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Lifting adapted connections from foliated manifolds to higher order adapted frame bundles

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Abstract. Let $(M,\mathcal{F})$ be a foliated manifold. We describe all natural operators $\mathcal{A}$ lifting $\mathcal{F}$-adapted (i.e. projectable in adapted coordinates) classical linear connections ∇ on $(M,\mathcal{F})$ into classical linear connections $\mathcal{A}(\nabla)$ on the rth order adapted frame bundle $P^r(M,\mathcal{F})$.

Introduction. All manifolds and maps are assumed to be of class $\mathcal{C}^\infty$.

A *classical linear connection* on a manifold N is a $\mathbb{R}$-bilinear map $\nabla : \mathcal{X}(N)\times\mathcal{X}(N)\to\mathcal{X}(N)$, where $\mathcal{X}(N)$ is the vector space of all vector fields on N, such that (1) $\nabla_{fX}Y = f\nabla_X Y$ and (2) $\nabla_X fY = XfY + f\nabla_X Y$ for any vector fields $X,Y\in\mathcal{X}(N)$ and all maps $f : N\to\mathbb{R}$. It is well-known that classical linear connections ∇ on N are in canonical bijection with sections $\nabla : N\to QN$ of the so-called *connection bundle* $Q(N) = (\mathrm{id}_{T^*N}\otimes\pi_1)^{-1}(\mathrm{id}_{TN})\subset T^*N\otimes J^1TN$ over N, where $\pi_1 : J^1TN\to TN$ is the jet projection from the first jet prolongation J^1TN of the tangent bundle TN. Every local diffeomorphism $f : N_1\to N_2$ induces canonically a fibred local diffeomorphism $Qf : Q(N_1)\to Q(N_2)$ covering f (the restriction of $T^*f\otimes J^1Tf$).

Let $\mathcal{F}ol_{m,n}$ be the category of all $m+n$-dimensional foliated manifolds $(M,\mathcal{F})$ with n-dimensional foliations and their foliation respecting local diffeomorphisms. Given a $\mathcal{F}ol_{m,n}$-object $(M,\mathcal{F})$ we have the rth order *adapted frame bundle*

$$P^r(M,\mathcal{F}) = \{j^r_0\varphi \mid \varphi : (\mathbb{R}^{m+n},\mathcal{F}^{m,n})\to(M,\mathcal{F}) \text{ is a } \mathcal{F}ol_{m,n}\text{-map}\}$$

over M of $(M\,\mathcal{F})$, where $\mathcal{F}^{m,n} = \{\{a\}\times\mathbb{R}^n\}_{a\in\mathbb{R}^m}$ is the standard n-dimensional foliation on $\mathbb{R}^{m+n}$. We see that $P^r(M,\mathcal{F})$ is a principal fibre bundle with the standard group $G^r_{m,n} = P^r(\mathbb{R}^{m,n},\mathcal{F}^{m,n})_0$ (with multiplication being composition of jets) acting on the right on $P^r(M,\mathcal{F})$ by the

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composition of jets. Every $\mathcal{F}ol_{m,n}$-map $\psi : (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$ induces a local fibred diffeomorphism (even a principal bundle local isomorphism) $P^r\psi : P^r(M_1, \mathcal{F}_1) \to P^r(M_2, \mathcal{F}_2)$ over ψ given by $P^r\psi(j^r_0\varphi) = j^r_0(\psi \circ \varphi)$, $j^r_0\varphi \in P^r(M_1, \mathcal{F}_1)$.

