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On the Kuratowski convergence of analytic sets

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Abstract. We discuss some conditions which guarantee that the Kuratowski limit of a sequence of analytic sets is a Nash set.

1. Preliminaries. Let $X \subset \mathbb{C}^n$ be a locally closed set. Let $\mathcal{F}_X$ denote the family of all closed subsets of X. In $\mathcal{F}_X$ we have the topology generated by the sets

$$\mathcal{U}(S, K) = \{F \in \mathcal{F}_X : F \cap K = \emptyset,\ F \cap U \neq \emptyset \text{ for } U \in S\},$$

where $K \subset X$ is compact and S is a finite family of open subsets of X. This topology has the following properties:

(1) $\mathcal{F}_X$ is a metrizable compact space. The convergence in this topology is called the *Kuratowski convergence*.
(2) $F_\nu \to F$ in $\mathcal{F}_X$ if and only if for any $x \in F$ there exist $x_\nu \in F_\nu$ such that $x_\nu \to x$ and for every compact subset $K \subset X \setminus F$ we have $K \cap F_\nu = \emptyset$ for all sufficiently large ν.
(3) The map $\mathcal{F}_X \times \mathcal{F}_X \ni (A, B) \mapsto A \cup B \in \mathcal{F}_X$ is continuous.
(4) If $Y \subset X$ is open, then the map $\mathcal{F}_X \ni F \mapsto F \cap Y \in \mathcal{F}_Y$ is continuous.

For (1) and (2), see [10, Lemmas 1 and 2]. The remaining two properties follow easily from (2).

Remark 1.1. *Let $\Omega \subset \mathbb{C}^n$ be open. If $A_\nu \to A$ in $\mathcal{F}_\Omega$ and A_ν are analytic and irreducible, then*

(1) *A need not be analytic.*
(2) *If A is analytic, it need not be irreducible.*

Proof. Put $A_\nu := \{z^{p_\nu} = y^{q_\nu}\} \subset \mathbb{C}^2$, where p_ν, q_ν are positive integers such that $\gcd(p_\nu, q_\nu) = 1$ and $p_\nu/q_\nu \to \sqrt{2}$. Passing to a subsequence if necessary we may assume that $A_\nu \to A$. Suppose that A is analytic. Note that $(0,0) \notin \operatorname{Int} A$. Clearly, if a convergent power series $f(z, y) = \sum a_{ij} z^i y^j$

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vanishes on the curve $\Gamma = \{(t, t^{\sqrt{2}}) \in \mathbb{R}^2 : t \in [0, \varepsilon]\}$, where $\varepsilon > 0$, then $f \equiv 0$ (because the map $\mathbb{N}^2 \ni (i,j) \mapsto i + j\sqrt{2} \in \mathbb{R}$ is injective). Since $\Gamma \subset A$, we easily obtain a contradiction.

For the second part of the remark, we put $A_\nu := \{zy = 1/\nu\} \subset \mathbb{C}^2$ and $A := \{zy = 0\}$.

Let $G'_k(\mathbb{C}^n)$ denote the set of all k-dimensional affine subspaces of $\mathbb{C}^n$ (cf. [6]). Łojasiewicz defines a topology (we call it τ) in $G'_k(\mathbb{C}^n)$ in such a way that it is metrizable and has the following property: for any $L = z_0 + \sum_{j=1}^k \mathbb{C}z_j \in G'_k(\mathbb{C}^n)$ and $L_\nu \in G'_k(\mathbb{C}^n)$ ($\nu \in \mathbb{N}$),

$$L_\nu \xrightarrow{\tau} L \ \Leftrightarrow\ L_\nu = z_0^\nu + \sum_{j=1}^k \mathbb{C}z_j^\nu \text{ for some } z_j^\nu \to z_j \ \ (j = 0, 1, \ldots, k).$$

On the other hand, since $G'_k(\mathbb{C}^n) \subset \mathcal{F}_{\mathbb{C}^n}$, we have the subspace topology in $G'_k(\mathbb{C}^n)$ (see also [12]). The following lemma states that these two topologies coincide.

