0$. 
Proof. Similar to the proof of Proposition 1. ■

8. Non-existence of natural operators $C^r_0 \rightsquigarrow Q(\tilde{T}_k^{r*} \to K_k^r)$. The definition of natural operators $C^r_0 \rightsquigarrow Q(\tilde{T}_k^{r*} \to K_k^r)$ is similar to that of natural operators $C^r_0 \rightsquigarrow Q(\tilde{T}_k^r \to K_k^r)$.

The second main result of this paper is the following theorem.

THEOREM 3. Let $n$, $k$ and $r$ be natural numbers such that $n \geq k + 1$. There are no natural operators $C^r_0 \rightsquigarrow Q(\tilde{T}_k^{r*} \to K_k^r)$ over $n$-manifolds.

Proof. The proof is similar to that of Theorem 1. One uses Proposition 2 instead of Proposition 1. ■

We have the following obvious corollaries of Theorem 3.

COROLLARY 4. Let $n$, $k$ and $r$ be natural numbers such that $n \geq k + 1$. There are no natural operators $C^r \rightsquigarrow Q(\tilde{T}_k^{r*} \to K_k^r)$ over $n$-manifolds.

COROLLARY 5. Let $n$, $k$ and $r$ be natural numbers such that $n \geq k + 1$. There are no canonical connections $\omega : T\tilde{T}_k^{r*}M \to l_k^r$ on $\tilde{T}_k^{r*}M$ over $n$-manifolds.

9. A generalization. Using the same method as in the proof of Theorem 2 we obtain the following general fact.

THEOREM 4. Let $n, r, k$ be natural numbers such that $n \geq k + 1$. Let $F : \mathcal{M}_f \to \mathcal{F}M$ be a bundle functor such that there is a $GL(\mathbb{R}^n)$-invariant element $\mu_0^0 \in F_0 \mathbb{R}^n$. Then there are no natural operators $\Lambda : F \rightsquigarrow Q(\tilde{T}_k^{r*} \to K_k^r)$.

COROLLARY 6. Let $n, r, k$ be natural numbers such that $n \geq k + 1$. There are no natural operators $\Lambda$ sending a generalized connection $\Gamma$ on $\tilde{T}_k^{r*}M \to M$ (or $K_k^{r*}M \to M$) and a linear connection $\nabla$ on $M$ to a connection $\Lambda(\Gamma, \nabla) : T\tilde{T}_k^{r*}M \to l_k^r$.

REMARK 3. Modifying Example 1 we can show that Theorem 3 is not true for an arbitrary natural bundle $F$ instead of $C^r_0$.

References


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