Let $(M, \mathcal{F})$ be a $\mathcal{F}ol_{m,n}$-object. A vector field X on M is called an *infinitesimal automorphism* of $(M, \mathcal{F})$ if the flow $\{\operatorname{Expt} X\}$ of X is formed by (locally defined) $\mathcal{F}ol_{m,n}$-maps $(M, \mathcal{F}) \to (M, \mathcal{F})$. Equivalently, a vector field on M is an infinitesimal automorphism of $(M, \mathcal{F})$ iff $[X, Y]$ is tangent to $\mathcal{F}$ for any vector field Y on M tangent to $\mathcal{F}$. We denote by $\mathcal{X}(M, \mathcal{F})$ the Lie algebra of all infinitesimal automorphisms of $(M, \mathcal{F})$.

A classical linear connection ∇ on M is called *$\mathcal{F}$-adapted* if $\nabla_X Y \in \mathcal{X}(M, \mathcal{F})$ for any $X, Y \in \mathcal{X}(M, \mathcal{F})$ and $\nabla_U W$ is tangent to $\mathcal{F}$ for any $U, W \in \mathcal{X}(M, \mathcal{F})$ with U or W tangent to $\mathcal{F}$. (We observe in the proof of Lemma 1 that ∇ is $\mathcal{F}$-adapted iff it is projectable in adapted coordinates.)

In the present paper we study how an $\mathcal{F}$-adapted classical linear connection ∇ on a $\mathcal{F}ol_{m,n}$-object $(M, \mathcal{F})$ can induce a classical linear connection $\mathcal{A}(\nabla)$ on the rth order adapted frame bundle $P^r(M, \mathcal{F})$. This problem is reflected in the concept of $\mathcal{F}ol_{m,n}$-natural operators $\mathcal{A} : Q_{\mathcal{F}ol_{m,n}} \rightsquigarrow QP^r$ in the sense of [1]. We recall that a *$\mathcal{F}ol_{m,n}$-natural operator* $\mathcal{A} : Q_{\mathcal{F}ol_{m,n}} \rightsquigarrow QP^r$ is a family of $\mathcal{F}ol_{m,n}$-invariant regular operators (functions)

$$\mathcal{A} = \mathcal{A}_{(M,\mathcal{F})} : Q_{\mathcal{F}ol}(M, \mathcal{F}) \to Q(P^r(M, \mathcal{F}))$$

for any $\mathcal{F}ol_{m,n}$-object $(M, \mathcal{F})$, where $Q_{\mathcal{F}ol}(M, \mathcal{F})$ is the set of all $\mathcal{F}$-adapted classical linear connections on $(M, \mathcal{F})$ and $Q(P^r(M, \mathcal{F}))$ is the set of all classical linear connections on $P^r(M, \mathcal{F})$. The invariance means that if $\nabla_1 \in Q_{\mathcal{F}ol}(M_1, \mathcal{F}_1)$ and $\nabla_2 \in Q_{\mathcal{F}ol}(M_2, \mathcal{F}_2)$ are φ-related for a $\mathcal{F}ol_{m,n}$-map $\varphi : (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$, then $\mathcal{A}(\nabla_1)$ and $\mathcal{A}(\nabla_2)$ are $P^r\varphi$-related. The regularity means that $\mathcal{A}$ transforms smoothly parametrized families of $\mathcal{F}$-adapted classical linear connections into smoothly parametrized families of classical linear connections.

In Section 1 we give an example $\mathcal{A}^0 : Q_{\mathcal{F}ol_{m,n}} \rightsquigarrow QP^r$ of such a $\mathcal{F}ol_{m,n}$-natural operator. Then we have

Theorem 1. *Any $\mathcal{F}ol_{m,n}$-natural operator $\mathcal{A} : Q_{\mathcal{F}ol_{m,n}} \to QP^r$ is of the form*

$$\mathcal{A}(\nabla) = \mathcal{A}^0(\nabla) + \mathcal{C}(\nabla)$$

for some unique $\mathcal{F}ol_{m,n}$-natural operator $\mathcal{C} : Q_{\mathcal{F}ol_{m,n}} \to (T^* \otimes T^* \otimes T)P^r$ transforming $\mathcal{F}$-adapted classical linear connections ∇ on $(M, \mathcal{F})$ into tensor fields $\mathcal{C}(\nabla)$ of type $(1, 2)$ on $P^r(M, \mathcal{F})$.