Lemma 1.2. *Suppose that (L_ν) is a sequence in $G'_k(\mathbb{C}^n)$ and let $L \in G'_k(\mathbb{C}^n)$. Then*

$$L_\nu \xrightarrow{\tau} L \ \Leftrightarrow\ L_\nu \xrightarrow{\mathcal{F}_{\mathbb{C}^n}} L.$$

Proof. Assume that $L_\nu \xrightarrow{\tau} L$, where $L_\nu, L \in G'_k(\mathbb{C}^n)$. Let $L = z_0 + \sum_{j=1}^k \mathbb{C}z_j$. We know that $L_\nu = z_0^\nu + \sum_{j=1}^k \mathbb{C}z_j^\nu$ for some $z_j^\nu \to z_j$ ($j = 0, 1, \ldots, k$). Obviously, for any $x \in L$ there exist $x^\nu \in L_\nu$ such that $x^\nu \to x$. Take now a compact set $K \subset \mathbb{C}^n$ such that $L \cap K = \emptyset$. We need to show that for some $\nu_0 \in \mathbb{N}$ we have $L_\nu \cap K = \emptyset$ whenever $\nu \geq \nu_0$. Suppose that this is not the case. Passing to a subsequence if necessary we may assume that $L_\nu \cap K \neq \emptyset$ for any ν. So let $y^\nu \in L_\nu \cap K$. We have

$$(\star) \qquad y^\nu = z_0^\nu + \sum_{j=1}^k \alpha_j^\nu z_j^\nu$$

for some $\alpha_j^\nu \in \mathbb{C}$ ($j = 1, \ldots, k$). Take $1 \leq l_1 < \cdots < l_k \leq n$ such that $\det[\pi(z_1), \ldots, \pi(z_k))] \neq 0$, where

$$\pi : \mathbb{C}^n \ni (t_1, \ldots, t_n) \mapsto (t_{l_1}, \ldots, t_{l_k}) \in \mathbb{C}^k.$$

Then for each ν large enough we have $|\det[\pi(z_1^\nu), \ldots, \pi(z_k^\nu))]| \geq M$, where M is a positive constant. We can treat $(\star)$ as a system of linear equations, where α_j^ν are the unknowns. By Cramer's rule applied to the equation $\pi(y^\nu) = \pi(z_0^\nu) + \sum_{j=1}^k \alpha_j^\nu \pi(z_j^\nu)$, all α_j^ν are bounded. Passing again to a subsequence if necessary we can assume that $\alpha_j^\nu \to \alpha_j \in \mathbb{C}$. This is a contradiction, since $y := z_0 + \sum_{j=1}^k \alpha_j z_j \in L \cap K$.

Assume now that $L_\nu \xrightarrow{\mathcal{F}_{\mathbb{C}^n}} L$, where $L_\nu, L \in G'_k(\mathbb{C}^n)$ and $L = z_0 + \sum_{j=1}^k \mathbb{C}z_j$. Take $z_0^\nu \in L_\nu$ such that $z_0^\nu \to z_0$, and $a_j^\nu \in L_\nu$ such that $a_j^\nu \to z_0 + z_j$ for $j = 1, \ldots, k$. Put $z_j^\nu := a_j^\nu - z_0^\nu$ $(j = 1, \ldots, k)$. Obviously, $z_j^\nu \to z_j$. Moreover, $z_1^\nu, \ldots, z_k^\nu \in L_\nu - z_0^\nu$ are linearly independent for all ν large enough and thus $L_\nu = z_0^\nu + \sum_{j=1}^k \mathbb{C}z_j^\nu$.

Let $\Omega \subset \mathbb{C}^n$ be an open set. We denote by $\mathcal{A}_k(\Omega)$ the set of all analytic sets in Ω of pure dimension k (1). Since $\mathcal{A}_k(\Omega) \subset \mathcal{F}_\Omega$, we have the subspace topology in $\mathcal{A}_k(\Omega)$. Recall the following theorem due to Tworzewski and Winiarski.

THEOREM 1.3 (Tworzewski, Winiarski). *Let $W \in \mathcal{A}_k(\Omega)$ and $V_0 \in \mathcal{A}_{n-k}(\Omega)$ be such that $V_0 \cap W \in \mathcal{A}_0(\Omega)$. Then the mapping*

$$\mathcal{A}_{n-k}(\Omega) \ni V \mapsto V \cap W \in \mathcal{F}_W$$

is continuous at V_0.

Proof. Cf. [10, Corollary 2].