In Sections 2 and 4 we describe explicitly all $\mathcal{F}ol_{m,n}$-natural operators $\mathcal{C} : Q_{\mathcal{F}ol_{m,n}} \rightsquigarrow (T^* \otimes T^* \otimes T)P^r$. (The definition of these operators is a direct modification of the one for $\mathcal{F}ol_{m,n}$-natural operators $Q_{\mathcal{F}ol_{m,n}} \rightsquigarrow QP^r$.)

In the case $n = 0$ we observe that $\mathcal{F}ol_{m,0}$ is (in an obvious way) equivalent to the category $\mathcal{M}f_m$ of m-dimensional manifolds and their local diffeomorphisms. Namely, we identify an m-manifold M (an $\mathcal{M}f_m$-object) with the foliated manifold $(M, \{\{a\}\}_{a\in M})$ foliated by points (a $\mathcal{F}ol_{m,0}$-object). Clearly, the bundle $P^r(M, \{\{a\}\}_{a\in M})$ is (again in an obvious way) equivalent to the rth order frame bundle $P^rM = \operatorname{inv} J^r_0(\mathbb{R}^m, M)$. Moreover, the $\{\{a\}\}_{a\in M}$-adapted classical linear connections ∇ on $(M, \{\{a\}\}_{a\in M})$ are exactly the classical linear connections ∇ on M. Thus in the case $n = 0$ the result of the present paper (almost) coincides with the result of the second author in [2]. In other words, the present paper is an (almost) extension of the result from [2] to foliated manifolds. We write "almost" because if ∇ is a classical linear connection on M then we have the complete lift ∇^C of ∇ to $P^r(M)$ (we note that $P^r(M)$ is an open subset in $T^r_mM = J^r_0(\mathbb{R}^m, M)$ and we have the restriction of the complete lift ∇^C of ∇ to the m^r-velocities bundle T^r_mM of M in the sense of Morimoto [3]). In [2], we use ∇^C instead of $\mathcal{A}^0(\nabla)$.

Up till now, for $n \geq 1$ no connections $\mathcal{A}(\nabla)$ on $P^r(M, \mathcal{F})$ coming from an $\mathcal{F}$-adapted one ∇ on $(M, \mathcal{F})$ have been known. Thus the first main difficulty of the present paper is to construct a connection $\mathcal{A}^0(\nabla)$ on $P^r(M, \mathcal{F})$ from an $\mathcal{F}$-adapted classical linear connection ∇ on $(M, \mathcal{F})$. In the construction of $\mathcal{A}^0(\nabla)$ we will use Lemma 1, which we will also apply many times in next sections.

In the last section we present an alternative version of the main result.

1. An example of a $\mathcal{F}ol_{m,n}$-natural operator $\mathcal{A}^0 : Q_{\mathcal{F}ol_{m,n}} \rightsquigarrow QP^r$. To present such an example we need the following lemma.

LEMMA 1. *Let ∇ be an $\mathcal{F}$-adapted classical linear connection on a $\mathcal{F}ol_{m,n}$-object $(M, \mathcal{F})$. Let $p = j^r_0\varphi \in (P^r(M, \mathcal{F}))_x$, $x \in M$. There exists a unique (germ of) ∇-normal coordinate system $\psi^{\nabla,p}$ on M with centre x such that $\psi^{\nabla,p} : (M, \mathcal{F}) \to (\mathbb{R}^{m+n}, \mathcal{F}^{m,n})$ is a locally defined $\mathcal{F}ol_{m,n}$-map and we have $P^1(\psi^{\nabla,p})(j^1_0\varphi) = j^1_0(\mathrm{id}_{\mathbb{R}^{m+n}})$.*

Proof. Clearly, for a vector field $X \in \mathcal{X}(\mathbb{R}^{m+n})$ we have $X \in \mathcal{X}(\mathbb{R}^{m+n}, \mathcal{F}^{m+n})$ iff the flow of X is formed by local fibred isomorphisms of the trivial bundle $q : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$, or (equivalently) iff X is q-projectable (i.e. there exists a unique vector field $\underline{X}$ on $\mathbb{R}^m$ q-related to X). Then a classical linear connection ∇ on $\mathbb{R}^{m+n}$ is $\mathcal{F}^{m,n}$-adapted iff ∇_XY is q-projectable for any q-projectable vector fields X, Y on $\mathbb{R}^m \times \mathbb{R}^n$ and ∇_UW is vertical if U, V are q-projectable and U or W is vertical, or (equivalently) iff ∇ is q-projectable (i.e. there exists a unique classical linear connection $\underline{\nabla}$ on $\mathbb{R}^m$ q-related to ∇).

We can assume that $(M, \mathcal{F}) = (\mathbb{R}^{m+n}, \mathcal{F}^{m,n})$, $x = 0$ and $j^1_0\varphi = j^1_0(\mathrm{id}_{\mathbb{R}^{m+n}})$. By the above considerations, the exponent $\mathrm{Exp}_{\nabla,0}$ of ∇ at $0 \in \mathbb{R}^{m+n}$ is

q-related to $\mathrm{Exp}_{\underline{\nabla},0}$ of $\underline{\nabla}$ at $0 \in \mathbb{R}^m$ (because ∇-geodesics project via q onto $\underline{\nabla}$-geodesics as ∇ and $\underline{\nabla}$ are q-related and q is a surjective submersion). Then $\psi^{\nabla,p} = \mathrm{Exp}_{\nabla,0}^{-1}$ is a unique ∇-normal coordinate system on $\mathbb{R}^{m+n}$ with centre 0 such that $P^1(\psi^{\nabla,p})(j_0^1(\mathrm{id})) = j_0^1(\mathrm{id})$ and $\psi^{\nabla,p} : (\mathbb{R}^{m+n}, \mathcal{F}^{m,n}) \to (\mathbb{R}^{m+n}, \mathcal{F}^{m,n})$ is a locally defined $\mathcal{F}ol_{m,n}$-map. ∎

We are in a position to present an example of a $\mathcal{F}ol_{m,n}$-natural operator $\mathcal{A}^0 : Q_{\mathcal{F}ol_{m,n}} \rightsquigarrow QP^r$. We fix an arbitrary classical linear connection λ^0 on $P^r(\mathbb{R}^{m+n}, \mathcal{F}^{m,n})$ (a section of the connection bundle $Q(P^r(\mathbb{R}^{m+n}, \mathcal{F}^{m,n})) \to P^r(\mathbb{R}^{m+n}, \mathcal{F}^{m,n})$).

EXAMPLE 1. Let $\nabla \in Q(M, \mathcal{F})$ be an $\mathcal{F}$-adapted classical linear connection on a $\mathcal{F}ol_{m,n}$-object $(M, \mathcal{F})$. We define a classical linear connection $\mathcal{A}^0(\nabla)$ on $P^r(M, \mathcal{F})$ (a section $\mathcal{A}^0(\nabla) : P^r(M, \mathcal{F}) \to Q(P^r(M, \mathcal{F}))$) as follows. Let $p = j_x^r\varphi \in (P^r(M, \mathcal{F}))_x$, $x \in M$. Let $\psi^{\nabla,p}$ be the unique (germ of) ∇-normal coordinate system on M with centre x such that $\psi^{\nabla,p} : (M, \mathcal{F}) \to (\mathbb{R}^{m+n}, \mathcal{F}^{m,n})$ is a locally defined $\mathcal{F}ol_{m,n}$-map and $P^1(\psi^{\nabla,p})(j_0^1\varphi) = j_0^1(\mathrm{id})$ (see Lemma 1). We define

$$\mathcal{A}^0(\nabla)(p) := Q(P^r((\psi^{\nabla,p})^{-1}))(\lambda^0(P^r(\psi^{\nabla,p})(p))).$$