COROLLARY 1.4. *Let $L \in G'_{n-k}(\mathbb{C}^n)$ and $A \in \mathcal{A}_k(U)$, where $U \subset \mathbb{C}^n$ is open. Suppose that $L \cap A$ contains (at least) p isolated points. Then*

- *There exists an open neighbourhood of L in $G'_{n-k}(\mathbb{C}^n)$ such that for any H from this neighbourhood we have*

$$\#(H \cap A) \geq p.$$

- *There exists an open neighbourhood of A in $\mathcal{A}_k(U)$ such that for any V from this neighbourhood we have*

$$\#(L \cap V) \geq p.$$

Proof. We prove only the first part, since the proof of the second is quite similar. Let $\Omega \subset U$ be open, bounded and such that $\#(L \cap \Omega \cap A) = p$. Suppose that there exist $H_\nu \in G'_{n-k}(\mathbb{C}^n)$ such that $H_\nu \to L$ and $\#(H_\nu \cap A) < p$. Obviously, $H_\nu \cap \Omega \to L \cap \Omega$ in $\mathcal{F}_\Omega$. By the previous theorem, we get $H_\nu \cap \Omega \cap A \to L \cap \Omega \cap A$ in $\mathcal{F}_{\Omega \cap A}$, but this is impossible.

2. Degree of analytic set. Let $V \subset \mathbb{C}^n$ be an algebraic set of pure dimension k. Recall that we define the degree $\deg V$ of V as the unique number p such that $\#(L \cap V) = p$ for each L from some open dense (2) subset of $G'_{n-k}(\mathbb{C}^n)$ (cf. [6]).

REMARK 2.1. Corollary 1.4 implies that

$$\deg V = \sup\{\#(L \cap V) : L \in G'_{n-k}(\mathbb{C}^n),\ \#(L \cap V) < \infty\}.$$

(1) We assume that $\emptyset$ is of pure dimension k for any $k \in \mathbb{N}$.

(2) In the topology τ which is equivalent by Lemma 1.2 to the topology induced from $\mathcal{F}_{\mathbb{C}^n}$.

Recall the following

THEOREM 2.2. *If A is an analytic set of dimension k in some open set $\Omega \subset \mathbb{C}^n$, then $L \cap A$ is discrete for each L from a dense subset of $G_{n-k}(\mathbb{C}^n)$* (3).

Proof. Cf. [6, p. 185].

Assume now that $\Omega \subset \mathbb{C}^n$ is an open set and let A be an analytic subset of Ω. We define $\mathrm{dg}(A;\Omega)$ as follows:

(1) If A is of pure dimension k, then we put
$$\mathrm{dg}(A;\Omega) := \sup\{\#(L\cap A) : L \in G'_{n-k}(\mathbb{C}^n),\ L\cap A \text{ is a discrete set}\}\ (^4).$$

(2) If $A = \bigcup A_l$, where A_l is the union of all irreducible components of A of dimension l, then we put
$$\mathrm{dg}(A;\Omega) := \sum_l \mathrm{dg}(A_l;\Omega) \in \mathbb{N} \cup \{\infty\}.$$

REMARK 2.3. *We have the following properties*:

- *If A is an analytic subset of Ω, then*
$$\mathrm{dg}(A;\Omega) \le \sum_i \mathrm{dg}(C_i;\Omega),$$
where $A = \bigcup C_i$ is the decomposition of A into irreducible components.
- *If $V \subset \mathbb{C}^n$ is algebraic of pure dimension, then*
$$\mathrm{dg}(V;\mathbb{C}^n) = \deg V.$$

Proof. The first part is trivial. The second follows immediately from Remark 2.1.

Let us add that for any analytic set $A \subset \mathbb{C}^n$ we have:
$$A \text{ is algebraic} \ \Leftrightarrow\ \mathrm{dg}(A;\mathbb{C}^n) < \infty.$$
This is a consequence of the following two theorems. The first one is due to Gruman. The second is a more precise version of Theorem 2.2.

THEOREM 2.4 (Gruman). *Suppose that $A \in \mathcal{A}_k(\mathbb{C}^n)$. If $L \cap A$ is finite for any $L \in E$, where $E \subset G_{n-k}(\mathbb{C}^n)$ is a set of positive volume, then A is algebraic.*

Proof. Cf. [7, Corollary 4.4].

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(3) $G_{n-k}(\mathbb{C}^n)$ denotes the Grassmann manifold of all $(n-k)$-dimensional subspaces of $\mathbb{C}^n$.

(4) Note that Theorem 2.2 implies in particular that the family of all $L \in G'_{n-k}(\mathbb{C}^n)$ such that $L \cap A$ is a discrete set is nonempty.

THEOREM 2.5. *Assume that A is an analytic set of dimension k in some open set $\Omega \subset \mathbb{C}^n$. Then there exists a subset $S \subset G_{n-k}(\mathbb{C}^n)$ of measure zero such that $L \cap A$ is discrete for each $L \in G_{n-k}(\mathbb{C}^n) \setminus S$.*

Proof. This follows from a more general result stated in the Appendix.