This definition is correct because the germ of $\psi^{\nabla,p}$ at x is uniquely determined and we can apply the functor P^r to $\psi^{\nabla,p}$ because $\psi^{\nabla,p}$ is a $\mathcal{F}ol_{m,n}$-map. Therefore the family $\mathcal{A}^0 : Q_{\mathcal{F}ol_{m,n}} \rightsquigarrow QP^r$ is a $\mathcal{F}ol_{m,n}$-natural operator.

2. The $\mathcal{F}ol_{m,n}$-natural operators $Q_{\mathcal{F}ol_{m,n}} \rightsquigarrow T^{(0,0)}P^r$. Let

$$\theta := j_0^1(\mathrm{id}_{\mathbb{R}^{m+n}}) \in (P^1(\mathbb{R}^{m+n}, \mathcal{F}^{m,n}))_0.$$

Let $(P^r(\mathbb{R}^{m+n}, \mathcal{F}^{m,n}))_\theta = \{j_0^r\varphi \in (P^r(\mathbb{R}^{m+n}, \mathcal{F}^{m,n}))_0 \mid j_0^1\varphi = \theta\}$. Let S^s ($s \in \mathbb{N} \cup \{\infty\}$) be the vector space of all s-jets at $0 \in \mathbb{R}^{m+n}$ of $\mathcal{F}^{m,n}$-adapted classical linear connections ∇ on $(\mathbb{R}^{m+n}, \mathcal{F}^{m,n})$ (or equivalently, of all s-jets at $0 \in \mathbb{R}^{m+n}$ of projectable classical linear connections on the trivial bundle $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$) given by the Christoffel symbols $\Gamma^i_{jk} : \mathbb{R}^{m+n} \to \mathbb{R}$ satisfying

$$\sum_{j,k=1}^{m+n} \Gamma^i_{jk}(x)x^jx^k = 0 \quad \text{for } i = 1, \ldots, m+n.$$

Equivalently, S^s is the space of all s-jets at 0 of $\mathcal{F}^{m,n}$-adapted classical linear connections ∇ on $(\mathbb{R}^{m+n}, \mathcal{F}^{m,n})$ such that the usual coordinate system $x^1, \ldots, x^{m+n}$ on $\mathbb{R}^{m+n}$ is a normal coordinate system for ∇ with centre 0. (The equivalence is almost clear if we remember the well-known differential equations for geodesics and apply the fact that ∇-geodesics passing through the centre of ∇-normal coordinates are straight lines in these coordinates.)

Let us consider a function $\mu : S^\infty \times (P^r(\mathbb{R}^{m+n}, \mathcal{F}^{m,n}))_\theta \to \mathbb{R}$ satisfying the following local finite determination property:

($*$) For any $\varrho \in S^\infty$ and $\sigma \in (P^r(\mathbb{R}^{m+n}, \mathcal{F}^{m,n}))_\theta$, we can find an open neighbourhood $V \subset (P^r(\mathbb{R}^{m+n}, \mathcal{F}^{m,n}))_\theta$ of σ, an open neighbourhood $U \subset S^\infty$ of ϱ, a natural number s and a smooth map $f : \pi_s(U) \times V \to \mathbb{R}$ such that $\mu = f \circ (\pi_s \times \mathrm{id}_V)$ on $U \times V$, where $\pi_s : J^\infty \to J^s$ is the jet projection.

An example of a μ with property ($*$) is the pull-back of a map $\widetilde{\mu} : S^s \times (P^r(\mathbb{R}^{m+n}, \mathcal{F}^{m,n}))_\theta \to \mathbb{R}$ for finite s with respect to the projection $\pi_s \times \mathrm{id} : S^\infty \times (P^r(\mathbb{R}^{m+n}, \mathcal{F}^{m,n}))_\theta \to S^s \times (P^r(\mathbb{R}^{m,n}, \mathcal{F}^{m,n}))_\theta$.

EXAMPLE 2. Given an $\mathcal{F}$-adapted classical linear connection ∇ on a $\mathcal{F}ol_{m,n}$-object $(M, \mathcal{F})$, we define a smooth map $\mathcal{B}^{\langle\mu\rangle}(\nabla) : P^r(M, \mathcal{F}) \to \mathbb{R}$ by

$$\mathcal{B}^{\langle\mu\rangle}(\nabla)(p) := \mu(j^\infty_0(\psi^{\nabla,p}_*\nabla), P^r(\psi^{\nabla,p})(p)) ,$$

$p = j^r_0\varphi \in (P^r(M, \mathcal{F}))_x$, $x \in M$, where $\psi^{\nabla,p} : (M, \mathcal{F}) \to (\mathbb{R}^{m+n}, \mathcal{F}^{m,n})$ is as in Lemma 1 for ∇ and p. The definition is correct because $\mathrm{germ}_x(\psi^{\nabla,p})$ is uniquely determined. The correspondence $\mathcal{B}^{\langle\mu\rangle} : Q_{\mathcal{F}ol_{m,n}} \rightsquigarrow T^{(0,0)}P^r$ is a $\mathcal{F}ol_{m,n}$-natural operator transforming $\mathcal{F}$-adapted classical linear connections on $\mathcal{F}ol_{m,n}$-objects $(M, \mathcal{F})$ into maps $\mathcal{B}^{\langle\mu\rangle}(\nabla) : P^r(M, \mathcal{F}) \to \mathbb{R}$. (The definition of $\mathcal{F}ol_{m,n}$-natural operators $Q_{\mathcal{F}ol_{m,n}} \rightsquigarrow T^{(0,0)}P^r$ is a direct modification of the one of $\mathcal{F}_{m,n}$-natural operators $Q_{\mathcal{F}ol_{m,n}} \rightsquigarrow QP^r$.)

PROPOSITION 1. *Any $\mathcal{F}ol_{m,n}$-natural operator $\mathcal{B} : Q_{\mathcal{F}ol_{m,n}} \rightsquigarrow T^{(0,0)}P^r$ is equal to $\mathcal{B}^{\langle\mu\rangle}$ for a unique map $\mu : S^\infty \times (P^r(\mathbb{R}^{m+n}, \mathcal{F}^{m,n}))_\theta \to \mathbb{R}$ with the above-mentioned property.*

Proof. Let $\mathcal{B}$ be an operator in question. Define a map $\mu : S^\infty \times (P^r(\mathbb{R}^{m+m}, \mathcal{F}^{m,n}))_\theta \to \mathbb{R}$ by

$$\mu(j^\infty_0(\nabla), \sigma) = \mathcal{B}(\nabla)_\sigma .$$

Then by the non-linear Peetre theorem [1], μ has property ($*$). Clearly, $\mathcal{B} = \mathcal{B}^{\langle\mu\rangle}$. ∎

3. The parallelism on $P^r(M, \mathcal{F})$ from an $\mathcal{F}$-adapted classical linear connection ∇ on $(M, \mathcal{F})$. Let ∇ be an $\mathcal{F}$-adapted classical linear connection ∇ on $(M, \mathcal{F})$.