Let N be a Nash subset of an open set $\Omega \subset \mathbb{C}^n$ (see [9] for the definition and properties of Nash sets). Recall a characterization of Nash sets:

THEOREM 2.6 (Tworzewski). *If $N \neq \emptyset$ is irreducible, then there exists an irreducible algebraic set $V \subset \mathbb{C}^n$ such that N is an irreducible component of $V \cap \Omega$.*

Proof. Cf. [9, Theorem 2.10].

REMARK 2.7. It is easy to see that the algebraic set V in the above theorem is unique.

We define $\deg(N; \Omega)$ as follows:

(1) If $N \neq \emptyset$ is irreducible, then we put

$$\deg(N; \Omega) := \deg V,$$

where V is from Theorem 2.6. Moreover, we put $\deg(\emptyset; \Omega) := 0$.

(2) If $N = \bigcup N_i$ is the decomposition into irreducible components ([5]), then we put

$$\deg(N; \Omega) := \sum_i \deg(N_i; \Omega) \in \mathbb{N} \cup \{\infty\}.$$

REMARK 2.8. We have the following properties:

(1) If $V \subset \mathbb{C}^n$ is algebraic of pure dimension, then

$$\deg(V; \mathbb{C}^n) = \deg V.$$

(2) If N is a Nash subset of $\Omega \subset \mathbb{C}^n$, then

$$\mathrm{dg}(N; \Omega) \leq \deg(N; \Omega).$$

The above inequality may be strict even if N is irreducible.

Proof. Part (1) follows from the following fact: if $V = V_1 \cup \cdots \cup V_p$ is the decomposition into irreducible components, then $\deg V = \deg V_1 + \cdots + \deg V_p$ (cf. [6, p. 310]).

Let us pass to (2). By Remark 2.3, we may restrict ourselves to the case when $N \neq \emptyset$ is irreducible. Put $k := \dim N$. Take $L \in G'_{n-k}(\mathbb{C}^n)$ such that $L \cap N$ is a discrete set. We need to show that $\#(L \cap N) \leq \deg(N; \Omega) = \deg V$, where V is the unique algebraic set as in Theorem 2.6. Suppose that this is not the case. Corollary 1.4 implies that $\#(H \cap N) > \deg V$ for each H in

([5]) They are Nash subsets of Ω (cf. [9, Theorem 2.11]).

some nonempty open set in $G'_{n-k}(\mathbb{C}^n)$. Hence $\#(H \cap V) > \deg V$. This gives a contradiction.

To see that the inequality in (2) may be strict consider the set $A := \{(z, y) \in \mathbb{C}^2 : y = z^2 + z^3\}$. Note that $\deg(A \cap \Omega; \Omega) \geq 3$ whenever $\Omega \subset \mathbb{C}^2$ is an open set such that $A \cap \Omega \neq \emptyset$. Put $\Omega := \{|z| < 1/3\} \times \mathbb{C}$. Suppose that $\mathrm{dg}(A \cap \Omega; \Omega) \geq 3$. Since $A \cap \Omega$ is of pure dimension 1, it follows that there exist some $a, b \in \mathbb{C}$ such that the equation $az + b = z^2 + z^3$ has three different roots z_1, z_2, z_3 in $\{|z| < 1/3\}$. We obtain a contradiction, because $z_1 + z_2 + z_3 = -1$.

3. Limits of analytic sets. Recall the main result of [11].

THEOREM 3.1 (Tworzewski, Winiarski). *The set* $\mathcal{H}_d^k(\mathbb{C}^n) := \{V \in \mathcal{A}_k(\mathbb{C}^n) :$ *V is algebraic*, $\deg V \leq d\}$ *is compact in* $\mathcal{F}_{\mathbb{C}^n}$.

Proof. Cf. [11, Theorem 2].

Tworzewski and Winiarski use among others the following tools:

THEOREM 3.2 (Bishop). *Let $\Omega \subset \mathbb{C}^n$ be open and let $A_\nu \in \mathcal{A}_k(\Omega)$, $\nu = 1, 2, \ldots$. If the $2k$-dimensional volumes of A_ν are finite and bounded by a positive constant independent of ν and $A_\nu \to A$ in $\mathcal{F}_\Omega$, then $A \in \mathcal{A}_k(\Omega)$.*

Proof. Cf. [2, Theorem 1]. See also [8, Theorem C].

THEOREM 3.3 (Griffiths). *Let $V \subset \mathbb{C}^n$ be an algebraic set of pure dimension k. Then*

$$\mathrm{vol}_{2k}(V \cap B[r]) \leq \alpha(k) \cdot \deg V \cdot r^{2k}$$

for any $r > 0$, where $B[r] := \{z \in \mathbb{C}^n : \|z\| < r\}$ and $\alpha(k)$ is a positive constant independent of V.

Proof. Cf. [4, Theorems 1.3 and 1.8].

We will now prove a refined version of Theorem 3.1.

THEOREM 3.4. *Suppose that $A \subset \mathbb{C}^n$ and let Ω be open in $\mathbb{C}^n$. Assume that there exists a sequence (N_ν) of Nash subsets of Ω such that $N_\nu \to A \cap \Omega$ in $\mathcal{F}_\Omega$ and* $\deg(N_\nu; \Omega) \leq d$. *Then $A \cap \Omega$ is a Nash subset of Ω. Moreover, if additionally $\Omega = \mathbb{C}^n$ or A is analytic irreducible such that $A \cap \Omega \neq \emptyset$, then A is algebraic.*

Proof. Since $\deg(N_\nu; \Omega) \leq d$, each N_ν contains at most d irreducible components. Passing to a subsequence if necessary we may assume that for each ν we have the following decomposition into irreducible components:

$$N_\nu = \bigcup_{i=1}^{s} \bigcup_{j=1}^{p_i} N_\nu^{i,j},$$

where for any $i \in \{1, \dots, s\}$, $N_\nu^{i,j}$ $(j = 1, \dots, p_i)$ denote the components of dimension k_i. Obviously,

$$\sum_{i=1}^{s} \sum_{j=1}^{p_i} \deg(N_\nu^{i,j}; \Omega) \le d.$$

For each $N_\nu^{i,j}$ take the unique algebraic set $V_\nu^{i,j}$ as in Theorem 2.6. Then:

- $N_\nu^{i,j}$ is an irreducible component of $V_\nu^{i,j} \cap \Omega$,
- $\deg V_\nu^{i,j} = \deg(N_\nu^{i,j}; \Omega) \le d$,
- $V_\nu^{i,j}$ is irreducible of dimension k_i.

Again passing to a subsequence if necessary we may assume that $N_\nu^{i,j} \to N^{i,j}$ in $\mathcal{F}_\Omega$ and $V_\nu^{i,j} \to V^{i,j}$ in $\mathcal{F}_{\mathbb{C}^n}$. It follows from Theorem 3.1 that $V^{i,j}$ is an algebraic set of pure dimension k_i. By Theorem 3.3, we get

$$\mathrm{vol}_{2k_i}(V_\nu^{i,j} \cap B[r]) \le \alpha(k_i) \cdot d \cdot r^{2k_i}$$

and thus

$$\mathrm{vol}_{2k_i}(N_\nu^{i,j} \cap B[r]) \le \alpha(k_i) \cdot d \cdot r^{2k_i}.$$

Theorem 3.2 implies that $N^{i,j} \in \mathcal{A}_{k_i}(\Omega)$. Since $N^{i,j} \subset V^{i,j} \cap \Omega$ and $V^{i,j}$ is of pure dimension k_i, $N^{i,j}$ is a union of some irreducible components of $V^{i,j} \cap \Omega$. Therefore $N^{i,j}$ is a locally finite (in Ω) union of Nash sets. By Proposition 2.6 in [9], $N^{i,j}$ is a Nash subset of Ω. Now it is enough to see that

$$A \cap \Omega = \bigcup_{i=1}^{s} \bigcup_{j=1}^{p_i} N^{i,j}.$$

It follows from the proof that for $\Omega = \mathbb{C}^n$, A is algebraic. If A is analytic irreducible and $A \cap \Omega \ne \emptyset$, then A is algebraic as well (cf. [9, Theorems 2.9 and 2.12]).

COROLLARY 3.5. *Let $\Omega \subset \mathbb{C}^n$ be open and let $N \subset \Omega \times \mathbb{C}^m$ be a Nash set with a finite number of irreducible components. Then $\overline{\pi(N)}^{\Omega}$ is Nash in Ω, where $\pi : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^n$ denotes the natural projection.*

Proof. It is enough to repeat the proof of Corollary in [11].

REMARK 3.6. One cannot omit the assumption that N has a finite number of irreducible components. To see this consider $N := \{(1/\nu, \nu) : \nu \in \mathbb{N}\}$ $(n = m = 1)$. In this case we have $\overline{\pi(N)} = \{1/\nu : \nu \in \mathbb{N}\} \cup \{0\}$, which is not even analytic.