EXAMPLE 3. For $i = 1, \ldots, m+n$, we define a vector field $A^i(\nabla)$ on $P^r(M, \mathcal{F})$ as follows. Let $p = j^r_0\varphi \in (P^r(M, \mathcal{F}))_x$, $x \in M$. Let $\psi^{\nabla,p} : (M, \mathcal{F}) \to (\mathbb{R}^{m+n}, \mathcal{F}^{m,n})$ be as in Lemma 1 for ∇ and p. We put

$$A^i(\nabla)(p) = TP^r((\psi^{\nabla,p})^{-1})\left(\mathcal{P}^r\left(\frac{\partial}{\partial x^i}\right)(P^r(\psi^{\nabla,p})(p))\right),$$

where $\partial/\partial x^i$ are the canonical vector fields on $\mathbb{R}^{m+n}$ (they are infinitesimal automorphisms of $(\mathbb{R}^{m+n}, \mathcal{F}^{m,n})$) and where $\mathcal{P}^r X$ means the flow lifting of an infinitesimal automorphism $X \in \mathcal{X}(N, \mathcal{F}_1)$ to $P^r(N, \mathcal{F}_1)$. ($\mathcal{P}^r X$ is given by the flow $\{P^r(\operatorname{Expt} X)\}$, where $\{\operatorname{Expt} X\}$ is the flow of X. We can apply the functor P^r to $\operatorname{Expt} X$ because $\operatorname{Expt} X$ is a $\mathcal{F}ol_{m,n}$-map as X is an infinitesimal automorphism.) The correspondence $A^i : Q_{\mathcal{F}ol_{m,n}} \rightsquigarrow TP^r$ is a $\mathcal{F}ol_{m,n}$-natural operator (transforming $\mathcal{F}$-adapted classical linear connections on $(M,\mathcal{F})$ into vector fields on $P^r(M,\mathcal{F})$).

Example 4. Let $G^r_{m,n} = P^r(\mathbb{R}^{m+n},\mathcal{F}^{m,n})_0$ be the Lie group as in the Introduction. Let $\{E_\alpha\}$ be a basis in the Lie algebra $\mathcal{L}(G^r_{m,n})$. Let $(E_\alpha)^*$ be the fundamental vector field corresponding to E_α on the principal $G^r_{m,n}$-bundle $P^r(M,\mathcal{F})$. The correspondence $(E_\alpha)^* : Q_{\mathcal{F}ol_{m,n}} \rightsquigarrow TP^r$ is a $\mathcal{F}ol_{m,n}$-natural operator (independent of ∇).

Proposition 2. *Given an $\mathcal{F}$-adapted classical linear connection ∇ on $(M,\mathcal{F})$, the vector fields $A^i(\nabla)$ and $(E_\alpha)^*$ for $i = 1,\dots,m+n$ and $\alpha = 1,\dots,\dim(G^r_{m,n})$ form a basis over $\mathcal{C}^\infty(P^r(M,\mathcal{F}))$ of the module of all vector fields on $P^r(M,\mathcal{F})$.*

Proof. This is a simple observation. ■

4. The $\mathcal{F}ol_{m,n}$-natural operators $Q_{\mathcal{F}ol_{m,n}} \rightsquigarrow (T^*\otimes T^*\otimes T)P^r$. The space of all $\mathcal{F}ol_{m,n}$-natural operators $Q_{\mathcal{F}ol_{m,n}} \rightsquigarrow (T^*\otimes T^*\otimes T)P^r$ transforming $\mathcal{F}$-adapted classical linear connections on $(M,\mathcal{F})$ into tensor fields of type $(1,2)$ on $P^r(M,\mathcal{F})$ is (in an obvious way) a module over the algebra of all $\mathcal{F}ol_{m,n}$-natural operators $Q_{\mathcal{F}ol_{m,n}} \rightsquigarrow T^{(0,0)}P^r$ (classified in Section 2).