If we replace $\deg(N_\nu; \Omega)$ in Theorem 3.4 by $\mathrm{dg}(N_\nu; \Omega)$, then the conclusion is no longer true, even if we assume that N_ν are irreducible and A is analytic irreducible. We give an example in the next section. However, we can prove the following theorem.

THEOREM 3.7. *Let $A \in \mathcal{A}_k(\mathbb{C}^n)$ and let $U_1 \subset U_2 \subset \cdots$ be a sequence of open subsets of $\mathbb{C}^n$ such that $\bigcup U_i = \mathbb{C}^n$. Assume that there is some positive integer d such that for each i there exists a sequence (A_{ij}) in $\mathcal{A}_k(U_i)$ with the following properties*:

- $A_{ij} \to A \cap U_i$ *in* $\mathcal{F}_{U_i}$,
- $\mathrm{dg}(A_{ij}; U_i) \le d$.

Then A is algebraic.

Proof. Fix $i \in \mathbb{N}$. By Theorem 2.5, we get a subset $S_i \subset G_{n-k}(\mathbb{C}^n)$ of measure zero such that the sets $L \cap A_{ij}$ $(j = 1, 2, \ldots)$ and $L \cap A \cap U_i$ are discrete for each $L \in G_{n-k}(\mathbb{C}^n) \setminus S_i$. Fix $L \in G_{n-k}(\mathbb{C}^n) \setminus S_i$. Since $\mathrm{dg}(A_{ij}; U_i) \le d$, we have $\#(L \cap A_{ij}) \le d$. Assume that $\#(L \cap A \cap U_i) > d$. Then $L \cap A \cap U_i$ contains $d + 1$ isolated points. Now it is enough to use Corollary 1.4 to get a contradiction. To summarize, for each $i \in \mathbb{N}$ and $L \in G_{n-k}(\mathbb{C}^n) \setminus S_i$ we have

$$\#(L \cap A \cap U_i) \le d.$$

Put $S := \bigcup S_i$. For any $L \in G_{n-k}(\mathbb{C}^n) \setminus S$ we have $\#(L \cap A) \le d$. Now the result follows from Theorem 2.4.

4. An example. We will now give an example announced in the previous section. First, we need some lemmata.

For any $\nu \in \mathbb{N}$ put

$$h_\nu(z) := \frac{z^\nu}{\nu!} + \frac{z^{\nu-1}}{(\nu-1)!} + \cdots + z + 1.$$

LEMMA 4.1. *If the equation $h_\nu(z) = az + b$, where $a, b \in \mathbb{C}$, has two different roots in $\{|z| \le 1\}$, then $|a| \le e$ and $|b| \le 2e$. The same is true for the equation $e^z = az + b$.*

Proof. Let $u, w \in \{|z| \le 1\}$ be two different roots of the equation $h_\nu(z) = az + b$. Then

$$a(u - w) = \sum_{j=1}^{\nu} \frac{1}{j!}\,(u^j - w^j).$$

Therefore we have

$$|a| \le 1 + \frac{1}{2!} \cdot 2 + \frac{1}{3!} \cdot 3 + \cdots + \frac{1}{\nu!} \cdot \nu < e$$

and

$$|b| \le |h_\nu(u)| + |au| \le 2e.$$

Just in the same way we prove the second statement.

LEMMA 4.2. *There exists a positive integer d such that for any $a, b \in \mathbb{C}$,*

$$\#\{z \in \mathbb{C} : |z| \le 1,\ e^z = az + b\} \le d.$$

Proof. By Lemma 4.1, we may assume that $|a| \le e$ and $|b| \le 2e$. Consider the set

$$K := \{u \in \mathbb{R}^6 : e^{x_1+iy_1} = (x_2+iy_2)(x_1+iy_1)+x_3+iy_3\} \cap T,$$

where $T := \{\|u_1\| \le 1,\ \|u_2\| \le e,\ \|u_3\| \le 2e\} \subset \mathbb{R}^6$, $u = (u_1, u_2, u_3)$ and $u_j = (x_j, y_j) \in \mathbb{R}^2$ for $j = 1, 2, 3$. Note that K is a semianalytic (6) compact subset of $\mathbb{R}^6$. Since for any $(u_2, u_3) \in \mathbb{R}^2 \times \mathbb{R}^2$ the fibre $K_{(u_2,u_3)}$ is finite, there exists a positive integer d such that $\# K_{(u_2,u_3)} \le d$.

LEMMA 4.3. *There exists $\nu_0 \in \mathbb{N}$ such that for any $\nu \ge \nu_0$ and any $a, b \in \mathbb{C}$ the equation $h_\nu(z) = az + b$ has at most $2d$ distinct roots in the set $\{|z| \le 1\}$, where d is from the previous lemma.*

Proof. Suppose that this is not the case. Then we can find a subsequence $\nu_k \to \infty$ and sequences a_{ν_k}, b_{ν_k} of complex numbers such that for any k the equation $h_{\nu_k}(z) = a_{\nu_k} z + b_{\nu_k}$ has at least $2d+1$ distinct roots in the set $\{|z| \le 1\}$. Denote these roots by $y_1^{\nu_k}, \ldots, y_{2d+1}^{\nu_k}$. Lemma 4.1 implies that a_{ν_k}, b_{ν_k} are bounded. Passing to a subsequence if necessary we may assume that

$$y_j^{\nu_k} \to y_j \ (j = 1, \ldots, 2d+1), \qquad a_{\nu_k} \to a, \qquad b_{\nu_k} \to b,$$

for some $y_j \in \{|z| \le 1\}$ and $a, b \in \mathbb{C}$. Put $g(z) = e^z - az - b$ and $g_{\nu_k}(z) = h_{\nu_k}(z) - a_{\nu_k} z - b_{\nu_k}$ for $z \in \mathbb{C}$. Obviously, $g_{\nu_k} \to g$ uniformly in $\{|z| \le 2\}$. It is easy to see that $g(y_j) = 0$ for $j = 1, \ldots, 2d+1$. By the previous lemma, at least three numbers among $y_1, \ldots, y_{2d+1}$ are equal, say $y_1 = y_2 = y_3$. By the Hurwitz theorem, y_1 is a zero of g of multiplicity at least 3. This is impossible, because $g''(y_1) = e^{y_1} \ne 0$.

EXAMPLE 4.4. Let $\Omega := \{|z| < 1\} \times \mathbb{C} \subset \mathbb{C}^2$. For any $\nu \in \mathbb{N}$ put

$$N_\nu := \{(z, y) \in \mathbb{C}^2 : y = h_\nu(z)\} \cap \Omega$$

and

$$A := \{(z, y) \in \mathbb{C}^2 : y = e^z\}.$$

Note that each N_ν is an irreducible Nash subset of Ω and A is an irreducible analytic set in $\mathbb{C}^2$. Moreover, $\mathrm{dg}(N_\nu; \Omega) \le 2d$ for $\nu \ge \nu_0$ (cf. Lemma 4.3) and $N_\nu \to A \cap \Omega$ in $\mathcal{F}_\Omega$. Nevertheless, A is not algebraic.

5. Appendix.

In this section we present a general result concerning o-minimal structures (7) from which Theorem 2.5 follows immediately.

(6) See [5] and [1] for the definition and properties of semianalytic sets.

(7) The definition and properties of o-minimal structures can be found in [3].

Let J denote the set of all increasing functions $\alpha : \{1,\dots,n-k\} \to \{1,\dots,n\}$. For each $\alpha \in J$ put

$$V_\alpha := \sum_{j=1}^{n-k} \mathbb{C}e_{\alpha(j)}, \qquad W_\alpha := \sum_{\nu\notin \operatorname{im}\alpha} \mathbb{C}e_\nu,$$

where $e_1,\dots,e_n$ is the usual basis in $\mathbb{C}^n$. We denote by $\Omega(V_\alpha)$ the subset of $G_k(\mathbb{C}^n)$ consisting of all linear complements of V_α. Recall that the family of maps

$$\varphi_\alpha : \mathcal{L}(W_\alpha, V_\alpha) \ni L \mapsto \widehat{L} \in \Omega(V_\alpha),$$

where $\alpha \in J$ and $\widehat{L} := \{y+L(y) : y \in W_\alpha\}$, is an inverse atlas on $G_k(\mathbb{C}^n)$ (8).

We will use the following result:

THEOREM 5.1. *Let $f : X \to \mathbb{R}^m$ be definable in some o-minimal structure, where $X \subset \mathbb{R}^p$. For each $j \in \{0,1,\dots,p\}$ put $A_j := \{x \in \mathbb{R}^m : \dim f^{-1}(x) = j\}$. Then*

$$\dim f^{-1}(A_j) = \dim A_j + j.$$

Proof. Cf. [3, p. 66].