Proposition 3. *The module of all $\mathcal{F}ol_{m,n}$-natural operators $Q_{\mathcal{F}ol_{m,n}} \rightsquigarrow (T^*\otimes T^*\otimes T)P^r$ is free and $(m+n+\dim(G^r_{m,n}))^3$-dimensional. All (suitable) tensor products of $A^i, (E_\alpha)^*, (A^i)^D$ and $((E_\alpha)^*)^D$ form a basis in this module, where given an $\mathcal{F}$-adapted classical linear connection ∇ on $(M,\mathcal{F})$, $(A^i(\nabla),(E_\alpha)^*)$ is a basis of vector fields on $P^r(M,\mathcal{F})$ as in Section 3 and $(A^i(\nabla)^D,((E_\alpha)^*)^D)$ is the dual basis of 1-forms on $P^r(M,\mathcal{F})$.*

Proof. Let $\mathcal{C} : Q_{\mathcal{F}ol_{m,n}} \rightsquigarrow (T^*\otimes T^*\otimes T)P^r$ be a $\mathcal{F}ol_{m,n}$-natural operator. For any $\mathcal{F}$-adapted classical linear connection ∇ on $(M,\mathcal{F})$ we can write

$$\mathcal{C}(\nabla) = \sum_k \lambda_k(\nabla)F^k(\nabla),$$

where $(F^k(\nabla))$ is the obvious basis of $(1,2)$-tensor fields on $P^r(M,\mathcal{F})$ induced by the basis $(A^i(\nabla),(E_\alpha)^*)$, and the maps $\lambda_k(\nabla) : P^r(M,\mathcal{F}) \to \mathbb{R}$ are uniquely determined. Because of the invariance of $\mathcal{C}$ with respect to $\mathcal{F}ol_{m,n}$-maps, $\lambda_k : Q_{\mathcal{F}ol_{m,n}} \rightsquigarrow T^{(0,0)}P^r$ are $\mathcal{F}ol_{m,n}$-natural operators. ■

5. Another version of the main theorem. We end this note by the following alternative description of all $\mathcal{F}ol_{m,n}$-natural operators $\mathcal{A} : Q_{\mathcal{F}ol_{m,n}} \rightsquigarrow QP^r$.

We use the notation of Section 2. Let $\nu : S^\infty \times (P^r(\mathbb{R}^{m+n}, \mathcal{F}^{m,n}))_\theta \to Q(P(\mathbb{R}^{m+n}, \mathcal{F}^{m,n}))$ be a map such that a local finite determination property quite similar to $(*)$ for μ from Section 2 is satisfied. Additionally we assume that $\pi \circ \nu(\sigma, p) = p$ for any (σ, p), where π is the projection of the connection bundle.

EXAMPLE 5. Let ∇ be an $\mathcal{F}$-adapted classical linear connection on a $\mathcal{F}ol_{m,n}$-object $(M, \mathcal{F})$. We define a classical linear connection $\mathcal{A}^{\langle\nu\rangle}(\nabla)$ on $P^r(M, \mathcal{F})$ as follows. Let $p = j^r_0\varphi \in (P^r(M, \mathcal{F}))_x$, $x \in M$. Let $\psi^{\nabla,p} : (M, \mathcal{F}) \to (\mathbb{R}^{m+n}, \mathcal{F}^{m,n})$ be as in Lemma 1 for ∇ and p. We put

$$\mathcal{A}^{\langle\nu\rangle}(\nabla)(p) = Q(P^r((\psi^{\nabla,p})^{-1}))(\nu(j^\infty_0(\psi^{\nabla,p}_*\nabla), P^r(\psi^{\nabla,p})(p))).$$

Clearly, $\mathcal{A}^{\langle\nu\rangle} : Q_{\mathcal{F}ol_{m,n}} \rightsquigarrow QP^r$ is a $\mathcal{F}ol_{m,n}$-natural operator.

THEOREM 2. *Any $\mathcal{F}ol_{m,n}$-natural operator $\mathcal{A} : Q_{\mathcal{F}ol_{m,n}} \rightsquigarrow QP^r$ is equal to $\mathcal{A}^{\langle\nu\rangle}$ for some ν as in Example* 5.

Proof. The proof is quite similar to the one of Proposition 1. ∎

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