THEOREM 5.2. *Suppose that $B \subset \mathbb{C}^n$ is definable (as a subset of $\mathbb{R}^{2n}$) in some o-minimal structure and $\dim B \le 2(n-k)$. Then there exists a positive integer d and a set $S \subset G_k(\mathbb{C}^n)$ of measure zero such that $\#(H\cap B) \le d$ for any $H \in G_k(\mathbb{C}^n)\setminus S$.*

Proof. It is enough to show that for any $\alpha \in J$ we can find a positive integer d_α and a subset $S_\alpha \subset \mathcal{L}(W_\alpha, V_\alpha)$ of measure zero such that $\#(\widehat{L}\cap B) \le d_\alpha$ whenever $L \in \mathcal{L}(W_\alpha, V_\alpha)\setminus S_\alpha$. We will prove this for $V_\alpha = \{0\}\times\mathbb{C}^{n-k} \subset \mathbb{C}^n$ (the general case is analogous). Put $V := V_\alpha$, $W := W_\alpha = \mathbb{C}^k \times \{0\} \subset \mathbb{C}^n$. Define

$$F : \mathbb{C}^k \times \mathcal{L}(W,V) \ni (u,L) \mapsto (u,0) + L(u,0) \in \mathbb{C}^n.$$

Since $\mathbb{C}^k = \mathbb{R}^{2k}$ and $\mathcal{L}(W,V)$ can be identified with $\mathbb{R}^{2k(n-k)}$, $F^{-1}(B)$ may be regarded as a subset of $\mathbb{R}^{2k}\times\mathbb{R}^{2k(n-k)}$. Obviously, $F^{-1}(B)$ is then definable, for F and B are. For fixed $u \in \mathbb{C}^k$ we have the linear mapping

$$h_u : \mathcal{L}(W,V) \ni L \mapsto L(u,0) \in V.$$

Note the following facts:

(1) If $u \neq 0$, then for any $v \in V$ the set $h_u^{-1}(v)$ is a $(k-1)(n-k)$-dimensional affine complex subspace.
(2) $h_0^{-1}(0) = \mathcal{L}(W,V)$ and $h_0^{-1}(v) = \emptyset$ if $v \neq 0$.

(8) $\mathcal{L}(X,Y)$ denotes the space of all linear maps $X \to Y$.

It is easy to see that for any $x = (x', x'') \in \mathbb{C}^k \times \mathbb{C}^{n-k} = \mathbb{C}^n$ we have $F^{-1}(x) = \{x'\} \times h_{x'}^{-1}(0, x'')$. Therefore

$$\dim F^{-1}(x) \le 2(k-1)(n-k) \quad \text{if } x \neq 0, \quad \dim F^{-1}(0) = 2k(n-k).\ (^9)$$

By Theorem 5.1, we get $\dim F^{-1}(B \setminus \{0\}) \le 2k(n-k)$. Combining this with the fact that $\dim F^{-1}(0) = 2k(n-k)$ we have

$$\dim F^{-1}(B) \le 2k(n-k).$$

Hence there exists a positive integer s and a nowhere dense and definable ([10]) set $Z \subset \mathcal{L}(W, V)$ such that $\#F^{-1}(B)_L \le s$ for any $L \in \mathcal{L}(W, V) \setminus Z$, where $F^{-1}(B)_L$ is the fibre of $F^{-1}(B)$ over L. Consider the bijection

$$g_L : \mathbb{C}^k \ni u \mapsto (u, 0) + L(u, 0) \in \widehat{L}.$$

Since $F^{-1}(B)_L = g_L^{-1}(\widehat{L} \cap B)$, we obtain

$$\#F^{-1}(B)_L = \#(\widehat{L} \cap B).$$

To summarize, for any $L \in \mathcal{L}(W, V) \setminus Z$ we have $\#(\widehat{L} \cap B) \le s$.

THEOREM 5.3. *Let $X \subset \mathbb{C}^n$. Suppose that for each $x \in X$ there exists an open neighbourhood $U_x \subset \mathbb{C}^n$ of x such that $U_x \cap X$ is definable in some o-minimal structure (which may depend on x) and* $\dim(U_x \cap X) \le 2(n-k)$. *Then there exists a subset $S \subset G_k(\mathbb{C}^n)$ of measure zero such that $H \cap X$ is a discrete set for any $H \in G_k(\mathbb{C}^n) \setminus S$.*

Proof. It is enough to use the previous theorem and the following facts:

- X is a second-countable space,
- a countable union of sets of measure zero is of measure zero.

REMARK 5.4. Theorem 2.5 follows immediately from Theorem 5.3.

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([9]) Here dim denotes the dimension of a definable set.

([10]) And thus of measure zero.

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